Stability properties of linear systems in phase-variable form

B. D. O. Anderson, Ph.D.

Synopsis

At least two types of stability may be defined for linear, finite-dimensional, dynamic systems: Lyapunov stability, reflecting the internal stability of the system, and bounded-input, bounded-output stability, reflecting the external stability of the system. Although connections between these two types of stability are well understood for time-invariant systems, this is not the case for time-varying systems, as witnessed by past controversy and present interest in the topic. A connection between the two types of stability is established for systems which may be represented in a special canonical form; the extent to which systems can be put into this canonical form has been discussed by others and is reviewed at the end of the paper.

1 Description of problem

We shall consider systems described by

$$\dot{x} = F(t)x + g(t)u \quad . \quad . \quad . \quad (1)$$

where $F()$ is an $n \times n$ matrix, $g()$ is an $n \times p$ matrix, $x(t)$ is the system state vector and $u(t)$ is the system input.

We term the system 'uniformly asymptotically stable' if the transition matrix $\phi(t, \tau)$ associated with eqn. 1 satisfies the inequality

$$||\phi(t, \tau)|| \leqslant K_1 e^{-K_2(t-\tau)} \quad . \quad . \quad . \quad (2)$$

for some positive constants K_1 and K_2. We term the system 'bounded-input, bounded-output stable' if for any u with

$$||u(t)|| \leqslant K_3 \quad . \quad . \quad . \quad (3)$$

for all t and some positive constant K_3 the resulting x for $t \geqslant t_0$ satisfies

$$||x(t)|| < K_4(K_3, ||x(t_0)||) \quad . \quad . \quad . \quad (4)$$

This statement is equivalent to the conditions[4]

$$\int_{-\infty}^{t} ||\phi(t, \lambda)g(\lambda)||d\lambda \leqslant K_5 \quad . \quad . \quad . \quad (5)$$

$$||\phi(t, t_0)|| \leqslant K_6 \quad . \quad . \quad . \quad (6)$$

holding for all t and some positive constants K_5 and K_6.

The problem arises as to when one type of stability implies the other. As shown in References 1–3, there is no connection between the two types of stability, unless extra conditions are placed on $F()$ and $g()$. An example of such conditions is provided by the theorem of Perron,[5] which, in effect, says that boundedness of F and g and nonsingularity of g, together with boundedness of its inverse, guarantee the equivalence of the two types of stability. Naturally the restriction of g is a severe one; here we relax it but impose a more severe restriction on F. We shall assume that

(*a*) *The matrix F*(*t*) *is a companion matrix, i.e.*

$$F(t) = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 \\ -a_1 & -a_2 & \cdot & \cdot & -a_n \end{bmatrix}$$

with each $a_i()$ *bounded*

and

(*b*) $$g(t) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

We shall establish the following result:

Theorem. Consider the system of eqn. 1 with $F(t)$ and $g(t)$ satisfying (*a*) and (*b*). Then eqn. 1 is uniformly asymptotically stable if and only if it is bounded-input, bounded-output stable.

2 Proof of theorem

The 'only if' part of the theorem is very easily established. To establish the 'if' part, we shall proceed with the aid of two lemmas:

Lemma 1

Given the system of eqn. 1, (*a*) and (*b*), there exists a feedback law

$$u(t) = k'(t) \times (t) \quad . \quad . \quad . \quad (7)$$

such that the eigenvalues $\lambda_i(t)$ of $F(t)$ and the eigenvalues $u_i(t)$ of $F(t) - g(t)k'(t)$ are related by

$$\mu_i(t) = \lambda_i(t) + \alpha \, (i = 1, 2, \ldots, n) \quad . \quad . \quad . \quad (8)$$

where α is an arbitrary constant (independent of time); moreover $||k'(t)||$ approaches zero uniformly in t as α approaches zero.

Proof. The eigenvalues of $F(t)$ are the zeros of the equation

$$s^n + a_n s^{n-1} + \ldots + a_1 = 0 \quad . \quad . \quad . \quad (9)$$

as may be readily checked; hence the entries $k_i(t)(i = 1, 2, \ldots, n)$ of the row vector $k'(t)$ are determined simply from the equality

$$(s - \alpha)^n + a_n(s - \alpha)^{n-1} + \ldots + a_1$$
$$= s^n + (a_n + k_n)s^{n-1} + \ldots + (a_1 + k_1) \quad . \quad (10)$$

which equates the eigenvalues of $F - gk'$ to the zeros of eqn. 9 shifted by α. The second part of the lemma statement follows from the boundedness of a_i for each i [see (*a*)], and from elementary properties of polynomials and their roots. This completes the proof of lemma 1.

Lemma 2

Given the system of eqn. 1, (*a*) and (*b*), let $k(t)$ be the feedback law existing by lemma 1, so that the eigenvalues $\lambda_i(t)$ of $F(t)$ and the eigenvalues $\mu_i(t)$ of $F(t) - g(t)k'(t)$ satisfy eqn. 8, i.e.

$$\mu_i(t) = \lambda_i(t) + \alpha$$

for some positive constant α.

Define the $n \times n$ matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ -\alpha & 1 & 0 & \cdot & \cdot & \cdot \\ \alpha^2 & -2\alpha & 1 & 0 & \cdot & \cdot \\ -\alpha^3 & 3\alpha^2 & -3\alpha & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad . \quad . \quad (11)$$

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Prof. Anderson is with the Department of Electrical Engineering, University of Newcastle, Newcastle, NSW, Australia

Then the following formula holds:

$$T(F - gk')T^{-1} = \alpha I + F \quad (12)$$

Proof. Evaluation of $TF - Tgk'$ and $\alpha T + FT$ is straightforward, and equality of these two matrices follows by using the equality of the coefficients on both sides of eqn. 10.

The proof of the theorem is now straightforward. First note that it is permissible to restrict attention to initial states $x(t_0)$ for which $||x(t_0)|| \leqslant 1$. The assumed bounded-input bounded-output stability then allows us to guarantee that $||u|| < K_3$ implies $||x|| < K_4(K_3)$ (with $K_4 > 1$).

With the aid of lemma 1, we suppose that α is chosen positive and sufficiently small that, for a particular K_3 and the associated K_4, $||x(t)|| \leqslant K_4$ implies $||k'(t) \times (t)|| < K_3$ for all t. The closed-loop system formed with the control law $k'(t)$

$$\dot{x} = \{F(t) - g(t)k'(t)\}x \quad (13)$$

is then Lyapunov stable, for if not, there exists an initial state $x(t_0)$ with $||x(t_0)|| \leqslant 1$ and a first-time $T > t_0$ for which $||x(T)|| = K_4$. Over the interval $[t_0, T]$, the state of the closed-loop system behaves like the state of the system of eqn. 1 when both systems are started in state $x(t_0)$ and the latter system has external input $u(t)$, with (eqn. 7)

$$u(t) = -k'(t) \times (t)$$

Moreover, the input $u(t)$ over this interval obeys the inequality of eqn. 3

$$||u(t)|| < K_3$$

because of the choice of T as the first time for which $||x(T)|| = K_4$ and because the choice of $||k'||$ is suitably small. The bounded-input assumption on eqn. 1 then guarantees that $||x(t)|| < K_4$ on $[t_0, T]$ and in particular $||x(T)|| < K_4$, contradicting the definition of T.

Since the system of eqn. 13 is Lyapunov-stable, so is the system

$$\dot{x} = \{\alpha I + F(t)\}x \quad (14)$$

by virtue of eqn. 12, for the states of eqn. 14 are related to the states of eqn. 13 by a constant nonsingular co-ordinate transformation defined by the matrix T.

It follows immediately that the states of the system

$$\dot{x} = F(t)x \quad (15)$$

are exponentially asymptotically stable, because the states of eqn. 15 behave like those of eqn. 14 multiplied by $\exp\{-\alpha(t - t_0)\}$. Equivalently, eqn. 2 holds. This proves the theorem.

3 Reduction of an arbitrary system to the canonical form

The assumption that in eqn. 1 the F and g matrices have the special form of eqns. 4 and 5 is at first glance severe, and it is of interest to know when an arbitrary system such as eqn. 1 is, in some sense, equivalent to one where the F and g matrices have the special form.

This equivalence has been considered by Silverman,[6] who has given explicit conditions under which a change of co-ordinate basis can be used to generate from a general F and g new matrices in the special form; Reference 6 also states explicitly what the basis-change matrix is. Provided that this matrix is a Lyapunov transformation,[4] the untransformed and transformed systems have the same stability properties.

It should also be noted that a natural rewriting of an nth-order linear differential equation with a single forcing function is in the form of eqn. 1 with the F and g matrices as in eqns. 4 and 5; thus the main result of the paper carries over very simply to this situation.

4 Extension to more general linear system

It is possible to consider the situation where the system output is not simply the state vector of eqn. 1, but rather a linear transformation of it, i.e.

$$y = h'(t)x \quad (16)$$

The notion of bounded-input, bounded-output stability extends naturally, while the notion of Lyapunov stability is of course unaltered. If the system defined by eqns. 1 and 16 is uniformly completely observable,[7] bounded-input, bounded-output stability of eqns. 1 and 16, regarded as defining one system, is equivalent to bounded-input, bounded-output stability of eqn. 1, regarded as a second system;* thus similar remarks can be made concerning the equivalence of bounded-input, bounded-output and Lyapunov stability for the more general situation as for the situation earlier considered.

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6 References

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