Liapunov Function Generation for a Class of Time-Varying Systems

Abstract—Finite dimensional systems with time-varying feedback, which satisfy the conditions of the circle criterion for stability, are considered. A recent result giving a system theory description of positive real matrices is used to generate Liapunov functions.

This correspondence considers the generation of Liapunov functions for systems in the form of Fig. 1. The forward part of the closed-loop system is itself a linear time-invariant finite-dimensional system, which is described by a transfer function \( w(\cdot) \). The function \( w(\cdot) \) is a ratio of two polynomials and will be assumed to satisfy

\[
w(s) = \frac{1}{\alpha}.
\]

In this equation, the pair \([F, g]\) is completely controllable, and the pair \([F, h]\) is completely observable. The time-varying feedback \( h(\cdot) \) is assumed to satisfy

\[
a < h(\cdot) < \beta
\]

for some positive constants \( a \) and \( \beta \).

The stability of such systems has been examined using circle criteria. The circle criterion may be described with the aid of Fig. 2. We denote by \( D(\alpha, \beta) \) the open disk with boundary points \(-1/\alpha\) and \(-1/\beta\) on the real axis. The associated closed disk is denoted by \( D[\alpha, \beta] \), and the graph of \( w(j\omega) \) is denoted by \( \Gamma \). Then, the circle criteria are as follows:

The closed-loop system is stable in the sense that all sets of initial conditions lead to outputs which are bounded as \( t \) approaches infinity if:

1) \( T \) does not intersect the disk \( D(\alpha, \beta) \) and encircles it \( n \) times in the counterclockwise direction where \( w(\cdot) \) has \( p \) poles in the half plane \( Re \{s\} > 0 \).

The closed-loop system is asymptotically stable in the sense that all sets of initial conditions lead to outputs that approach zero as \( t \) approaches infinity if:

2) \( T \) does not intersect the disk \( D[\alpha, \beta] \) and encircles it \( n \) times in the counterclockwise direction.

The complete observability of \([F, h]\) means that 1) is equivalent to stability in the sense of Liapunov, and that 2) is equivalent to asymptotic stability in the sense of Liapunov.

To generate Liapunov functions for a system satisfying one of the two circle criteria, we make use of a result in Brckett and Lee (see Lemmas 3 and 4). This result yields that 1) may be replaced by

1a) The function \( z(\cdot) \) where

\[
\begin{align*}
w(s) + \frac{1}{\alpha} & = z(s) + \frac{1}{\beta} \\
\frac{1}{\alpha} & = \frac{1}{\beta}
\end{align*}
\]

is positive real. Likewise, condition 2) may be replaced by:

2a) The function \( z(\cdot) \) as in 3) above is strictly positive real, i.e., \( z(\cdot) \) has no \( j\alpha \) axis poles, and \( Re z(j\omega) \geq 0 \) for all \( \omega \) and some positive \( \delta \).

Condition 2a) is equivalent to

2b) There exists a positive constant \( \sigma \) such that the function \( z(\cdot) \) given by

\[
\begin{align*}
\delta(\cdot) & = \delta(\cdot - \sigma) \\
\delta(\cdot) & = \delta(\cdot + \sigma)
\end{align*}
\]

is positive real. [This follows from the continuity of \( \delta(\cdot) \) and \( Re \delta(\cdot) \).]

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We shall generate a Liapunov function for the system of Fig. 1 by applying to \( z(\cdot) \) or \( \delta(\cdot) \) the positive real lemma of Anderson. In order to apply this lemma, it is necessary to determine a minimal realization of \( z(\cdot) \) or \( \delta(\cdot) \). This may be done as follows.

By explicit calculation,

\[
w(s) + \frac{1}{\alpha} = \frac{1}{\beta}.
\]

\[
\begin{align*}
\delta(\cdot) & = \delta(\cdot - \sigma) \\
\delta(\cdot) & = \delta(\cdot + \sigma)
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\]

Noting that if \( z(\cdot) \) is positive real, so is \( \langle \alpha/\beta \rangle \delta(\cdot) \), we see that a realization for \( \langle \alpha/\beta \rangle \delta(\cdot) \) is given by the quadruple \( \{F - \alpha h, P, \delta(\cdot) \} \) then by Anderson, there exists a matrix \( P = P^* \geq 0 \), and a vector \( l \) such that

\[
(PP' - \alpha h) + (P' - \alpha h')P = -\delta(\cdot) e^2(\cdot)
\]

We claim that

\[
\delta w(s) = \dot{\theta}(P - \alpha h)\delta w(s) + (P' - \alpha h')dP = -\delta(\cdot) e^2(\cdot)
\]

We observe that in this case

\[
\delta w(s) = \dot{\theta} (P - \alpha h) \delta w(s) + (P' - \alpha h') e^2(\cdot)
\]

We have that the choice of the tentative Liapunov function is indeed a correct one, implying stability or asymptotic stability, as the case may be.

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