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Application of the Second Method of Lyapunov to the Proof of the Markov Stability Criterion†

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ABSTRACT

The paper indicates a number of approaches to checking the roots of a polynomial to determine whether they have negative real parts. Interrelations between the various approaches are discussed, and the stability criterion in terms of Markov determinants is related to the Hermite criterion with the aid of Lyapunov theory.

§ 1. INTRODUCTION

THE objects of this paper are several: first, to summarize a number of the techniques for checking the Hurwitz character of a polynomial, i.e. for checking whether the roots have negative real part; second, to indicate references where interrelations between these various techniques are discussed; and finally, to present a derivation of a stability criterion involving Markov determinants using the second method of Lyapunov.

Section 2 reviews various stability criteria, in particular those associated with the names of Hermite, Hurwitz, Routh, Bilharz, Schwarz, Markov and Lyapunov. In this section there is also indicated a number of connections between these criteria, which in the main have been recently developed. It is of course also possible to reformulate the stability problem in terms of finding conditions on a polynomial for it to have roots with modulus less than unity; associated with this problem are a number of sets of conditions, each in general equivalent to some technique employed for checking the negative real part character of polynomial roots.

In § 3, Guillemin's interpretation of the Routh criterion as a test for reactance functions is examined; this is to permit connection with earlier work by Parks relating the ideas of Lyapunov and Hermite. Finally, in § 4, the stability criterion in terms of Markov determinants is established, using the ideas of Lyapunov.

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§ 2. Review of Stability Criteria

We shall henceforth assume that we are interested in checking whether the roots of the following polynomial have negative real part zeros:

$$p(s) = s^n + a_1 s^{n-1} + \dots + a_n. \tag{1}$$

All the coefficients a_i are real.

An entirely equivalent problem is to check the asymptotic stability of the system:

$$\dot{x} = Fx, \tag{2}$$

where F is any matrix such that

$$\det(sI - F) = p(s). \tag{3}$$

The earliest successful attack on (1) appears to be in Hermite (1854), Hermite established:

(A) A necessary and sufficient condition for (1) to have zeros with negative real parts is that the following matrix should be positive definite:

$$H = \begin{bmatrix} & & & & & & \cdot \\ & & & \cdot & & \cdot & 0 \\ & & \cdot & \cdot & & 0 & a_5 \\ & \cdot & \cdot & 0 & & -a_5 + a_1 a_4 & 0 \\ & \cdot & 0 & \begin{matrix} a_5 - a_1 a_4 \\ + a_2 a_3 \end{matrix} & & 0 & a_3 \\ \cdot\ \cdot & 0 & -a_5 + a_1 a_4 & 0 & & -a_3 + a_1 a_2 & 0 \\ \cdot\ 0 & a_5 & 0 & a_3 & & 0 & a_1 \end{bmatrix} \tag{4}$$

Independently of Hermite Routh (1877) stated:

(B) A necessary and sufficient condition for (1) to have zeros with negative real parts is that the entries c_{i1} of the first column of the following array should be positive:

$$\left.\begin{array}{lllll} c_{11} = 1 & c_{12} = a_2 & c_{13} = a_4 & c_{14} = a_6 & \cdot \\ c_{21} = a_1 & c_{22} = a_3 & c_{23} = a_5 & c_{24} = a_7 & \cdot \\ c_{31} = a_2 - \dfrac{a_3}{a_1} & c_{32} = a_4 - \dfrac{a_5}{a_1} & c_{33} = a_6 - \dfrac{a_7}{a_1} & \cdot & \\ c_{41} = c_{22} - \dfrac{c_{21} c_{32}}{c_{31}} & c_{42} = c_{23} - \dfrac{c_{21} c_{33}}{c_{31}} & \cdot & \cdot & \\ c_{51} = c_{32} - \dfrac{c_{31} c_{42}}{c_{41}} & \cdot & \cdot & & \\ \cdot & \cdot & & & \end{array}\right\} \tag{5}$$

A third well-known test is that of Hurwitz (1895):

(C) A necessary and sufficient condition for (1) to have zeros with negative real part is that the leading minors $\Delta_1, \Delta_2, \dots \Delta_n$ of the following $n \times n$ matrix should all be positive:

$$\Delta = \begin{bmatrix} a_1 & 1 & 0 & . & . & . & . & . \\ a_3 & a_2 & a_1 & 1 & 0 & . & . & . \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \tag{6}$$

A modification of this test is due to Bilharz (1944):

(D) A necessary and sufficient condition for (1) to have zeros with negative real parts is that all even order leading minors of the following matrix should be positive:

$$B = \begin{bmatrix} 1 & 0 & -a_2 & 0 & a_4 & 0 & . & . \\ 0 & a_1 & 0 & -a_3 & 0 & a_5 & . & . \\ 0 & 1 & 0 & -a_2 & 0 & a_4 & . & . \\ 0 & 0 & a_1 & 0 & -a_3 & 0 & . & . \\ 0 & 0 & 1 & 0 & -a_2 & 0 & . & . \\ 0 & 0 & 0 & a_1 & 0 & -a_3 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \tag{7}$$

Recently Schwarz (1956) established:

(E) There exists a matrix S of the form:

$$S = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ -b_n & 0 & 1 & . & . & 0 \\ 0 & -b_{n-1} & 0 & 1 & . & 0 \\ . & . & . & & & 0 \\ . & & & -b_3 & 0 & 1 \\ . & & & 0 & -b_2 & -b_1 \end{bmatrix} \tag{8}$$

and such that

$$\det (sI - S) = p(s). \tag{9}$$

Moreover, a necessary and sufficient condition for $p(s)$ to have roots with negative real part is that

$$b_i > 0, \quad i = 1, 2, \dots n. \tag{10}$$

To define the Markov determinants (Markov 1948) associated with $p(s)$, the procedure is to write $p(s)$ as the sum of a polynomial of even powers of s and one of odd powers of s:

$$p(s) = Evp(s) + 0dp(s). \tag{11}$$

Then there is formed the ratio:

$$w(s) = \frac{Evp(s)}{0dp(s)} \quad \text{or} \quad \frac{0dp(s)}{Evp(s)}, \tag{12}$$

the alternative being adopted which yields $w(\infty) = 0$. Then $w(s)$ is expanded in a power series in $1/s$, to yield coefficients (Markov parameters) m_i:

$$w(s) = \frac{m_0}{s} - \frac{m_1}{s^2} + \frac{m_2}{s^3} - \frac{m_3}{s^4} + \dots . \tag{13}$$

(Actually here, $m_1 = m_3 = m_5 = \dots = 0$. The Markov parameters may be defined for any rational $w(s)$ such that $w(\infty)$ is zero as distinct from merely for a $w(s)$ which is a ratio of an even order to odd order polynomial; in the general case of course, $m_1 = m_3 = \dots = 0$ would not hold.) Then (Gantmacher 1959):

(F) Let $m_0, m_2, m_4, \dots$ be the non-zero Markov parameters of a rational function $w(s)$, with $w(\infty) = 0$, formed from $p(s)$ in (1) by the technique illustrated in eqns. (12) and (13). Then a necessary and sufficient condition that $p(s)$ should have roots with negative real part is that the sequence of kth-order determinants:

$$C_k = \begin{vmatrix} m_0 & -m_2 & m_4 & \cdot & \cdot \\ -m_2 & m_4 & -m_6 & \cdot & \cdot \\ m_4 & -m_6 & m_8 & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{vmatrix} \quad k = 1, 2, \dots [n/2] \tag{14}$$

and

$$D_k = \begin{vmatrix} -m_2 & m_4 & -m_6 & \cdot & \cdot \\ m_4 & -m_6 & m_8 & \cdot & \cdot \\ -m_6 & m_8 & -m_{10} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{vmatrix} \quad k = 1, 2, \dots n - [n/2] \tag{15}$$

should all be positive.

The final test which we wish to indicate, and the one that can be viewed as underpinning all the others, is that due to Lyapunov (1892) in the slightly amended form of Kalman (1963):

(G) Let h be an arbitrary n-vector such that $[F, h]$ is completely observable (Zadeh and Desoer 1963). Then a necessary and sufficient condition for (2) to be asymptotically stable is that there exist a symmetric positive definite matrix P such that

$$PF+F'P=-hh'. \tag{16}$$

Tests (A) to (F) are actually all recoverable from (G), as would be expected; at the same time there are of course interrelations independent of (G).

Connections to (A). Lehnigk (1966) relates the Bilharz test (D) to the Hermite test (A) by prescribing formulas for the Bilharz minors in terms of minors of the Hermite matrix. Ralston (1962) gave what he termed a 'symmetric formulation' of the Hurwitz test which in effect amounted to a statement of the Hermite test; thus Ralston established a connection between (A) and (C). Finally, Parks (1964) used a special form for F and h in (16) so that the solution of (16), P, turned out to be the Hermite matrix, H. The special choice was:

$$F=\begin{bmatrix} 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & I & & \\ 0 & & & & \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \tag{17}$$

(which is easily verified to have characteristic polynomial $p(s)$), together with

$$h=\sqrt{2}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ a_3 \\ 0 \\ a_1 \end{bmatrix} \tag{18}$$

This choice of F and h is discussed further in § 3.

Connections to (B). The Routh test (B) has been known for a long time to be equivalent to the Hurwitz test (C), and formulae relating the Routh coefficients c_{i1} to the Hurwitz determinants Δ_j may be found in, for example, Gantmacher (1959). This reference also relates the Markov stability criterion to the Routh criterion. Another connection to (B) was noticed by Guillemin (1957) who gave a reactance function interpretation to the Routh test. This is discussed further in the next section.

Connections to (C). Lienard and Chipart (1914) gave a significant improvement to the Hurwitz test (C), by showing, that the Hurwitz determinants were not all independent. They concluded that, provided all the a_i in (1) were positive, stability could be checked merely by examining either all odd order or all even order Hurwitz determinants. Parks (1962) related the Hurwitz test to the Schwarz test (E), and presented formulas for the Schwarz coefficients in terms of the Hurwitz determinants and vice versa. Jarominek (1960) succeeded in expressing the Hurwitz determinants in terms of the Markov determinants C_k and D_k of (14) and (15).

Connections to (D) have already been discussed.

Connections to (E). One further connection to (E) was provided by Kalman and Bertram (1960) who applied the Lyapunov theory (G) to proving the result of Schwarz. They did this by solving (16) and demonstrating the positive definiteness of the solution when F in (16) was identified with S in (8), and h in (16) was taken as:

$$h=\sqrt{2b_1}\begin{bmatrix}0\\0\\ \cdot\\ \cdot\\0\\1\end{bmatrix} \tag{19}$$

It is also of interest to note a conjecture (Chen and Chu 1966) that a certain matrix T involving the coefficients of the Routh array will transform the matrix F of (17) into the matrix S of (8) according to

$$TFT^{-1}=S. \tag{20}$$

§ 3. Reactance Function Interpretation of the Routh Test

As pointed out by Routh, one way of carrying out this test is to form the function $w(s)$ in (12) and then to do a continued fraction expansion. The coefficients in the continued fraction expansion are then precisely the Routh coefficients c_{i1}.

In Guillemin (1957) the connection is discussed between this procedure and the synthesizing of reactance functions (i.e. the determination of LC networks given an LC driving point impedance or admittance). The continued fraction method is one such synthesis procedure, and it is indeed easy to establish:

Lemma 1. With $p(s)$ as in (1), $w(s)$ as in (12), $w(s)$ is a reactance function if and only if $p(s)$ has roots with negative real part.

An alternative characterization of reactance functions in terms of the existence of a positive definite symmetric matrix can be found in Anderson (1967).

Lemma 2. Let $w(s)$ be a rational function, with $w(\infty)=0$. Let F_L, g_L and h_L be such that

$$w(s)=h_L'(sI-F_L)^{-1}g_L, \tag{21}$$

with $[F_L, g_L]$ completely controllable and $[F_L, h_L]$ completely observable (Zadeh and Desoer 1963). Then $w(s)$ is a reactance function if and only if there exists a unique symmetric positive definite matrix P such that

$$PF_L+F_L'P=0, \tag{22}$$

$$Pg_L=h_L. \tag{23}$$

It is now possible to show simply that the Hermite matrix H can be made to be the unique solution of (22) and (23), by appropriate choice of F_L, g_L and h_L. From the definition (12) of $w(s)$, it is evident that

$$w(s)=\frac{a_1s^{n-1}+a_3s^{n-3}+\dots}{s^n+a_2s^{n-1}+\dots}, \tag{24}$$

and it is easy to check that the following triple F_L, g_L h_L satisfies (21) and the controllability and observability restrictions:

$$F_L=\left[\begin{array}{c:ccccc} 0 & & & & & \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & I & & \\ \cdot & & & & & \\ 0 & & & & & \\ \hdashline \cdot\ \ \cdot & 0 & -a_4 & 0 & -a_2 & 0 \end{array}\right] \tag{25}$$

$$g_L=\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{26}$$

$$h_{\mathrm{L}} = \begin{bmatrix} \cdot \\ \cdot \\ 0 \\ a_3 \\ 0 \\ a_1 \end{bmatrix} \tag{27}$$

Observe that the matrix F of (17), as used by Parks in the proof of the Hermite criterion, is related to F_{L}, g_{L} and h_{L} by:

$$F = F_{\mathrm{L}} - g_{\mathrm{L}} h_{\mathrm{L}}', \tag{28}$$

while Parks' vector h is related to h_{L} by:

$$h = \sqrt{2} h_{\mathrm{L}}. \tag{29}$$

Consequently, with P defined by (22) and (23):

$$\begin{aligned} PF + F'P &= PF_{\mathrm{L}} + F_{\mathrm{L}}'P + Pg_{\mathrm{L}}h_{\mathrm{L}}' - h_{\mathrm{L}}g_{\mathrm{L}}'P \quad \text{by (28)} \\ &= -2h_{\mathrm{L}}h_{\mathrm{L}}' \quad \text{by (22) and (23)} \\ &= -hh' \quad \text{by (29).} \end{aligned}$$

This equality may be regarded as an equation for P, with F and h both known. As remarked earlier, the equation has a unique solution, which Parks has shown to be the Hermite matrix H. Thus the Hermite criterion can be recovered from the system theory characterization of reactance functions, and its recovery by this means is intimately related to its recovery from the lemma of Lyapunov.

§ 4. Connection with the Markov Parameters

We consider again the matrix P defined by lemma 2. Observe that, using (22) and (23):

$$\begin{aligned} PF_{\mathrm{L}}g_{\mathrm{L}} &= -F_{\mathrm{L}}'Pg_{\mathrm{L}} \\ &= -F_{\mathrm{L}}'h_{\mathrm{L}} \end{aligned} \tag{30}$$

Further,

$$\begin{aligned} PF_{\mathrm{L}}^2 &= -F_{\mathrm{L}}'PF_{\mathrm{L}} = +(F_{\mathrm{L}}')^2P, \\ PF_{\mathrm{L}}^2g_{\mathrm{L}} &= (F_{\mathrm{L}}')^2h_{\mathrm{L}}. \end{aligned} \tag{31}$$

More generally for non-negative r:

$$PF_{\mathrm{L}}^r g_{\mathrm{L}} = (-1)^r (F_{\mathrm{L}}')^r h_{\mathrm{L}}. \tag{32}$$

From (32), it follows that

$$PW = V, \tag{33}$$

where

$$W = [g_{\mathrm{L}}, F_{\mathrm{L}} g_{\mathrm{L}}, F_{\mathrm{L}}^2 g_{\mathrm{L}}, \ldots, F_{\mathrm{L}}^{n-1} g_{\mathrm{L}}] \tag{34}$$

and

$$V = (h_L, -F_L'h_L, (F_L')^2h_L, \ldots (-1)^{n-1}(F_L')^{n-1}h_L]. \tag{35}$$

Because of the complete controllability assumption, W is non-singular, and because of the complete observability assumption, V is non-singular. Thus:

$$W'PW = W'V \tag{36}$$

is symmetric positive definite when P is. The i–j entry of $W'V$ is:

$$\begin{aligned}(WV)_{ij} &= (-1)^{j-1}g_L'(F_L')^{i+j-2}h_L \\ &= (-1)^{j-1}h_L'(F_L)^{i+j-2}g_L.\end{aligned} \tag{37}$$

which can be readily identified using (13) and (21) as the Markov parameter $(-1)^{j-1}m_{i+j-2}$. In other words:

$$Q = W'PW = \begin{bmatrix} m_0 & 0 & m_2 & 0 & m_4 & 0 & \cdot & \cdot & \cdot \\ 0 & -m_2 & 0 & -m_4 & 0 & -m_6 & \cdot & \cdot & \cdot \\ m_2 & 0 & m_4 & 0 & m_6 & 0 & \cdot & \cdot & \cdot \\ 0 & -m_4 & 0 & -m_6 & 0 & -m_8 & \cdot & \cdot & \cdot \\ m_4 & 0 & m_6 & 0 & m_8 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{38}$$

The matrix Q will be positive definite if and only if the following matrix Q is positive definite, as may easily be checked:

$$\hat{Q} = \begin{bmatrix} m_0 & 0 & -m_2 & 0 & m_4 & 0 & \cdot & \cdot \\ 0 & -m_2 & 0 & m_4 & 0 & -m_6 & \cdot & \cdot \\ -m_2 & 0 & m_4 & 0 & -m_6 & 0 & \cdot & \cdot \\ 0 & m_4 & 0 & -m_6 & 0 & m_8 & \cdot & \cdot \\ m_4 & 0 & -m_6 & 0 & m_8 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{39}$$

The reference (Wall, 1948) establishes that the leading minor $\det \hat{Q}_i$ of Q is given by:

$$\det \hat{Q}_i = C_{[i/2]}D_{i-[i/2]}, \tag{40}$$

where C_k and D_k are the Markov determinants of eqns. (14) and (15). That $\det \hat{Q}_i$ is positive for all i if and only if all the C_k and D_k are positive, follows simply from (40). It is also actually true that with $\det \hat{Q}_i$ the ith minor of Q, $\det Q_i$ and $\det \hat{Q}_i$ are identical.

Our conclusions are therefore as follows. The Hermite matrix P above may be put through a congruency transformation, see (38), to yield Q, where the matrix W defining the congruency transformation is given by (34) and the entries of (34) are in turn calculated from (25). The positive definite nature of Q is directly associated with the positive nature of the Markov determinants; eqn. (40) allows computation of the leading minors of Q from the Markov determinants, and vice versa.

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