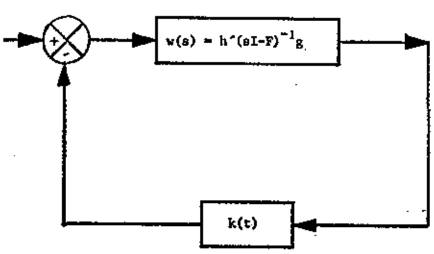

Fig. 1. Closed-loop system.

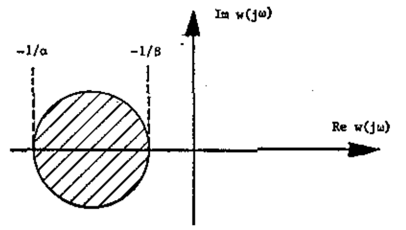

Fig. 2. The critical disk.

Liapunov Function Generation for a Class of Time-Varying Systems

Abstract—**Finite dimensional systems with time-varying feedback, which satisfy the conditions of the circle criterion for stability, are considered. A recent result giving a system theory description of positive real matrices is used to generate Liapunov functions.**

This correspondence considers the generation of Liapunov functions for systems in the form of Fig. 1. The forward part of the closed-loop system is itself a linear time-invariant finite-dimensional system, which is described by a transfer function $w(\cdot)$. The function $w(\cdot)$ is a ratio of two polynomials and will be assumed to satisfy:

$$w(s) = h'(sI - F)^{-1}g. \qquad (1)$$

In this equation, the pair $[F, g]$ is completely controllable, and the pair $[F, h]$ is completely observable. The time-varying feedback $k(t)$ is assumed to satisfy

$$\alpha < k(t) < \beta \qquad (2)$$

for some positive constants α and β.

The stability of such systems has been examined using circle criteria.[1]–[4] The criteria may be described with the aid of Fig. 2. We denote by $D(\alpha, \beta)$ the open disk with boundary points $-1/\alpha$ and $-1/\beta$ on the real axis. The associated closed disk is denoted by $D[\alpha, \beta]$, and the graph of $w(j\omega)$ is denoted by Γ. Then, the circle criteria are as follows:

The closed-loop system is stable in the sense that all sets of initial conditions lead to outputs which are bounded as t approaches infinity if:

1) Γ does not intersect the disk $D(\alpha, \beta)$ and encircles it ρ times in the counterclockwise direction where $w(\cdot)$ has ρ poles in the half plane Re $[s]>0$.

The closed-loop system is asymptotically stable in the sense that all sets of initial conditions lead to outputs that approach zero as t approaches infinity if:

2) Γ does not intersect the disk $D[\alpha, \beta]$ and encircles it ρ times in the counterclockwise direction.

Manuscript received October 3, 1967.

The complete observability of $[F, h]$ means that 1) is equivalent to stability in the sense of Liapunov, and that 2) is equivalent to asymptotic stability in the sense of Liapunov.

To generate Liapunov functions for a system satisfying one of the two circle criteria, we make use of a result in Brockett and Lee[4] (see Lemmas 3 and 4). This result yields that 1) may be replaced by

1a) the function $z(\cdot)$ where

$$z(s) = \frac{w(s) + \dfrac{1}{\alpha}}{w(s) + \dfrac{1}{\beta}} \qquad (3)$$

is positive real.

Likewise, condition 2) may be replaced by:

2a) The function $z(\cdot)$, as in (3) above is strictly positive real, i.e., $z(\cdot)$ has no $j\omega$ axis poles, and Re $z(j\omega) \geq \delta > 0$ for all ω and some positive δ.

Condition 2a) is equivalent to

2b) There exists a positive constant σ such that the function $\hat{z}(\cdot)$ given by

$$\hat{z}(s) = z(s - \sigma) \qquad (4)$$

is positive real. [This follows from the continuity of z and Re z away from the neighborhood of any pole of $z(\cdot)$.]

We shall generate a Liapunov function for the system of Fig. 1 by applying to $z(\cdot)$ or $\hat{z}(\cdot)$ the positive real lemma of Anderson.[5] In order to apply this lemma, it is necessary to determine a minimal realization[6] of z or $\hat{z}$. This may be done as follows.

By explicit calculation,

$$\frac{w(s) + \dfrac{1}{\alpha}}{w(s) + \dfrac{1}{\beta}} = \frac{h'(sI - F)^{-1}g + \dfrac{1}{\alpha}}{h'(sI - F)^{-1}g + \dfrac{1}{\beta}} = \frac{\beta}{\alpha}\left[1 - (\beta - \alpha)h'(sI - F + gh')^{-1}g\right]. \qquad (5)$$

Noting that if $z(s)$ is positive real, so is $(\alpha/\beta)z(s)$, we see that a realization for $(\alpha/\beta)z(s)$ is given by the quadruple $\{F - \beta gh', g, -(\beta - \alpha)h, 1\}$; then by Anderson,[5] there exists a matrix $P = P' \geq 0$, and a vector l such that

$$P(F - \beta gh') + (F' - \beta hg')P = -ll' \qquad (6a)$$

$$Pg = -(\beta - \alpha)h - \sqrt{2}l. \qquad (6b)$$

We claim that

$$V(x) = x'Px \qquad (7)$$

is a Liapunov function for the closed-loop system, when x is the system state vector satisfying

$$\dot{x} = Fx - gk(t)h'x. \qquad (8)$$

Clearly, $V(x)$ is positive definite. Also,

$$\begin{aligned}\dot{V} &= x'(PF + F'P)x - 2k(t)h'xx'Pg \\ &= -(l'x)^2 - 2[\beta - k(t)]h'x[(\beta - \alpha)(h'x) - \sqrt{2}(l'x)] \\ &= -\{l'x + \sqrt{2}[\beta - k(t)]h'x\}^2 - 2[\beta - k(t)][k(t) - \alpha](h'x)^2. \end{aligned} \qquad (9)$$

Evidently (2) guarantees the nonpositive nature of $\dot{V}$.

The determination of a Liapunov function establishing asymptotic stability when 2b) is satisfied proceeds similarly. In this case, we observe that a minimal realization for $\hat{z}(\cdot)$ is the same as a minimal realization for $z(\cdot)$ with F replaced by $F + \sigma I$. Equations (6) then hold with the same replacement of F by $F + \sigma I$. A Liapunov function is again given by (7), but the time derivative is now given by

$$\begin{aligned}\dot{V} = &-\{l'x + \sqrt{2}[\beta - k(t)]h'x\}^2 \\ &- 2[\beta - k(t)][k(t) - \alpha](h'x)^2 \\ &- 2\sigma x'Px. \end{aligned} \qquad (10)$$

We observe from (10) that in this case

$$\dot{V} \leq -2\sigma V \qquad (11)$$

which demonstrates the decay of the states of the system at a rate of at least $\exp[-\sigma t]$.

In summary, we have noted the interpretation of the circle criterion provided by Brockett and Lee[4] as a requirement that a certain function, derivable from the transfer function associated with Fig. 1, be positive real or strictly positive real. By then using the known result concerning the algebraic structure of positive real functions (see Anderson[5]), a tentative Liapunov function may be deduced. Some simple calculations then demonstrate that the choice of the tentative Liapunov functions is indeed a correct one, implying stability or asymptotic stability, as the case may be.

B. D. O. Anderson
J. B. Moore
Dept. of Elec. Engrg.
University of Newcastle,
New South Wales, Australia

REFERENCES

[1] K. S. Narendra and R. M. Goldwyn, "A geometrical criterion for the stability of certain nonlinear nonautonomous systems," *IEEE Trans. Circuit Theory*, vol. CT-11, pp. 406–408, September 1964.
[2] J. J. Bongiorno, "An extension of the Nyquist–Barkhausen stability criterion to linear lumped parameter systems with time-varying elements," *IEEE Trans. Automatic Control*, vol. AC-8, pp. 166–172, April 1963.
[3] I. W. Sandberg, "A frequency domain condition for the stability of systems containing a single time-varying nonlinear element," *Bell Sys. Tech. J.*, vol. 43, pp. 1601–1638, July 1964.
[4] R. W. Brockett and H. B. Lee, "Frequency domain instability criteria for time-varying and nonlinear systems," *Proc. IEEE*, vol. 55, pp. 604–619, May 1967.
[5] B. D. O. Anderson, "A system theory criterion for positive real matrices," *J. SIAM Control*, vol. 5, pp. 171–182, May 1967.

Reprinted from IEEE TRANSACTIONS ON *AUTOMATIC CONTROL*
Volume AC-13, Number 2, April, 1968
pp. 205-206