Simulation of stationary stochastic processes

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Synopsis

A 'spectral-factorisation' procedure involving the solution of a Riccati matrix differential equation is considered to determine systems which, with white-noise input signals, may be used in the simulation of stochastic processes having prescribed stationary covariances. More specifically, the specification of a system is made so that the covariance of the system output is a prescribed stationary covariance $R(t-\tau)$ for all t and τ greater than or equal to the 'switch-on' time of the system. The advantage of the 'spectral-factorisation' procedure described compared with those previously given is that, assuming an initial-state mean of zero, a suitable initial-state covariance is calculated as an intermediate result in the procedure. The calculation of an appropriate initial-state covariance is of interest since, if zero initial conditions are used in an attempted simulation, an undesirable time lapse may be necessary for the output covariance to be acceptable as a simulation of the prescribed stationary covariance. For the case when the system is given or is determined using alternative procedures to those described in the paper, the initial-state covariance is calculated from the solution of a linear matrix equation.

1 Introduction

The problem considered in the paper is the simulation of stationary stochastic processes with prescribed covariances using linear, finite-dimensional, time-invariant systems with white-noise input. Of particular interest is the selection of an initial-state covariance, so that the covariance of the outputs will be indistinguishable from that observed over the same time interval for the hypothetical limiting case as the initial time approaches $-\infty$.

Systems which may be used in the simulation of stationary stochastic processes with prescribed covariances may be determined from any of a number of spectral-factorisation procedures.[1,2,*] With regard to the initial conditions, current practice is to set these to zero and ignore the outputs for a period corresponding to a few time constants of the system. The inadequacy of this procedure has been recognised.[3]

In the paper two results are presented. The first is a method for selecting an initial-state covariance for a given system, so that the application of white noise at the input results in outputs that may be considered, after the switch-on, as sample functions of a stationary stochastic process; this is the best possible real-time simulation for a stationary stochastic process. All that is required in order to obtain the result is the solution of a linear matrix equation.

The second result of the paper is a spectral factorisation of a specified covariance matrix using theorems from Anderson.† The procedure gives a system having a stable transfer-function matrix with a stable inverse (often required in certain optimisation problems), together with the initial-state covariance; the advantage of the particular approach presented is that all the information necessary for the simulation is given in one procedure.

The key step in the procedure is the solution of a quadratic matrix equation which satisfies certain constraints. This solution, which is unique, may be found using algebraic means similar to those of Reference 4 or by determining the steady-state solution of a Riccati matrix differential equation.† The method avoids the need to carry out any of the procedures in References 1, 2 or *, which prove very complex in cases where the covariance is a matrix rather than scalar.

2 Review of system-theory results

The systems under consideration are linear, time-invariant, finite-dimensional and asymptotically stable, and thus can be described by the state-space equations

$$\dot{x} = Fx + Gu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$y = H'x + Ju \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

* BROCKETT, R. W.: 'Spectral factorization of rational matrices', to be published
† ANDERSON, B. D. O.: 'Quadratic minimisation, positive real matrices and spectral factorisation', to be published

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where x is an $n$vector (the state), u is a $p$vector (the input) y is an $m$vector (the output) and the matrices F, G, H and J are of the appropriate dimension. The system input is white Gaussian noise with zero mean and a covariance matrix:

$$\text{cov}\,\{u(t), u(\tau)\} = I\delta(t-\tau) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and is first applied at time t_0.

Some results to be used in the following Sections are reviewed. For the system of eqn. 1, when u is deterministic,

$$x(t) = \Phi(t-t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t-\lambda)Gu(\lambda)d\lambda \quad . \quad (3)$$

where $x(t_0)$ is the initial state and $\Phi(.)$ is the system matrix satisfying

$$\dot{\Phi}(t) = F\Phi(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\Phi(0) = I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, since the system of eqn. 1 is asymptotically stable,

$$\lim_{t\to\infty} \Phi(t) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the inputs to the system of eqn. 1 are white Gaussian noise with zero mean and a covariance matrix given by eqn. 2, the covariance of the states at time t is given by

$$\text{cov}\,\{x(t), x(t)\} = E\{x(t)x'(t)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the covariance of the outputs is given by

$$\text{cov}\,\{y(t), y(\tau)\} = E[\{H'x(t) + Ju(t)\}\{x'(\tau)H + u'(\tau)J'\}] \quad . \quad . \quad . \quad . \quad (8)$$

Employing the notation

$$P(t_0, t) = \text{cov}\,\{x(t), x(t)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and expanding eqn. 7, using eqns. 2 and 3, gives

$$P(t_0, t) = \Phi(t-t_0)P(t_0, t_0)\Phi'(t-t_0) + \int_{t_0}^{t} \Phi(t-\lambda)GG'\Phi'(t-\lambda)d\lambda \quad . \quad (10)$$

For the limiting case as $t_0 \to -\infty$, this may be written, using eqn. 6, as

$$\lim_{t_0\to-\infty} P(t_0, t) = \lim_{t_0\to-\infty} \int_{t_0}^{t} \Phi(t-\lambda)GG'\Phi'(t-\lambda)d\lambda \quad . \quad . \quad . \quad . \quad (11)$$

and it may be noted that this limit is independent of any initial-state covariance and is independent of the time t. Thus, in reference to this limit, the notation P will be used.

An expansion of eqn. 8 using eqns. 3 and 10 gives the covariance of the system output as

$$R_s\{(t, \tau) = JJ'\delta(t-\tau) + H'e^{F(t-\tau)}\{P(t_0, \tau)H + GJ'\}1(t-\tau) + \{H'P(t_0, t)$$

where $1(t)$ is a unit step function at time t. For the limiting case as $t_0 \to -\infty$, eqn. 12 may be written as the stationary covariance

$$R(t-\tau) = JJ'\delta(t-\tau) + H'e^{F(t-\tau)}(PH + GJ')1(t-\tau) + (H'P + JG')e^{F'(\tau-t)}H1(\tau-t) \quad (13)$$

A prior step to the spectral-factorisation procedure of Section 4 is to arrange the specified covariance matrix in the form of eqn. 13:

$$R(t-\tau) = D\delta(t-\tau) + C'e^{A(t-\tau)}B1(t-\tau) + B'e^{A'(\tau-t)}C1(\tau-t) \quad (14)$$

where A is square and of minimal dimension. In this respect, it may be noted that the Laplace transform of eqn. 14 is

$$R_T(s) = Z(s) + Z'(-s) \quad (15)$$

where $R_T(s)$ is the Laplace transform of the covariance matrix $R(t-\tau)$ and $Z(s)$ is a positive real matrix given by

$$Z(s) = \frac{D}{2} + C'(s1 - A)^{-1}B \quad (16)$$

If the covariance is specified in terms of the Fourier transform of $R(t-\tau)$, it may first be arranged as a Laplace transform $R_T(s)$, and then, from a determination of $Z(s)$, a quadruple A, B, C, $D/2$ (not unique) may be calculated.[5]

Another result to be used in Section 4 given by explicit calculation is that, for the system of eqn. 1, the inverse of the associated transfer-function matrix

$$W(s) = J + H'(s1 - F)^{-1}G \quad (17)$$

is

$$W^{-1}(s) = [1 - J^{-1}H'\{s1 - F + GJ^{-1}H'\}^{-1}G]J^{-1} \quad (18)$$

3 Selection of initial-state covariances

The importance of specifying both a system and an initial-state covariance for the simulation of a stationary stochastic process has been mentioned in Section 1. Intuitive reasoning suggests that an appropriate value for the initial-state covariance is $\lim_{t_0 \to -\infty} \text{cov}\{x(t), x(t)\}$. A lemma is now stated with proof which gives the required initial-state covariance in terms of the system matrices F and G. The first step in the proof is to verify that $\lim_{t_0 \to -\infty} \text{cov}\{x(t), x(t)\}$ is in fact the required covariance.

Lemma: For the system of eqn. 1 with input white Gaussian noise with zero mean and covariance of eqn. 2 applied at time t_0, the covariance of the system output $R_s(t, \tau)$ is a stationary covariance $R(t-\tau)$ for all $t > t_0$ and $\tau > t_0$ if and only if the initial-state mean is zero and its covariance is the unique nonnegative definite solution P^* of the linear matrix equation

$$FP^* + P^*F' + GG' = 0 \quad (19)$$

The first step in the proof of this lemma is to use eqns. 10 and 11 to verify the following result:

$$P(t_0, t)|\{t \geqslant t_0 \text{ and } P(t_0, t_0) = P\} = P \quad (20)$$

(i.e., with initial state covariance $P(t_0, t_0) = P$, the covariance of the state at any time t, i.e. $P(t_0, t)$, is also P). This result may now be stated in terms of the covariances of the output using eqns. 12, 13 and 20 as

$$\left.\begin{array}{l} R_s(t, \tau)|\{t \geqslant t_0,\ \tau \geqslant t_0 \\ \text{and} \quad P(t_0, t_0) = P\} = R(t-\tau)|(t \geqslant t_0, \tau \geqslant t_0) \end{array}\right\} \quad (21)$$

To complete the proof of the lemma, it remains to be shown that P is in fact the unique solution P^* of eqn. 19. To show this, we first conclude from eqn. 11 and the asymptotic stability of F that P exists, is unique and is bounded. Differentiating eqn. 11 with respect to time and using eqn. 4 gives

$$0 = FP + PF' + GG' \quad (22)$$

Thus, since for a stable system eqn. 19 has a unique solution, it is seen that

$$P^* = P \quad (23)$$

and the lemma is established.

4 Spectral-factorisation procedure

In this Section, a spectral-factorisation procedure is given which determines a system $\{F, G, H, J\}$ (eqn. 1) together with an appropriate initial-state covariance (assuming that the initial-state mean is zero), so that when the inputs of the system are excited by white Gaussian noise with a zero mean and covariance matrix given by eqn. 2, the output has a a prescribed covariance $R(t-\tau)$ for all t and τ greater than or equal to the switch-on time t_0 of the system.

It will be assumed throughout this Section that the prescribed $R(t-\tau)$ always includes 'nonsingular' white noise, in the sense that when $R(t-\tau)$ is given in the form of eqn. 14, D is a nonsingular (and, of course, nonnegative-definite-symmetric) matrix.

Since, as outlined in Section 2, any specified covariance matrix can be arranged in the form given by eqn. 14, where A is a square matrix of minimal dimension, the following derivations will assume that the covariance matrix is specified in this form. That is, the spectral-factorisation problem may now be considered as the following:

Given $R(t-r)$ in the form of eqn. 14, i.e. given the quadruple $\{A, B, C, D\}$, determine a quintuple $\{F, G, H, J, P\}$ from $\{A, B, C, D\}$ so that $R(t-\tau)$ has the form of eqn. 13 and so that with P^* replaced by P the constraint of eqn. 19 holds.

We claim that a convenient (but not the only possible) choice for $\{F, G, H, J, P\}$ may be derived as follows:

$$H = C,\ F = A,\ J = J' = D^{1/2} \quad (24)$$

where $D^{1/2}$ is the unique positive-definite square root of D. Since $D^{1/2}$ is square, this implies that u and y have the same dimension. The matrix P is any solution of

$$P(A' - CD^{-1}B') + (A - BD^{-1}C')P + PCD^{-1}C'P + BD^{-1}B' = 0 \quad (25)$$

and

$$G = (B - PC)D^{-1/2} \quad (26)$$

It is readily verified that, with the substitutions for $\{F, G, H, J, P\}$ in eqn. 13 suggested by eqns. 24, 25 and 26, eqns. 14 and 19 are recovered. It is important that a solution P of eqn. 25 not only gives G (eqn. 26), but also is the initial-state-covariance matrix for the system $\{F, G, H, J\}$ (eqns. 1, 24, 19 and the lemma of Section 3) when used for simulating the covariance matrix of eqn. 14.

There are a number of approaches to solving eqn. 25.[6] The approach presented is chosen so that the resulting system satisfies the property that, as well as the system itself being stable, the system corresponding to the transfer-function-matrix inverse $W^{-1}(s)$ is also stable (eqn. 18). That is, both $W(s)$ of eqn. 17 and $W^{-1}(s)$ of eqn. 18 are analytic in the right halfplane.

Since $Z(s)$ is positive-real, its transpose, given by taking the transpose of eqn. 16, i.e.

$$Z'(s) = \frac{D}{2} + B'(s1 - A')^{-1}C \quad (27)$$

is positive-real, and, since $\{A, B, C, D/2\}$ is a minimum realisation of $Z(s)$, $\{A', C, B, D/2\}$ is a minimum realisation for $Z'(s)$. This means that, using a theorem,† the solution $\Pi(t, t_1)$ of the Riccati matrix differential equation

$$-\dot{\Pi} = \Pi(A' - CD^{-1}B') + (A - BD^{-1}C')\Pi + \Pi CD^{-1}C'\Pi + BD^{-1}B' \quad (28)$$

with the boundary conditions

$$\Pi(t_1, t_1) = 0 \quad (29)$$

† ANDERSON, B. D. O.: *loc. cit.*

is well defined for all $t < t_1$, and the limit

$$\lim_{t_0 \to -\infty} \Pi(t_0, t_1) = \tilde{P} \quad . \quad . \quad . \quad (30)$$

exists. Further, $\tilde{P}$ is one solution of eqn. 25. If also the Fourier transform of R, namely $R_T(j\omega)$, is strictly positive at all frequencies, i.e.

$$R_T(j\omega) \geqslant \eta I \quad . \quad . \quad . \quad (31)$$

for all ω and some positive constant η, $\tilde{P}$ is also the unique positive-definite solution of eqn. 25† with the additional property

$$\mathrm{Re}\,\lambda_i\{A' - CD^{-1}(B - \tilde{P}C)'\} < 0 \quad . \quad . \quad . \quad (32)$$

This may be rewritten using eqns. 24 and 26 as

$$\mathrm{Re}\,\lambda_i\{F' - HJ^{-1}G'\} < 0 \quad . \quad . \quad . \quad (33)$$

and thus, for this case, $W^{-1}(s)$ (eqn. 18) is analytic in the right halfplane.

These results may be summarised as a lemma for the specification of systems to simulate stationary stochastic processes with prescribed covariances.

Lemma: A stationary stochastic process with a prescribed covariance arranged in the form of eqn. 14 may be simulated by a sudden application of white Gaussian noise with zero mean and covariance given by eqn. 2 to the system $\{F, G, H, J\}$ (eqn. 1) having an initial-state mean of zero and a covariance of P. The value for P is any solution of eqn. 25, where the quadruple $\{A, B, C, D/2\}$ characterises the specified covariance (eqn. 14) and the quadruple $\{F, G, H, J\}$ is given from eqns. 24 and 26. The particular P (i.e. $\tilde{P}$) which, for the case when A is of minimum dimension and the specified covariance has the property of eqn. 31, is the unique positive-definite solution of eqn. 25 satisfying the eigenvalue inequality (expr. 32), is given as the limiting solution (eqn. 30) of the Riccati matrix differential eqns. 28 and 29.

As an example, consider the simulation of the covariance

$$R(t - \tau) = \delta(t - \tau) + be^{-a(t-\tau)}1(t - \tau) + be^{-a(\tau-t)}1(\tau - t) \quad . \quad (34)$$

where a is a positive constant and $|b| < a$. Note that the Laplace transform $R_T(s)$ of $R(t)$ is

$$R_T(s) = 1 + \frac{b}{s + a} + \frac{b}{-s + a} \quad . \quad . \quad . \quad (35)$$

and that

$$R_T(j\omega) = \frac{\omega^2 + (a^2 + ab)}{\omega^2 + a^2} \quad . \quad . \quad . \quad (36)$$

which is nonnegative for all real ω, as required.

In terms of the earlier notation, one has

$$A = F = -a, B = 1, C = H = b, D = 1, J = 1 \quad . \quad (37)$$

The matrix quadratic equation, eqn. 25, becomes a scalar equation

$$b^2p^2 - 2(a + b)p + 1 = 0 \quad . \quad . \quad . \quad (38)$$

(where P of eqn. 25 has been replaced by p in eqn. 38). A solution of eqn. 38 is

$$p = \frac{(a + b) + \sqrt{\{(a + b)^2 - b^2\}}}{b^2} \quad . \quad . \quad . \quad (39)$$

which yields the covariance of the stochastic initial state, as well as the matrix G of eqn. 26.

If the correct F, G, H and J are used, but the initial state is taken as zero, eqn. 12 gives the associated covariance. The error which would result between the covariance of eqn. 12 and the correct covariance of eqn. 14 is

$$R(t - \tau) - R_s(t, \tau) = H'e^{F(t-\tau)}\{P - P(t_0, \tau)\}H1(t - \tau) + H'\{P - P(t_0, t)\}e^{F'(\tau-t)}H1(\tau - t) \quad . \quad . \quad . \quad (40)$$

where (eqn. 10)

$$P(t_0, t) = \int_{t_0}^{t} e^{F(t-\lambda)}GG'e^{F'(t-\lambda)}d\lambda \quad . \quad . \quad . \quad (41)$$

Making the appropriate identifications of F, G etc.,

$$P(t_0, t) = p\{1 - e^{-2a(t-t_0)}\} \quad . \quad . \quad . \quad (42)$$

and thus

$$R(t - \tau) - R_s(t, \tau) = b^2e^{-a(t-\tau)}e^{-2a(\tau-t_0)}1(t - \tau) + b^2e^{-a(\tau-t)}e^{-2a(t-t_0)}1(\tau - t) \quad . \quad (43)$$

Observe in particular that

$$R(0) - R_s(t, t) = b^2e^{-2a(t-t_0)} \quad . \quad . \quad . \quad (44)$$

and that the error between R and R_s decays exponentially, with time constant $1/2a$. This is as one might expect; the effect of an initial state, be it stochastic or deterministic, dies out after several time constants of the simulating system.

5 Conclusions

The two main results of the paper are applications of recent system-theory developments to the practical problem of the simulation of stochastic processes with prescribed stationary covariances. The determination of the initial-state covariance to use in a simulation means that the system outputs may be used from the switch-on time onwards, rather than after a delay of the order of a few time constants of the system. In the case where state-space equations are known for a system which would have the required covariance in the steady-state, the derivation of the initial conditions requires the solution of a linear matrix equation (or the calculation of an infinite integral). When such a system is not known and it is not desired to find it by the standard spectral-factorisation procedures, it may be found, together with the initial-state covariance, by solving a quadratic matrix equation. A particular solution of this equation may be derived as the limit of the solution of an associated Riccati equation; since such differential equations have been the subject of study by computer programmers in connection with optimal-control problems,[6] this approach to the whole spectral-factorisation problem may prove worthwhile.

Some of the results of the paper have been generalised to time-varying systems with nonstationary covariances; References 7 and 8 may be consulted.

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