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TIME-WEIGHTED PERFORMANCE-INDEX EVALUATION*

Consideration is given to the evaluation of performance indexes of the type $\int_0^\infty t^k f(t)dt$ where $f(t) = e_1(t)e_2(t)$, with e_1 and e_2 both possessing rational Laplace transforms. The method is algebraic in character and does not require factoring of the denominators of the Laplace transforms of $e_1(t)$ and $e_2(t)$.

Two recent correspondence items[1,2] have dealt with the problem of evaluating performance integrals of the type

$$J_k = \int_0^\infty t^k f(t)dt \qquad k = 0, 1, 2 \ldots \qquad (1)$$

where $f(t)$ has been identified with the square of a function representing an error:

$$f(t) = e^2(t) \qquad (2)$$

the Laplace transform of $e(t)$ being rational. Loo's method requires the carrying out of a contour integration, whereas Power's method requires the development of the Laplace transform of $e^2(t)$ in partial fractions. Since Loo's method essentially requires this too, a characteristic of both procedures is that the denominator of the Laplace transform of $e(t)$ or $f(t)$ be factored.

The object of this letter is to indicate that no factoring need be done, unless desired. We present an algebraic approach to the evaluation of

$$J_k = \int_0^\infty t^k e_1(t)e_2(t)dt \qquad k = 0, 1, 2 \ldots \qquad (3)$$

where e_1 and e_2 are functions with known rational Laplace transforms. Of course, by taking $e_1 = e_2$, one recovers the earlier results.

The procedure constitutes an extension of a technique due to Macfarlane[3] for the evaluation of the matrix

$$\int_0^\infty t^k e^{F't} Q e^{Ft} dt \qquad (4)$$

where F and Q are square matrices of the same order, and F has eigenvalues in the left-hand halfplane.

Denoting by $E_1(s)$ and $E_2(s)$ the Laplace transforms of $e_1(t)$ and $e_2(t)$, we begin by finding matrices F_1 and F_2, and vectors g_1, g_2, h_1 and h_2, such that

$$E_1(s) = h_1'(sI - F_1)^{-1}g_1 \qquad (5)$$

and $$E_2(s) = h_2'(sI - F_2)^{-1}g_2 \qquad (6)$$

It is important to realise that the desired matrices and vectors can be found by well known procedures (see, e.g., Reference 4) whether the denominators of $E_1(s)$ and $E_2(s)$ are factored or not. For simplicity, F_1 and F_2 should be taken to be of minimal dimension.

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To guarantee that eqn. 3 is finite, we suppose that $E_1(s)$ and $E_2(s)$ have poles in the halfplane $\mathrm{Re}\,(s) < 0$. Then the eigenvalues of the matrices F_1 and F_2 all possess negative real parts.

Now, in eqn. 3, we have

$$J_k = \int_0^\infty t^k h_1' e^{F_1 t} g_1 h_2' e^{F_2 t} g_2 dt \qquad (7)$$

(where we are not assuming the availability of an explicit expression for $e^{F_1 t}$). Define the matrix

$$L_k = \int_0^\infty t^k e^{F_1 t} g_1 h_2' e^{F_2 t} dt \qquad (8)$$

so that

$$J_k = h_1' L_k g_2 \qquad (9)$$

We can give a recursion formula for L_k as follows. From eqn. 8, we have, for $k \geqslant 1$,

$$\begin{aligned} F_1 L_k + L_k F_2 &= \int_0^\infty t^k (F_1 e^{F_1 t} g_1 h_2' e^{F_2 t} + e^{F_1 t} g_1 h_2' e^{F_2 t} F_2) dt \\ &= \int_0^\infty t^k \frac{d}{dt}(e^{F_1 t} g_1 h_2' e^{F_2 t}) dt \\ &= k \int_0^\infty t^{k-1} e^{F_1 t} g_1 h_2' e^{F_2 t} dt \\ &= k L_{k-1} \end{aligned} \qquad (10)$$

Note that we have used the fact that the eigenvalues of F_1 and F_2 have negative real parts in the above sequence of equalities. For $k = 0$, it follows in a similar manner that

$$F_1 L_0 + L_0 F_2 = -g_1 h_2' \qquad (11)$$

Eqn. 11, and then eqn. 10 for $k = 1, 2, \ldots$ give successively $L_0, L_1, L_2 \ldots$ Note that the fact that all eigenvalues of F_1 and F_2 possess negative real part guarantees the solvability of these equations by standard procedures,[5] even when the dimensions of F_1 and F_2 differ.

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