TIME-WEIGHTED PERFORMANCE-INDEX EVALUATION*

Consideration is given to the evaluation of performance indexes of the type \( \int_0^\infty e^{tf(t)}dt \) where \( f(t) = e_1(t)e_2(t) \), with \( e_1 \) and \( e_2 \) both possessing rational Laplace transforms. The method is algebraic in character and does not require factoring of the denominators of the Laplace transforms of \( e_1(t) \) and \( e_2(t) \).

Two recent correspondence items\(^1,2\) have dealt with the problem of evaluating performance integrals of the type

\[
J_k = \int_0^\infty t^k f(t) dt \quad k = 0, 1, 2 \ldots \quad (1)
\]

where \( f(t) \) has been identified with the square of a function representing an error:

\[
f(t) = e^2(t) \quad \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

the Laplace transform of \( e(t) \) being rational. Loo's method requires the carrying out of a contour integration, whereas Power's method requires the development of the Laplace transform of \( e^2(t) \) in partial fractions. Since Loo's method essentially requires this too, a characteristic of both procedures is that the denominator of the Laplace transform of \( e(t) \) or \( f(t) \) be factored.

The object of this letter is to indicate that no factoring need be done, unless desired. We present an algebraic approach to the evaluation of

\[
J_k = \int_0^\infty t^k e_1(t)e_2(t) dt \quad k = 0, 1, 2 \ldots \quad (3)
\]

where \( e_1 \) and \( e_2 \) are functions with known rational Laplace transforms. Of course, by taking \( e_1 = e_2 \), one recovers the earlier results.

The procedure constitutes an extension of a technique due to Macfarlane\(^3\) for the evaluation of the matrix

\[
\int_0^\infty e^{tQ} e^{tF} dt \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

where \( F \) and \( Q \) are square matrices of the same order, and \( F \) has eigenvalues in the left-hand halfplane.

Denoting by \( E_1(s) \) and \( E_2(s) \) the Laplace transforms of \( e_1(t) \) and \( e_2(t) \), we begin by finding matrices \( F_1 \) and \( F_2 \), and vectors \( g_1, g_2, h_1 \) and \( h_2 \) such that

\[
E_1(s) = h_1(sI - F_1)^{-1}g_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)
\]

and

\[
E_2(s) = h_2(sI - F_2)^{-1}g_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)
\]

It is important to realise that the desired matrices and vectors can be found by well known procedures (see, e.g., Reference 4) whether the denominators of \( E_1(s) \) and \( E_2(s) \) are factored or not. For simplicity, \( F_1 \) and \( F_2 \) should be taken to be of minimal dimension.

To guarantee that eqn. 3 is finite, we suppose that \( E_1(s) \) and \( E_2(s) \) have poles in the halfplane \( \text{Re}(s) < 0 \). Then the eigenvalues of the matrices \( F_1 \) and \( F_2 \) all possess negative real parts.

Now, in eqn. 3, we have

\[
J_k = \int_0^\infty t^k e^{tF_1}g_1 h_2 e^{F_2}g_2 dt \quad \ldots \ldots \ldots (7)
\]

(where we are not assuming the availability of an explicit expression for \( e^2(t) \)). Define the matrix

\[
L_k = \int_0^\infty t^k e^{tF_1}g_1 h_2 e^{F_2}g_2 dt \quad \ldots \ldots \ldots (8)
\]

so that

\[
J_k = h_1^T L_k h_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9)
\]

We can give a recursion formula for \( L_k \) as follows. From eqn. 8, we have, for \( k \geq 1 \),

\[
F_1 L_k + L_k F_2 = \int_0^\infty t^k (F_1 e^{tF_1} g_1 h_2 e^{F_2} g_2 + e^{tF_1} g_1 h_2 e^{F_2} F_2) dt
\]

\[
= \int_0^\infty \frac{d}{dt} (e^{tF_1} g_1 h_2 e^{F_2} g_2) dt
\]

\[
= \int_0^\infty \frac{d}{dt} (e^{tF_1} g_1 h_2 e^{F_2} g_2) dt
\]

\[
= k L_{k-1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10)
\]

Note that we have used the fact that the eigenvalues of \( F_1 \) and \( F_2 \) have negative real parts in the above sequence of equalities. For \( k = 0 \), it follows in a similar manner that

\[
F_1 L_0 + L_0 F_2 = -g_1 h_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (11)
\]

Eqn. 11, and then eqn. 10 for \( k = 1, 2 \ldots \) give successively \( L_0, L_1, L_2 \ldots \). Note that the fact that all eigenvalues of \( F_1 \) and \( F_2 \) possess negative real part guarantees the solvability of these equations by standard procedures,\(^2\) even when the dimensions of \( F_1 \) and \( F_2 \) differ.

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