INT. J. CONTROL, 1967, VOL. 5, NO. 4, 345–353.

# Applications of the Multivariable Popov Criterion†

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[Received May 2, 1967]

ABSTRACT

Two classes of systems are considered for the application of the multivariable Popov criterion. The first is obtained from a linear, finite-dimensional system with a state feedback law derived from a quadratic loss function minimization problem. It is shown that a non-critical part of the system is the set of transducers producing the inputs to the system, in the sense that stability is retained even when the transducers are far from ideal.

The second class of systems is derived from linear, finite-dimensional systems which are stable. It is shown that it is always possible to tolerate in general a small amount of non-linearity at virtually any point in the system without impairment of stability.

## § 1. INTRODUCTION

MANY so-called linear systems are in fact not so, and it is the aim of this paper to indicate situations where non-linearity may be tolerated in a system, with this system remaining stable if the linear one from which it is derived is also stable. The technique used for establishing stability of the non-linear system is the Popov theorem (Popov 1961) in multidimensional format (Anderson 1966 a, b, Tokumaru and Saito 1965).

Two classes of systems are considered. The first is obtained from a linear, finite-dimensional system with a state feedback law derived from a quadratic loss function minimization problem. Non-linearity is then permitted in the transducers producing the input to the linear system. As is shown below, and as is known for the single input case (Kalman, private communication) a substantial amount of non-linearity can be tolerated with stability being retained. In practical terms, this means that a non-critical part of a regulator system is the transducer driving the system.

The second class of systems is derived from linear, finite-dimensional systems which are stable. It is shown that it is always possible to tolerate in general a small amount of non-linearity at virtually any point in the system. Moreover, it is possible to apply a sufficiency test to determine whether a non-linearity lying within prescribed bounds will not affect the stability of the system.

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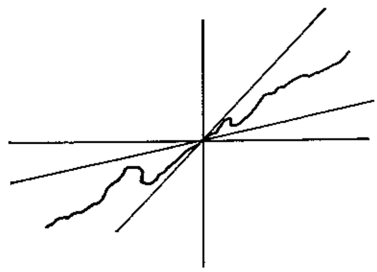
† Communicated by the Authors.

As in the standard Popov theory, all non-linearities considered are of 'sector type' and are memoryless. A memoryless non-linearity of sector type is one in which the non-linearity always lies within (perhaps strictly) a sector in a plane whose axes are associated with the input and output to the non-linearity, see fig. 1.

Fig. 1



Sector non-linearity.

It is clear from the figure that unless one (or both) of the sector boundaries coincides with a coordinate axis, saturating non-linearities cannot be considered. Neither can switching or relay-type non-linearities.

## § 2. Review of the Multidimensional Popov Criterion

The multidimensional Popov criterion is discussed in Anderson (1966 a, b) and Tokumaru and Saito (1965). We assume we are given a system with linear forward part, described by an $n \times n$ matrix $W(s)$ of rational functions. Feedback is provided by $n$ memoryless non-linearities, fig. 2, the sole restriction on which is that the $i$th non-linearity $(i=1, 2, \ldots, n)$ should be confined within a sector defined by the horizontal axis and a line of slope $k_i$ through the origin, fig. 3.

Defining the matrix $K = \operatorname{diag}\{k_1, k_2, \ldots, k_n\}$, the multidimensional Popov criterion may be stated as follows:

*Consider the system of fig. 2, where $W(s)$ is a stable transfer function. Suppose there exist constants $\alpha > 0$, $\beta > 0$, with $\alpha + \beta > 0$ and $-\alpha/\beta$ not a pole of $W(s)$ such that*

$$Z(s) = \alpha K^{-1} + (\alpha + \beta s) W(s) \tag{1}$$

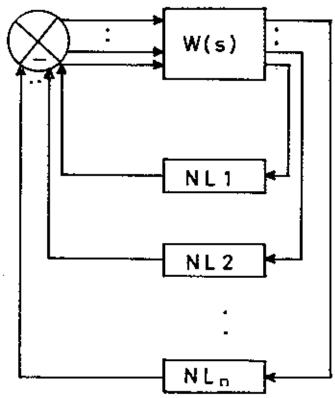
*is a positive real matrix. Then the closed loop system is Lyapunov stable.*

We comment that for a given stable $W(s)$, there always exists a set of slopes $k_1, k_2, \ldots, k_n$ which will guarantee the positive real nature of $Z(s)$ for $\beta = 0$. This follows from observing that if the elements of the matrix $K^{-1}$ are big enough

$$Z(j\omega) + Z'(-j\omega).$$

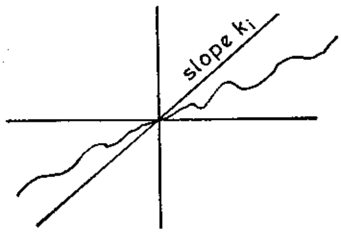
may be made strictly non-negative definite for all $\omega$. The other requirements to guarantee positive realness (Newcomb 1966), can be shown to be satisfied in a straightforward manner.

Fig. 2




System to which Popov theorem applies.

Fig. 3



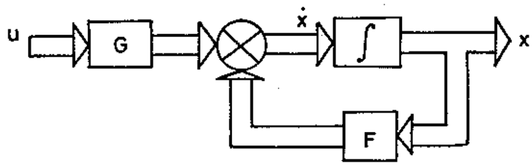

Restrictions on memoryless non linearities.

## § 3. Non-Linearities in otherwise Optimal Systems

We shall consider the system of fig. 4 described by the equation:

$$\dot{x} = Fx + Gu. \tag{2}$$

Fig. 4




Linear finite-dimensional system.

A common technique used for regulating such a system is to seek the control vector $u$ which will minimize the following performance index:

$$V(x_0; u) = \int_{t_0}^{\infty} (u'u + x'Qx)\, dt. \tag{3}$$

Here $Q$ is a symmetric non-negative definite matrix. (The $u'u$ term is often $u'Ru$, where $R$ is positive definite symmetric, but the form of (3) is quite general if the inputs are appropriately scaled.)

The result of the optimization is well known (Athans and Falb 1966), and is that:

$$u = -M'x, \tag{4}$$

where

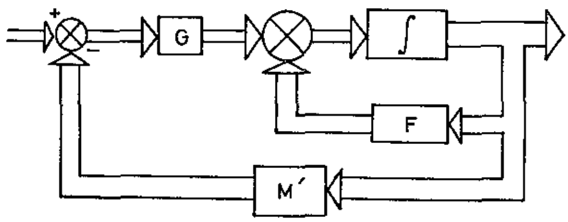
$$M = PG \tag{5}$$

and $P$ is the unique positive definite symmetric solution of:

$$PF + F'P - PGG'P + Q = 0. \tag{6}$$

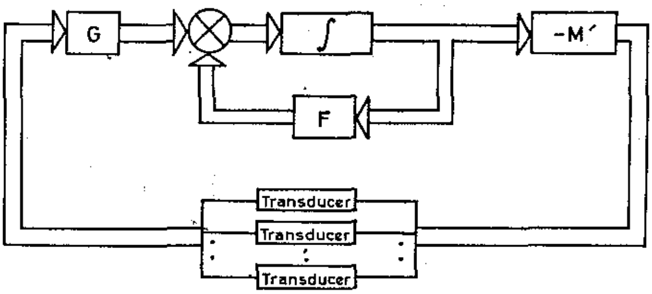
Implementation of the control law (4) is shown in fig. 5. Since we are primarily interested in Lyapunov stability, or stability of the system under zero input conditions, we shall neglect external inputs, and re-draw the system appropriately. In this re-drawing, see fig. 6, we introduce

Fig. 5




Implementation of optimal control

Fig. 6

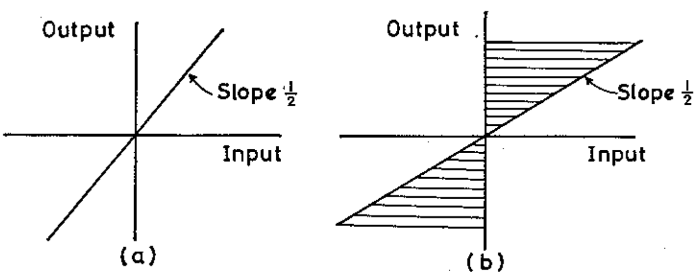



System in absence of inputs.

blocks labelled transducers. In a practical situation, the realization of the block $M'$ may well be done with electronics, while the generation of system inputs may require devices of large power-handling capability, e.g. motors, which inherently tend to contain non-linearities. Thus the transducers of fig. 6 are a generic representation of such devices.

Ideally, the transducer transfer characteristics should lead to identical input and output (fig. 7 (*a*)). But in practice, some non-linearity in each transducer will always exist, and we shall demonstrate that *any non-linearity lying in the shaded sector of fig.* 7 (*b*) *will lead to a stable system.*

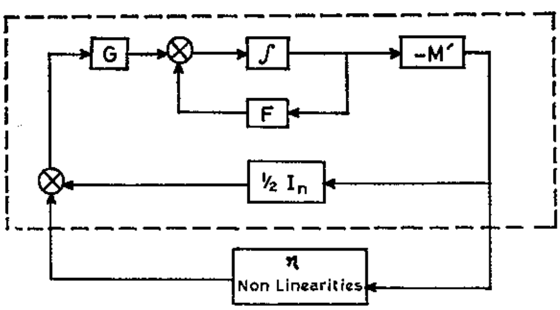
Fig. 7




(*a*) Ideal transducer characteristic. (*b*) Region of permissible transducer non-linearity retaining stability.

In order to prove this fact, it is necessary to transform the non-linearities so as to be amenable to the theory of §2. This may be done by observing that the non-linearity of fig. 7 (*b*) is equivalent to the sum of a linear transmission characteristic of slope $\frac{1}{2}$, and a non-linear transmission characteristic restricted to lying in the entire first and third quadrants. Taking the number of separate inputs to the system as $n$, this permits re-drawing of the system as in fig. 8.

Fig. 8




Nominally optimal non-linear system.

This figure shows enclosed in dashed lines the part of the system corresponding to that labelled $W(s)$ in fig. 2, save for the absence of the sign inverter at the input. It is easy to evaluate the transfer function matrix of the dashed linear section, and multiplying it by $-1$, there obtains:

$$W(s)=M'[sI-F+\tfrac{1}{2}GM']^{-1}G. \tag{7}$$

The non-linearities of fig. 8 are like those of figs. 2 and 3, with each $k_i$ set equal to infinity. Thus the matrix $K^{-1}$ is zero.

Demonstration of the stability of the system of fig. 8 can be achieved, using the ideas of §2, by showing the existence of $\alpha$ and $\beta$ for which $(\alpha+\beta s)W(s)$ is positive real. In fact $W(s)$ itself is positive real, as we show below, and a choice of zero for $\beta$ and any positive number for $\alpha$ will serve our purpose.

To check the positive real nature of $W(s)$, we observe that, from (5) and (6),

$$PG=M. \tag{8a}$$

and

$$P(F-\tfrac{1}{2}GM')+(F'-\tfrac{1}{2}MG')P=-Q. \tag{8b}$$

But with $P$ positive definite symmetric, these two conditions are precisely the ones required to guarantee the positive real property, by the fundamental theorem of Anderson (1967). Hence the stability of the non-linear system.

## § 4. Non-linearities in Nominally Linear, Stable Systems

In this section, we consider the situation depicted in fig. 5, with one change. Instead of requiring the feedback law $M'$ to be derived from a quadratic loss minimization, we require merely that the eigenvalues of $F-GM'$ all have negative real parts, i.e. the closed loop system must be stable.

Let us suppose that $M'$ is a $p\times n$ matrix; in other words the dimension of the system state vector is $p$, and the dimension of the system input vector is $n$. Defining $M=(m_{ij})$, it is evident that the $i$th component of the system input, $u_i$, resulting from feedback is:

$$u_i=-\sum_j m_{ji}x_j. \tag{9}$$

In practice, such a linear law is impossible to achieve exactly. Our purpose here is to show that some non-linearity may always be tolerated.

Thus we shall suppose that the true feedback law may be written, for each $i$:

$$u_i=-\sum_j m_{ji}x_j+\sum_j \mu_{ji}(x_j), \tag{10}$$

where each $\mu_{ji}$ is a sector non-linearity, the lower bound on the sector being the $x_j$ axis and the upper bound being a line whose slope has yet to be specified. Note that (10) forces, for positive $x_j$, the real $u_i$ to be always greater than the ideal $u_i$ of (9). The situation where a non-linearity causes the real $u_i$ to be smaller than the ideal one can be covered as follows.

Because the eigenvalues of $F-GM'$ are continuous functions of the entries of $F$, $G$ and $M$, it is true that if some of the entries $m_{ij}$ of $M$ are replaced by $m_{ij}+\epsilon_{ij}$, where $\epsilon_{ij}$ may have to be chosen sufficiently small, the eigenvalues of the modified $F-GM'$ will still be in the left half plane. But then the ideal $u_i$ is now:

$$u_i = -\sum_j m_{ji}x_j - \sum_j \epsilon_{ji}x_j \tag{11}$$

and the real $u_i$ is:

$$u_i = -\sum_j m_{ji}x_j + \sum_j [\mu_{ji}(x_j) - \epsilon_{ji}x_j]. \tag{12}$$

By still restricting the $\mu_{ji}$ to be first and third quadrant functions, we can, by the artifice of introducing a linear transformation with the $\epsilon_{ji}$, arrange to consider non-linearities that result in values of $u_i$ which are smaller than the ideal. All that is required is that $\mu_{ji}(x_j)-\epsilon_{ji}x_j$ be negative!

Because of the above remarks, we shall henceforth restrict attention to feedback laws as in (10), with the $\mu_{ij}$ first and third quadrant functions. Figure 9 illustrates the closed loop system incorporating the non-linearities.

To discuss the stability of the system with the aid of the Popov criterion, it is necessary to construct the $pn \times pn$ matrix $W(s)$, the transfer function matrix associated with the linear part of fig. 9, which maps the outputs of the non-linearities into the corresponding inputs, just like the $W(s)$ of fig. 2.

Neglecting the minus sign associated with the $-G$ block, $W(s)$ can be calculated explicitly by straightforward means as:

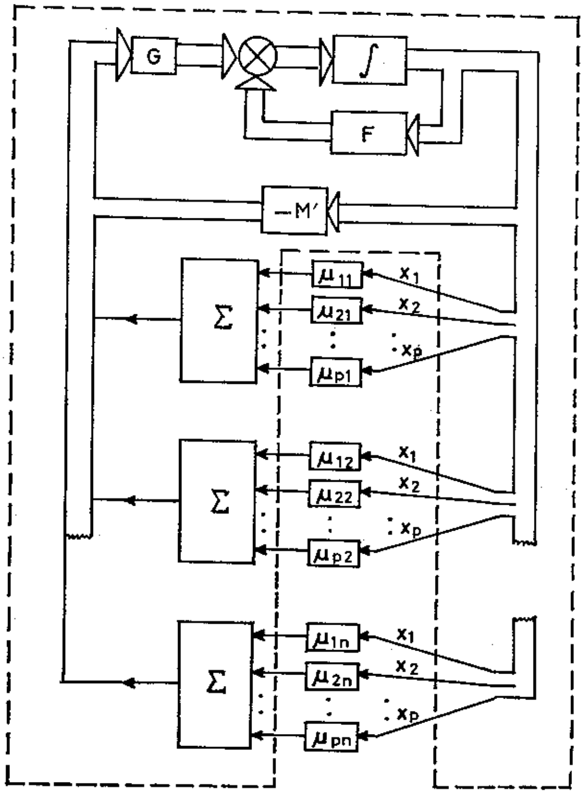
$$W(s) = \begin{bmatrix} I_p \\ I_p \\ \cdot \\ \cdot \\ I_p \end{bmatrix} (sI_p - F + GM')^{-1} G \begin{bmatrix} \overbrace{1\ 1\ \ldots\ 1}^{p} & \overbrace{0\ 0\ \ldots\ 0}^{p} & 0\ \ldots \\ 0\ 0\ \ldots\ 0 & 1\ 1\ \ldots\ 1 & 0\ \ldots \\ 0\ 0\ \ldots\ 0 & 0\ 0\ \ldots\ 0 & 1\ \ldots \\ \cdot & & \cdot \\ \cdot & & \cdot \end{bmatrix} \tag{13}$$

Stability of the closed loop non-linear system will follow if for some set of slopes $k_1, k_2 \ldots k_{pn}$ restricting the $\mu_{ij}$, these are non-negative constants $\alpha$ and $\beta$ for which:

$$Z(s) = \alpha K + (\alpha + \beta s) W(s), \tag{14}$$

is positive real. (Here $K$ is of course diag $\{k_1, k_2, \dots k_{pn}\}$). *But, as pointed out in* §2, *such* $\alpha$, $\beta$ *and* $k_i$ *always exist, assuming* $W(s)$ *is stable.* Quite evidently $W(s)$ is stable, by the earlier assumption on the eigenvalues of $F-GM'$; consequently we see that it is always possible to restrict the amount of the non-linearities $\mu_{ij}$ so that the closed loop non-linear system is stable.

Fig. 9




Non-linear version of nominally linear stable system.

Essentially the same arguments will carry through to show that sector non-linearities of restricted magnitude can always be tolerated in the realization of $G$ and $F$. Thus we conclude:

*In any 'linear' part of a stable system some sector non-linearity can always be tolerated without the system going unstable. A 'sufficiency' test allows the checking of a non-linearity or set of non-linearities to see if stability is retained; the test proceeds by forming a matrix of rational transfer functions and seeing if it is positive real.*

## § 5. Conclusions

The results presented here, though constituting two useful applications of the multidimensional Popov theorem, do apply to differing practical situations. The range of tolerable non-linearities are quite different, and it is perhaps fortunate that the situation where a large amount of non-linearity can be tolerated, the nominally optimal feedback system, is also the situation wherein large amounts of non-linearity may be experienced.

The result on nominally linear stable systems is reassuring, in that it shows such systems are structurally stable. But it does go further than this, for, at the expense of the difficult calculations necessary to check positive realness, it gives a sufficiency test to see whether components of prescribed specifications can be satisfactorily incorporated in a system.

Both results illustrate the often experienced fact that saturation-type non-linearities can lead to trouble. Such non-linearities of course fall well outside a sector non-linearity in general.

## References

Anderson, B. D. O., 1966 a, *Int. J. Control*, **3**, 535; 1966 b, *J. Franklin Inst.*, **281**, 155; 1967, *J. SIAM Control* Vol. 5, No 2. May '67.

Athans, M., and Falb, P. L., 1966, *Optimal Control* (New York; McGraw-Hill).

Newcomb, R. W., 1966, *Linear Multiport Synthesis* (New York: McGraw-Hill).

Popov, V. M., 1961, *Avomatika Telemekh.*, **22**, 961 (English translation: 1962, *Automn remote Control.*, **22**, 857).

Tokumaru, H., and Saito, N., 1965, *Mem. Fac. Engng Kyoto Univ.*, **28**, 347.