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Brief paper

Selection of a single uniquely specifiable $\mathscr{H}_\infty$ controller in the chain-scattering framework☆

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Abstract

This paper studies the notion of central $\mathscr{H}_\infty$ controllers in the chain-scattering framework. Since in the chain-scattering framework the $\mathscr{H}_\infty$ control problem is equivalent to a factorisation problem that yields non-unique factors, it is important to somehow pin down these factors so that the resulting central controller is uniquely defined and corresponds to the central and minimum-entropy controller frequently discussed in the literature. In so doing, we devise a procedure for the selection of a single uniquely specifiable $\mathscr{H}_\infty$ controller in the chain-scattering framework. The chain-scattering framework is important because it has a distinct advantage over state-space based methods for some applications in that it poses and solves the $\mathscr{H}_\infty$ control problem entirely in an operator-theoretic setting, thereby allowing us to contemplate direct changes of the frequency domain symbols rather than changes in the corresponding state-space matrices.

Keywords: $\mathscr{H}_\infty$-control; J-Lossless factorisation; Chain-scattering; Central controller; Minimum entropy controller

1. Introduction

The *normalised $\mathscr{H}_\infty$ control problem* can be stated as follows: "Find a controller K such that the closed-loop system of Fig. 1 is internally stable and the closed-loop transfer function $T_{zw} = \mathscr{F}_l(P,K)$ satisfies $\|T_{zw}\|_\infty < 1$", where $\mathscr{F}_l(.,.)$ denotes the lower Linear Fractional Transformation depicted in Fig. 1. A controller K is said to be *admissible* if it solves the normalised $\mathscr{H}_\infty$ control problem. We also usually seek to characterise the set of all admissible controllers. For such a set, a controller K_c is said to be a *central controller* if it is achieved by setting a certain free parameter characterising this set to zero.

In standard literature (Doyle, Glover, Khargonekar, & Francis 1989; Green & Limebeer, 1995; Zhou, Doyle, & Glover, 1996), the set of all admissible controllers is given by the set of all transfer function matrices $K = \mathscr{F}_l(M,S)$, where M is constructed from the plant state-space matrices and $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$ is a free parameter that characterises the set. Consequently, the central controller frequently discussed in the literature is simply given by $K_c = \mathscr{F}_l(M,0)$. It has been shown in Glover and Doyle (1988), Doyle et al. (1989), that K_c has the same McMillan degree as the plant P and in Mustafa and Glover (1988), Glover and Mustafa (1989), Mustafa and Glover (1990) and Mustafa, Glover, and Limebeer (1991) that K_c also minimises the entropy function. In fact, this central controller has some interesting interpretations and motivations in the literature.

It thus seems natural that if for any reason we wish to select a single uniquely specifiable controller from the set of all admissible $\mathscr{H}_\infty$ controllers, the central (or minimum entropy) controller should be our natural choice. For instance, one may be interested in making use of a single controller from the set of all admissible controllers to study how changes in the plant P (perhaps due to changes in the performance weights that are absorbed in P) manifest

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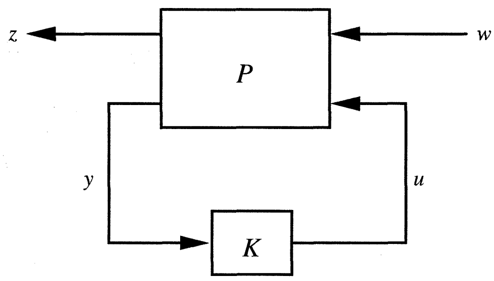

Fig. 1. Closed-loop system.

themselves as controller changes in an $\mathscr{H}_\infty$ design. In order for such a study to make sense, we must first be able to select a single uniquely specifiable controller from the admissible controller set and then simply investigate modifications or changes on this single controller. This kind of study was in fact first tackled in Bombois and Anderson (2002) and later generalised and extended in Lanzon, Bombois, and Anderson (2003b); both Bombois and Anderson (2002), and Lanzon et al. (2003b) motivated the development of the ideas of this paper.

Since their inception, $\mathscr{H}_\infty$ control problems have been amenable to a variety of solution techniques. While some techniques may be more powerful than others in particular design situations (say state-space methods for numerical solution, interpolation theory for fundamental limitation constraints, linear matrix inequalities for multiple objective control, etc.), it is important to make sure that we understand all the implications of any particular solution method on $\mathscr{H}_\infty$ control theory and to ensure that all solution techniques induce the exact same notions as other solution techniques. In this paper, we shall study the notions of central controller and minimum-entropy controller in the chain-scattering approach to $\mathscr{H}_\infty$ control (Kimura, 1997).

The J-lossless factorisations used in (Kimura, 1997) and this paper are closely related to the J-spectral factorisations used in Ball and Cohen (1987), Green, Glover, Limebeer, and Doyle (1990), Kimura, Lu, and Kawatani (1991), Green (1992) to derive equivalent results. This chain-scattering operator-theoretic framework possesses a distinct advantage over state-space based approaches in that it poses and solves the $\mathscr{H}_\infty$ control problem entirely in the frequency domain (i.e. Kimura, 1997 shows that the normalised $\mathscr{H}_\infty$ control problem is equivalent to a J-lossless factorisation problem and that the set of all admissible controllers can be completely characterised in terms of one of the resulting factors). This entirely operator-theoretic framework permits direct manipulation of the frequency domain symbols which may be very useful in some applications. For example, one may wish to consider changes in the frequency domain symbols that involve changes in the McMillan degree of the symbols, as in Bombois and Anderson (2002), and Lanzon et al. (2003b). This kind of manipulation would otherwise be more cumbersome and not easily cast in state-space descriptions. It is this kind of argument that motivates us to study the notion of central controller in the chain-scattering framework.

It is however well known that factorisation problems often do not have unique solutions and hence the admissible controller set is characterised in terms of a non-unique factor in this chain-scattering framework. This is clearly undesirable as the centre of the admissible controller set (i.e. the central controller for the particular parametrisation considered) may be different for every reparametrisation. One has to thus pin down the factors resulting from the J-lossless factorisation in order to ensure that the central controller is uniquely defined in this chain-scattering framework and to also ensure that it corresponds to the central and minimum-entropy controller so frequently discussed in the literature. This will be done by pinning down one of the factors in the J-lossless factorisation at infinite frequency. In so doing, we shall also demonstrate that the standard assumptions frequently adopted in the literature (Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996) bury some interesting features of $\mathscr{H}_\infty$ control, particularly associated with the central and minimum-entropy controller. These features will also be exposed and discussed in this paper.

2. Preliminaries

2.1. Chain-scattering representation of plant

Consider the following generalised plant

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A^P & B_1^P & B_2^P \\ \hline C_1^P & D_{11}^P & D_{12}^P \\ C_2^P & \underbrace{D_{21}^P}_{r} & \underbrace{D_{22}^P}_{p} \end{array}\right] \begin{array}{l} \\ \}m \\ \}q \end{array} \tag{1}$$

and assume that it satisfies the following assumption:

Assumption A1. $q \leqslant r$, $p \leqslant m$ and $\operatorname{rank}[P_{21}(j\omega)] = q$, $\operatorname{rank}[P_{12}(j\omega)] = p$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

If P_{21}^{-1} exists (i.e. if $r = q$), then the generalised plant P can be alternatively represented by

$$G := \mathrm{CHAIN}(P) = \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \tag{2}$$

with inputs and outputs swapped as shown in Fig. 2. Then, applying a controller $u = Ky$, the closed-loop transfer function matrix T_{zw} is given by

$$\begin{aligned} T_{zw} &= \mathscr{F}_l(P,K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= \mathrm{HM}(G,K) := (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}, \end{aligned}$$

where HM(.,.) denotes the "homographic transformation" frequently used in classical circuit theory.

Problems where neither $r = q$ nor $m = p$ hold are harder because (a) the plant needs to be augmented in order to

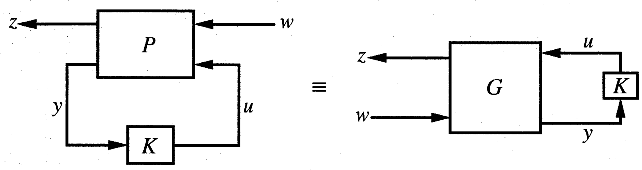

Fig. 2. Linear fractional and homographic transformations.

derive a chain-scattering representation, and (b) the results obtained need to be independent of the particular augmentation chosen. Such problems are usually referred to as four-block problems in the literature. In four-block problems, the plant P is augmented by a fictitious measured output y' of dimension $(r-q)$ given by

$$y' = P'_{21}w + P'_{22}u \tag{3}$$

to give

$$\begin{bmatrix} z \\ \begin{pmatrix} y \\ y' \end{pmatrix} \end{bmatrix} = \overbrace{\begin{bmatrix} P_{11} & P_{12} \\ \begin{pmatrix} P_{21} \\ P'_{21} \end{pmatrix} & \begin{pmatrix} P_{22} \\ P'_{22} \end{pmatrix} \end{bmatrix}}^{P_{\mathrm{aug}}} \begin{bmatrix} w \\ u \end{bmatrix}. \tag{4}$$

Assumption A2. P'_{21} is chosen in such cases so that $\operatorname{rank}\left[\begin{pmatrix} P_{21}(j\omega) \\ P'_{21}(j\omega) \end{pmatrix}\right] = r$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

If assumption (A2) holds, a chain-scattering representation G of the augmented plant P_{aug} exists.

2.2. Normalised $\mathscr{H}_\infty$ control problems

In this article, we will restrict attention to the four-block problem setting (i.e. when $q<r$) because the four-block problem is harder than the two-block problem (i.e. when $q=r$), and hence the reader can easily obtain the results for the two-block problem by simple modifications of the proofs presented in this paper.

The (J_{mr},J_{pr})-lossless factorisation defined below is a generalisation of the well known inner-outer factorisation for stable systems and the well known spectral factorisation for positive hermitian systems. Here, J_{mr} denotes the signature matrix $J_{mr} = \operatorname{diag}(I_m, -I_r)$.

Definition 1. The rational matrix $G \in \mathbb{R}^{(m+r)\times(p+r)}$ is said to have a (J_{mr},J_{pr})-lossless factorisation if G is represented as the product $G = \Theta\Pi$, where Θ is (J_{mr},J_{pr})-lossless and Π is unimodular in $\mathscr{RH}_\infty$.

Necessary and sufficient conditions for the existence of a (J_{mr},J_{pr})-lossless factorisation of G and the construction of factors Θ and Π can be found in Kimura (1997) and a detailed treatment of J-lossless systems is given in Dewilde and Dym (1981, 1984).

It should be clear that if factors Θ and Π exist, then they are not unique. In fact, any two solution pairs Θ_1, Π_1 and Θ_2, Π_2 to the (J_{mr},J_{pr})-lossless factorisation of G must be related by

$$\Theta_2 = \Theta_1\Psi^{-1} \quad \text{and} \quad \Pi_2 = \Psi\Pi_1, \tag{5}$$

where Ψ is a real nonsingular matrix satisfying $\Psi^{\mathrm{T}}J_{pr}\Psi = J_{pr}$. This is because

$$\Pi_2^{\sim}J_{pr}\Pi_2 = G^{\sim}J_{mr}G = \Pi_1^{\sim}J_{pr}\Pi_1$$

$$\Rightarrow J_{pr}(\Pi_2\Pi_1^{-1}) = (\Pi_2\Pi_1^{-1})^{-\sim}J_{pr}$$

and thus we see that $\Psi := \Pi_2\Pi_1^{-1}$ is a real constant matrix since the left side of the above equation is unimodular in $\mathscr{RH}_\infty$ and the right side is unimodular in $\mathscr{RH}_\infty^{\perp}$.

The following theorem reduces the normalised $\mathscr{H}_\infty$-control problem into a (J_{mr},J_{pr})-lossless factorisation problem.

Theorem 1 (Kimura, 1997). *Suppose that a plant $P \in \mathscr{RL}_\infty$ given by Eq.* (1) *satisfies assumption* (A1) *and is such that $q<r$. Then, the normalised $\mathscr{H}_\infty$ control problem is solvable for P if and only if there exists an output augmentation* (3) *with $P'_{21}, P'_{22} \in \mathscr{RL}_\infty$ such that the augmented plant P_{aug} in* Eq. (4) *satisfies assumption* (A2) *and* $G = \mathrm{CHAIN}(P_{\mathrm{aug}})$ *has a (J_{mr},J_{pr})-lossless factorisation*

$$G = \Theta\Pi$$

with Π of the form

$$\Pi = \begin{array}{c} \begin{bmatrix} \Pi_a & 0 \\ \Pi_{b1} & \Pi_{b2} \end{bmatrix} \begin{array}{l} \}p+q \\ \}r-q. \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{p+q} & \underbrace{\quad}_{r-q} \end{array} \end{array} \tag{6}$$

In that case, K is an admissible controller if and only if

$$K = \mathrm{HM}(\Pi_a^{-1}, S)$$

for an $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$.

In the above theorem, a controller K resulting from the choice $S=0$ is termed a *central controller obtained through the chain-scattering framework.*

As asserted earlier in this section, the particular choice of the plant augmentation (3) ought to play no role in the final solution of the normalised $\mathscr{H}_\infty$ control problem. That this is indeed the case can be seen from the following argument: let the chain-scattering plant $G=\mathrm{CHAIN}(P_{\mathrm{aug}})$ be written as

$$G = \begin{bmatrix} P_{12} & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \begin{pmatrix} P_{22} \\ P'_{22} \end{pmatrix} & \begin{pmatrix} P_{21} \\ P'_{21} \end{pmatrix} \end{bmatrix}^{-1}.$$

If the normalised $\mathscr{H}_\infty$ control problem is solvable, then there exists a Π of the form of Eq. (6) such that

$G^{\sim}J_{mr}G = \Pi_1^{\sim}J_{pr}\Pi$. We must show that Π_a (the only sub-block of Π that is used in characterising the controller set) is independent of the particular augmentation chosen. Towards this end, note that

$$(G^{\sim}J_{mr}G)^{-1} = \begin{bmatrix} I & 0 \\ P_{22} & P_{21} \\ P'_{22} & P'_{21} \end{bmatrix}$$

$$\times \begin{bmatrix} P_{12}^{\sim}P_{12} & P_{12}^{\sim}P_{11} \\ P_{11}^{\sim}P_{12} & P_{11}^{\sim}P_{11} - I \end{bmatrix}^{-1} \begin{bmatrix} I & P_{22}^{\sim} & P_{22}'^{\sim} \\ 0 & P_{21}^{\sim} & P_{21}'^{\sim} \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} \surd & \surd & \times \\ \surd & \surd & \times \\ \hline \times & \times & \times \end{array}\right].$$

Here, $\surd$ denotes terms that do not depend on the augmentation and $\times$ denotes terms that do depend of the particular augmentation chosen. Since

$$(G^{\sim}J_{mr}G)^{-1} = \Pi^{-1}J_{pr}\Pi^{-\sim} = \begin{bmatrix} \Pi_a^{-1}J_{pq}\Pi_a^{-\sim} & \blacklozenge \\ \blacklozenge & \blacklozenge \end{bmatrix},$$

where $\blacklozenge$ denotes "Don't care items", it follows that Π_a is independent of the particular augmentation chosen.

Furthermore, since Π in Theorem 1 is required to be of the lower triangular form of Eq. (6), the argument immediately after Definition 1 that establishes the non-uniqueness in the factors Θ and Π characterised by a real nonsingular matrix Ψ that satisfies $\Psi^{\mathrm{T}}J_{pr}\Psi = J_{pr}$ may be extended further to give other structural properties of Ψ. In fact, note that $\Psi := \Pi_2\Pi_1^{-1}$ is now also lower triangular and hence this new observation together with the constraint $J_{pr}\Psi = \Psi^{-\mathrm{T}}J_{pr}$ imply that Ψ has to have the following block diagonal form

$$\Psi = \begin{bmatrix} \Psi_a & 0 \\ 0 & \Psi_b \end{bmatrix} \tag{7}$$

with $\Psi_a \in \mathbb{R}^{(p+q)\times(p+q)}$ satisfying $\Psi_a^{\mathrm{T}}J_{pq}\Psi_a = J_{pq}$ and $\Psi_b \in \mathbb{R}^{(r-q)\times(r-q)}$ satisfying $\Psi_b^{\mathrm{T}}\Psi_b = I_{(r-q)}$. Note furthermore that Ψ_a expresses the non-uniqueness in the unimodular matrix Π_a.

3. Questions of interest

Since the unimodular matrix Π and the portion of interest Π_a are unique up to left multiplication by a constant J-unitary real nonsingular matrix, the questions of interest that will be addressed in this paper are:

A. Is $\mathrm{HM}(\Pi_a^{-1}, 0)$ a single controller or is there a family of such controllers obtained by considering all possible Π_a in the factorisation of G?
B. If there is a family of such controllers:
   a. Does the central controller advanced by the literature (Glover & Doyle, 1988; Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996; Kimura, 1997) belong to this family?
   b. Do all such controllers possess the same properties as the central controller given in the literature (Glover & Doyle, 1988; Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996; Kimura, 1997)? For instance, do they all minimise the entropy function, and are they all strictly proper when certain conditions are fulfilled?
   c. What properties need to be enforced in order to isolate just one member (i.e. select a single uniquely specifiable member) of this family?

4. Addressing the posed questions in a four-block setting

4.1. Reparametrisation of controller set

The first result presented here, which is as expected, shows that the extra freedom associated with the non-uniqueness of Π simply reparametrises the same controller set.

Lemma 2. *Given any two (J_{mr}, J_{pr})-lossless factorisations of $G = \Theta_1\Pi_1 = \Theta_2\Pi_2$ with Π_1 and Π_2 of the form in equation* (6), *the following two sets are identical*:

$$\{\mathrm{HM}(\Pi_{a,1}^{-1}, S) : S \in \mathcal{RH}_\infty, \|S\|_\infty < 1\}$$
$$\equiv \{\mathrm{HM}(\Pi_{a,2}^{-1}, S) : S \in \mathcal{RH}_\infty, \|S\|_\infty < 1\}.$$

Proof. It follows trivially from the fact that the non-uniqueness in Π_a has the form $\Pi_{a,2} = \Psi_a\Pi_{a,1}$ for some nonsingular real Ψ_a that satisfies $\Psi_a^{\mathrm{T}}J_{pq}\Psi_a = J_{pq}$. □

4.2. Uniqueness of strictly proper central controllers

As pointed out at the end of Section 2.2, the unimodular matrix Π_a is unique up to left multiplication by a constant real nonsingular matrix Ψ_a that satisfies $\Psi_a^{\mathrm{T}}J_{pq}\Psi_a = J_{pq}$. Consequently, there is evidently a family of central controllers described by $\mathrm{HM}(\Pi_a^{-1}, 0)$ for all Π_a arising in the (J_{mr}, J_{pr})-lossless factorisation of G. In this section, we will determine a condition that enforces uniqueness of $\mathrm{HM}(\Pi_a^{-1}, 0)$ and give necessary and sufficient conditions when such a unique central controller in the chain-scattering framework exists.

Theorem 3. *Suppose that a plant $P \in \mathcal{RL}_\infty$ given by Eq.* (1) *satisfies assumption* (A1) *and is such that $q < r$. Suppose furthermore that there exists an output augmentation* (3) *with $P'_{21}, P'_{22} \in \mathcal{RL}_\infty$ such that the augmented plant P_{aug} in Eq.* (4) *satisfies assumption* (A2) *and $G = \mathrm{CHAIN}(P_{\mathrm{aug}})$ admits a (J_{mr}, J_{pr})-lossless factorisation*

$G = \Theta\Pi$

with Π of the form

$$\Pi = = \begin{bmatrix} \Pi_a & 0 \\ \Pi_{b1} & \Pi_{b2} \end{bmatrix} \begin{matrix} \}p+q \\ \}r-q. \end{matrix}$$
$$\qquad \underbrace{}_{p+q} \underbrace{}_{r-q}$$

Then, the following statements are equivalent:

(a) $\bar{\sigma}(D_{11}^P) < 1$,
(b) *there exists a unique real nonsingular matrix* $E = \Pi(j\infty)$ *satisfying* $D^{\mathrm{T}} J_{mr} D = E^{\mathrm{T}} J_{pr} E$ *with* $D = G(j\infty)$ *of the form*

$$E = \begin{bmatrix} \begin{pmatrix} E_{a11} & 0 \\ E_{a21} & E_{a22} \end{pmatrix} & 0 \\ E_{b1} & E_{b2} \end{bmatrix}$$

with $0 < E_{a11} \in \mathbb{R}^{p\times p}$, $0 < E_{a22} \in \mathbb{R}^{q\times q}$ *and* $0 < E_{b2} \in \mathbb{R}^{(r-q)\times(r-q)}$,
(c) *there exists an admissible strictly proper central controller* K_{c},
(d) *there exists an admissible strictly proper non-central controller* K_{nc}.

Furthermore, if these conditions are satisfied, then the admissible strictly proper central controller K_{c} (*in statement* (c) *above*) *is uniquely defined as there exists no other admissible strictly proper controller that is also central in the sense of the chain-scattering framework considered.*

Proof. This equivalent conditions in the theorem statement will be shown by the following sequence of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a $\Rightarrow$ b): Since $G = \mathrm{CHAIN}(P_{\mathrm{aug}})$, it follows that

$$D = \begin{bmatrix} D_{12}^P - D_{11}^P (D_{21}^P)_{\mathrm{aug}}^{-1} (D_{22}^P)_{\mathrm{aug}} & D_{11}^P (D_{21}^P)_{\mathrm{aug}}^{-1} \\ -(D_{21}^P)_{\mathrm{aug}}^{-1} (D_{22}^P)_{\mathrm{aug}} & (D_{21}^P)_{\mathrm{aug}}^{-1} \end{bmatrix}.$$

Thus,

$$\bar{\sigma}(D_{11}^P) < 1 \Leftrightarrow \bar{\sigma}(D_{12} D_{22}^{-1}) < 1$$
$$\Leftrightarrow D_{12}^* D_{12} - D_{22}^* D_{22} < 0.$$

Now,

$$D^{\mathrm{T}} J_{mr} D = \begin{bmatrix} \blacklozenge & \blacklozenge \\ \blacklozenge & D_{12}^* D_{12} - D_{22}^* D_{22} \end{bmatrix} =: \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12}^{\mathrm{T}} & \varphi_{22} \end{bmatrix},$$

where $\blacklozenge$ denotes "Don't Care" elements. Consequently, the (22)-element of $D^{\mathrm{T}} J_{mr} D$ (i.e. φ_{22}) is strictly negative.

This implication is shown through a constructive proof as follows: First note that since $0 > \varphi_{22} \in \mathbb{R}^{r\times r}$ and

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12}^{\mathrm{T}} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} I_p & \varphi_{12}\varphi_{22}^{-1} \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \varphi_{11} - \varphi_{12}\varphi_{22}^{-1}\varphi_{12}^{\mathrm{T}} & 0 \\ 0 & \varphi_{22} \end{bmatrix} \times \begin{bmatrix} I_p & 0 \\ \varphi_{22}^{-1}\varphi_{12}^{\mathrm{T}} & I_r \end{bmatrix}$$

is nonsingular and has at most r negative eigenvalues, $0 < (\varphi_{11} - \varphi_{12}\varphi_{22}^{-1}\varphi_{12}^{\mathrm{T}}) \in \mathbb{R}^{p\times p}$. Furthermore, decompose $\varphi_{22} \in \mathbb{R}^{r\times r}$ as follows

$$\varphi_{22} =: \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12}^{\mathrm{T}} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} I_q & \alpha_{12}\alpha_{22}^{-1} \\ 0 & I_{(r-q)} \end{bmatrix} \times \begin{bmatrix} \alpha_{11} - \alpha_{12}\alpha_{22}^{-1}\alpha_{12}^{\mathrm{T}} & 0 \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ \alpha_{22}^{-1}\alpha_{12}^{\mathrm{T}} & I_{(r-q)} \end{bmatrix}$$

so that $0 > (\alpha_{11} - \alpha_{12}\alpha_{22}^{-1}\alpha_{12}^{\mathrm{T}}) \in \mathbb{R}^{q\times q}$ and $0 > \alpha_{22} \in \mathbb{R}^{(r-q)\times(r-q)}$. Consequently,

$$\exists E_{a22} \in \mathbb{R}^{q\times q} : E_{a22}^{\mathrm{T}} E_{a22} = -(\alpha_{11} - \alpha_{12}\alpha_{22}^{-1}\alpha_{12}^{\mathrm{T}})$$
$$\exists E_{b2} \in \mathbb{R}^{(r-q)\times(r-q)} : E_{b2}^{\mathrm{T}} E_{b2} = -\alpha_{22}$$
$$E_{b12} \in \mathbb{R}^{(r-q)\times q} : E_{b2}^{\mathrm{T}} E_{b12} = -\alpha_{12}^{\mathrm{T}}$$

with both E_{a22} and E_{b2} nonsingular such that

$$\varphi_{22} = \begin{bmatrix} E_{a22} & 0 \\ E_{b12} & E_{b2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -I_q & 0 \\ 0 & -I_{(r-q)} \end{bmatrix} \begin{bmatrix} E_{a22} & 0 \\ E_{b12} & E_{b2} \end{bmatrix}$$

and furthermore

$$\exists E_{a11} \in \mathbb{R}^{p\times p} : E_{a11}^{\mathrm{T}} E_{a11} = (\varphi_{11} - \varphi_{12}\varphi_{22}^{-1}\varphi_{12}^{\mathrm{T}}),$$

$$\exists \begin{bmatrix} E_{a21} \\ E_{b11} \end{bmatrix} \in \mathbb{R}^{r\times p} : \begin{bmatrix} E_{a22} & 0 \\ E_{b12} & E_{b2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} E_{a21} \\ E_{b11} \end{bmatrix} = -\varphi_{12}^{\mathrm{T}}$$

with E_{a11} nonsingular such that

$$D^{\mathrm{T}} J_{mr} D = \begin{bmatrix} E_{a11} & 0 & 0 \\ E_{a21} & E_{a22} & 0 \\ E_{b11} & E_{b12} & E_{b2} \end{bmatrix}^{\mathrm{T}} J_{pr} \begin{bmatrix} E_{a11} & 0 & 0 \\ E_{a21} & E_{a22} & 0 \\ E_{b11} & E_{b12} & E_{b2} \end{bmatrix}.$$

This part of the proof is concluded by restricting E_{a11}, E_{a22} and E_{b2} to be strictly positive definite matrices so as to remove all non-uniqueness in the above selections. This can always be done because a lower triangular matrix satisfying the above equation is unique up to left multiplication by a block-diagonal real orthogonal matrix.

(b $\Rightarrow$ c): The suppositions of the theorem guarantee that the normalised $\mathscr{H}_\infty$ control problem is solvable (by Theorem 1). Thus there exists a set of admissible controllers with the central member for a particular parametrisation of this admissible controller set referred to as the central controller K_{c}. Then note that $K_{\mathrm{c}}(j\infty) = \mathrm{HM}(\Pi_a(j\infty)^{-1}, 0) = \mathrm{HM}(E_a^{-1}, 0) = 0$ since E_a^{-1} is lower triangular.

(c $\Rightarrow$ d): If $K_c(j\infty) = \text{HM}(\Pi_a(j\infty)^{-1}, 0) = 0$, then this implication trivially follows from $K_{nc}(j\infty) = \text{HM}(\Pi_a(j\infty)^{-1}, S(j\infty)) = 0$ on choosing a non-zero $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$ and $S(j\infty) = 0$.

(d $\Rightarrow$ a): If $K_{nc}(j\infty) = 0$, then $T_{zw} = P_{11} + P_{12}K_{nc}(I - P_{22}K_{nc})^{-1}P_{21}$ reduces to $T_{zw}(j\infty) = D^P_{11}$ at infinite frequency. Thus, $\bar{\sigma}(D^P_{11}) < 1$ because the suppositions of the theorem guarantee that the normalised $\mathscr{H}_\infty$ control problem is solvable.

Now, we conclude the proof of this theorem by showing that, when it exists, a strictly proper central controller is always uniquely defined in the chain-scattering framework. Towards this end, note that $K_c(j\infty) = \text{HM}(\Pi_a^{-1}(j\infty), 0) = 0$ if and only if $E_a = \Pi_a(j\infty)$ is a lower triangular matrix. Furthermore, similarly to the discussion at the end of Section 2.2, such a matrix E_a is unique up to left multiplication by a block-diagonal real orthogonal matrix $\Psi_a = \text{diag}(\Psi_{a1}, \Psi_{a2})$. Then, since the non-uniqueness in Π_a comes from nonuniqueness in E_a, we can relate two unimodular matrices Π_a by $\Pi_{a,2} = \Psi_a \Pi_{a,1}$. This in turn implies

$$\begin{aligned}\text{HM}(\Pi_{a,2}^{-1}, 0) &= \text{HM}((\Psi_a \Pi_{a,1})^{-1}, 0)\\ &= \text{HM}(\Pi_{a,1}^{-1}, \text{HM}(\Psi_a^{-1}, 0))\\ &= \text{HM}(\Pi_{a,1}^{-1}, 0)\end{aligned}$$

which gives the required result. □

Thus, from this theorem, if there exists a strictly proper central controller, then (with the strict properness constraint imposed) it is unique. It should be pointed out that if we impose the same simplifying assumptions as in the literature (Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996), then the strictly proper central controller discussed here (which was shown in this article to be uniquely defined also in the chain-scattering framework) is identical to the central controller given in the literature. However, the simplifying assumptions of the literature bury the existential questions as there always exists a strictly proper central controller under the assumptions, as will be argued next. There are in fact generalised plants P with $\bar{\sigma}(D^P_{11}) \geqslant 1$ for which the normalised $\mathscr{H}_\infty$ control problem is solvable. For such generalised plants, there cannot exist an admissible strictly proper controller (as seen from Theorem 3). A method for dealing with such generalised plant is detailed in subsection 4.4. For the moment, let us turn our attention to the relation between minimum entropy and central controllers in the chain-scattering framework.

4.3. *Minimum entropy and central controllers*

Let T be a transfer function matrix such that $\|T\|_\infty < \gamma$. Then the entropy of $T(s)$ is defined by

$$I(T,\gamma) := -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\sum_i \ln|1 - \gamma^{-2}\sigma_i(T(j\omega))^2|\,\mathrm{d}\omega, \quad (8)$$

where $\sigma_i(T(j\omega))$ is the ith singular value of $T(j\omega)$. It should be clear that $I(T,\gamma) \geqslant 0$ and

$$\lim_{\gamma\to\infty} I(T,\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i(T(j\omega))^2\,\mathrm{d}\omega = \|T\|_2^2.$$

Thus the entropy $I(T,\gamma)$ is a performance index measuring the tradeoff between $\mathscr{H}_\infty$ optimality ($\gamma \to \|T\|_\infty$) and $\mathscr{H}_2$ optimality ($\gamma \to \infty$).

The entropy function $I(T,\gamma)$ has been studied in great detail (Mustafa & Glover, 1988; Glover & Mustafa, 1989; Mustafa & Glover, 1990; Mustafa et al., 1991) in the late 1980s and early 1990s. Letting $T(s)$ be our closed-loop transfer function matrix $T_{zw}(s)$ and $\gamma = 1$ for a normalised $\mathscr{H}_\infty$ control problem, it is not difficult to see that $I(T_{zw}, 1)$ is finite if and only if $T_{zw}(j\infty) = 0$. Since the simplifying assumptions in the literature (Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996) always require $D^P_{11} = 0$, we know from Theorem 3 that in this situation there always exists a strictly proper central controller that is uniquely defined in this chain-scattering framework. This controller obviously achieves $T_{zw}(j\infty) = 0$ and hence finite entropy. It was in fact shown in Mustafa and Glover (1988), Glover and Mustafa (1989), Mustafa and Glover (1990), Mustafa et al. (1991) that this unique strictly proper central controller also minimises the value of the entropy function. For this reason, when the simplifying assumptions of the literature (Doyle et al., 1989; Green & Limebeer, 1995; Zhou et al., 1996) are enforced, the central controller is often also called the minimum entropy controller.

The question that immediately arises is: Would the central controller (in the sense of this paper in the chain-scattering framework considered) still be the same as the minimum entropy controller if $D^P_{11} = 0$ is not assumed and if we are allowed to choose the characterisation of the set of all admissible controllers as we desire? In general, the answer to this question is no (again it is emphasised that the definition of central controller is taken in the sense of this paper). This is illustrated by the following example.

Example. Consider a plant P with $0 \neq \bar{\sigma}(D^P_{11}) < 1$ for which the normalised $\mathscr{H}_\infty$ control problem is solvable. Assume that there exists an admissible controller that makes the closed-loop transfer function matrix T_{zw} strictly proper (i.e. assume that there exists $D^K \in \mathbb{R}^{p\times q}$ such that $D^P_{11} + D^P_{12}D^K(I - D^P_{22}D^K)^{-1}D^P_{21} = 0$). Then for this controller, we get finite entropy. Thus the minimum entropy controller will certainly achieve finite entropy.

However, since $\bar{\sigma}(D^P_{11}) < 1$, we know from Theorem 3 that there always exists a unique strictly proper central controller. With this controller, T_{zw} is clearly not strictly proper and hence we get infinite entropy. Consequently, this unique strictly proper central controller does not minimise the entropy function in this situation since there is another admissible controller which achieves a smaller value of entropy.

The essence of this example and the preceeding discussion is captured in the following theorem.

Theorem 4. *Let the suppositions of Theorem* 3 *hold and furthermore let* $\bar{\sigma}(D_{11}^P) < 1$ (*which is necessary and sufficient for the existence of an admissible strictly proper central controller*) *and* $\min_Q \bar{\sigma}(D_{11}^P + D_{12}^P Q D_{21}^P) = 0$ (*which is necessary for there to be a controller yielding finite minimum entropy*).

I. *If* $D_{11}^P = 0$, *then the uniquely defined admissible strictly proper central controller* $K_c = \mathrm{HM}(\Pi_a^{-1}, 0)$ *also minimises the entropy function.*
II. *If* $D_{11}^P \neq 0$, *then the uniquely defined admissible strictly proper central controller* $K_c = \mathrm{HM}(\Pi_a^{-1}, 0)$ *does not minimise the entropy function, even though a minimum entropy controller exists.*

Recall that the principal problem we are addressing in this paper is to be able to specify a uniquely defined central controller in the chain-scattering framework. We have shown how to do this when $\bar{\sigma}(D_{11}^P) < 1$ but we have not yet shown what needs to be done when $\bar{\sigma}(D_{11}^P) \geqslant 1$. When $\bar{\sigma}(D_{11}^P) \geqslant 1$, the normalised $\mathscr{H}_\infty$ control problem may be solvable, but any central controller will necessarily be bi-proper, since a non-zero controller gain at infinite frequency is required in order to secure $\|T_{zw}\|_\infty < 1$. Therefore, in the next subsection, we will treat the case when $\bar{\sigma}(D_{11}^P) \geqslant 1$ via a loop-shifting argument. This loop-shifting argument transforms the original problem into one where a strictly proper central controller always exists. In the next subsection, we will also indirectly address the issues raised by Theorem 4 and the above example. Through careful selection of a loop-shifting gain matrix, we will achieve the desirable correspondence between the resulting uniquely defined central controller in this chain-scattering setting and the minimum entropy controller (Mustafa & Glover, 1988; Glover & Mustafa, 1989; Mustafa & Glover, 1990; Mustafa et al., 1991) advanced in the literature.

4.4. *A loop-shifting argument to select a uniquely defined controller*

One approach is to explain how to make the selection of a unique E satisfying $D^T J_{mr} D = E^T J_{pr} E$ with $D = G(j\infty)$, as this E will uniquely determine the central controller in this chain-scattering setting and the properties associated with this central controller. It was for example pointed out in the preceding subsection that we can make selections of E for which the resulting central controller, though unique, does not always correspond to the minimum entropy controller. This is clearly undesirable. Furthermore, in an earlier subsection, it was also shown that if the unique selection of E is obtained by requiring some special property on the central controller (such as strict-properness), then there are admissible situations (with $\bar{\sigma}(D_{11}^P) \geqslant 1$) that do not allow this kind of unique selection of E.

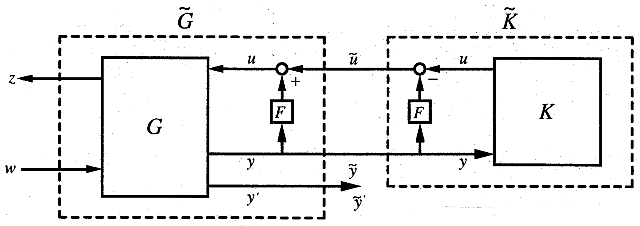

Fig. 3. Loop-shifting transformation for a four-block problem.

In this subsection, we will construct a unique E, show that this unique E always exists (provided the normalised $\mathscr{H}_\infty$ control problem is solvable), and show that it also reduces to the unique lower triangular matrix E of Theorem 3 when $D_{11}^P = 0$. The derivation of such a unique E relies on the following loop-shifting argument. The reader can refer to Safonov and Limebeer (1988), Safonov, Limebeer, and Chiang (1989), Green and Limebeer (1995) for extensive coverage of this topic.

Consider the feedback interconnection of Fig. 3. The process of loop-shifting can be conceptually viewed as extracting the controller gain at infinite frequency from the controller and putting it into the plant. This is done through the constant gain matrix F. When this is done, the original interconnection of G and K is replaced with an equivalent interconnection of $\tilde{G}$ and $\tilde{K}$. The relations between the original systems and the loop-shifted systems are given below

$$\tilde{G}(s) = G(s) \begin{bmatrix} \begin{pmatrix} I_p & F \\ 0 & I_q \end{pmatrix} & 0 \\ 0 & I_{(r-q)} \end{bmatrix}$$

and

$$\tilde{K}(s) = \mathrm{HM}\left(\begin{bmatrix} I_p & -F \\ 0 & I_r \end{bmatrix}, K(s) \right) = K(s) - F.$$

In the context of this paper, F will be chosen so as to minimise $\bar{\sigma}(\mathrm{HM}(D,F))$, where $D = G(j\infty)$. Note that $\mathrm{HM}(D,F)$ corresponds to the gain at infinite frequency of the transfer function from w to z. Since the normalised $\mathscr{H}_\infty$ control problem is assumed to be solvable, it will always be possible to select an F such that $\bar{\sigma}(\mathrm{HM}(D,F)) < 1$ (the argument here is that F is just the gain at infinite frequency of any controller achieving $\|T_{zw}\|_\infty < 1$). Then, applying Theorem 3 on the loop-shifted plant $\tilde{G}$, we see that there exists a unique lower triangular matrix $\tilde{E}$ that satisfies $\tilde{D}^T J_{mr} \tilde{D} = \tilde{E}^T J_{pr} \tilde{E}$ where $\tilde{D} = \tilde{G}(j\infty)$. Consequently, the matrix E that needs to be selected will be composed of the unique lower triangular matrix $\tilde{E}$ and the loop-shifting transformation.

The matrix E is only unique after we fix (a) the choice of plant augmentation, and (b) in the case when finite minimum entropy is not possible (i.e. $\min_F \bar{\sigma}(\mathrm{HM}(D,F)) \neq 0$), the choice of the minimising matrix F. However, the sub-block

E_a in the matrix E (i.e. the only sub-block of interest in constructing the controller set) is always independent of the choice of plant augmentation. It is also important to point out that if it is possible to achieve finite minimum entropy with an admissible controller, then $\min_F \bar{\sigma}(\mathrm{HM}(D,F)) = 0$ and hence the chosen minimising F is unique (Green & Limebeer, 1995). On the other hand, if it is not possible to achieve finite minimum entropy with an admissible controller, then $0 < \min_F \bar{\sigma}(\mathrm{HM}(D,F)) < 1$ and hence there is a set of matrices F that minimise the quantity $\bar{\sigma}(\mathrm{HM}(D,F))$. We simply pick a single uniquely specifiable member of this set for ease of selection.

All this argument is captured by the following important theorem.

Theorem 5. *Let the suppositions of Theorem 3 hold and let* $Q = \arg\min_Q \bar{\sigma}(D_{11}^P + D_{12}^P Q D_{21}^P) \in \mathbb{R}^{p\times q}$.[1] *Then there exists a unique real nonsingular matrix* E *satisfying* $D^{\mathrm{T}} J_{mr} D = E^{\mathrm{T}} J_{pr} E$ *with* $D = G(j\infty)$ *of the form*

$$E = \begin{bmatrix} \begin{pmatrix} \tilde{E}_{a11} & 0 \\ \tilde{E}_{a21} & \tilde{E}_{a22} \end{pmatrix} & 0 \\ \tilde{E}_{b1} & \tilde{E}_{b2} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} I_p & -F \\ 0 & I_q \end{pmatrix} & 0 \\ 0 & I_{(r-q)} \end{bmatrix} \tag{9}$$

with $0 < \tilde{E}_{a11} \in \mathbb{R}^{p\times p}$, $0 < \tilde{E}_{a22} \in \mathbb{R}^{q\times q}$, $0 < \tilde{E}_{b2} \in \mathbb{R}^{(r-q)\times(r-q)}$ *and*

$$F = \mathrm{HM}\left(\begin{bmatrix} I_p & 0 \\ D_{22}^P & I_q \end{bmatrix}, Q \right).$$

Proof. First note that the expression for F can be rearranged as $F = Q(D_{22}^P Q + I_q)^{-1}$ and Q is such that $\bar{\sigma}(D_{11}^P + D_{12}^P Q D_{21}^P) < 1$ since the suppositions of the theorem guarantee that the normalised $\mathscr{H}_\infty$ control problem is solvable (i.e. controller gain at infinite frequency (F) must make closed-loop gain at infinite frequency ($D_{11}^P + D_{12}^P Q D_{21}^P$) less than unity). For shorthand's sake, let $\Xi := (D_{11}^P + D_{12}^P Q D_{21}^P)$. Then, noting that

$$D \begin{bmatrix} I_p & (F \quad 0) \\ 0 & I_r \end{bmatrix}$$

$$= \mathrm{CHAIN}\left(\begin{bmatrix} D_{11}^P & D_{12}^P \\ \begin{pmatrix} D_{21}^P \\ D_{21}^{P\,\prime} \end{pmatrix} & \begin{pmatrix} D_{22}^P \\ D_{22}^{P\,\prime} \end{pmatrix} \end{bmatrix} \right) \begin{bmatrix} I_p & (F \quad 0) \\ 0 & I_r \end{bmatrix}$$

$$= \begin{bmatrix} \blacklozenge & D_{12}^P [F \quad 0] + D_{11}^P \begin{bmatrix} D_{21}^P \\ D_{21}^{P\,\prime} \end{bmatrix}^{-1} \left(I_r - \begin{bmatrix} D_{22}^P \\ D_{22}^{P\,\prime} \end{bmatrix} [F \quad 0] \right) \\ \blacklozenge & \begin{bmatrix} D_{21}^P \\ D_{21}^{P\,\prime} \end{bmatrix}^{-1} \left(I_r - \begin{bmatrix} D_{22}^P \\ D_{22}^{P\,\prime} \end{bmatrix} [F \quad 0] \right) \end{bmatrix}$$

$$= \begin{bmatrix} \blacklozenge & \Xi \begin{bmatrix} D_{21}^P \\ D_{21}^{P\,\prime} \end{bmatrix}^{-1} \begin{bmatrix} (D_{22}^P Q + I_q) & 0 \\ D_{22}^{P\,\prime} Q & I_{(r-q)} \end{bmatrix}^{-1} \\ \blacklozenge & \begin{bmatrix} D_{21}^P \\ D_{21}^{P\,\prime} \end{bmatrix}^{-1} \begin{bmatrix} (D_{22}^P Q + I_q) & 0 \\ D_{22}^{P\,\prime} Q & I_{(r-q)} \end{bmatrix}^{-1} \end{bmatrix}$$

and letting

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12}^{\mathrm{T}} & \varphi_{22} \end{bmatrix} := \begin{bmatrix} I_p & (F \quad 0) \\ 0 & I_r \end{bmatrix}^{\mathrm{T}} D^{\mathrm{T}} J_{mr} D \begin{bmatrix} I_p & (F \quad 0) \\ 0 & I_r \end{bmatrix},$$

it follows that

$$\varphi_{22} = \begin{bmatrix} (D_{22}^P Q + I_q) D_{21}^P \\ D_{22}^{P\,\prime} Q D_{21}^P + D_{21}^{P\,\prime} \end{bmatrix}^{-\mathrm{T}} \times [\Xi^{\mathrm{T}}\Xi - I_r] \begin{bmatrix} (D_{22}^P Q + I_q) D_{21}^P \\ D_{22}^{P\,\prime} Q D_{21}^P + D_{21}^{P\,\prime} \end{bmatrix}^{-1} < 0.$$

Since $0 > \varphi_{22} \in \mathbb{R}^{r\times r}$, a similar argument to the constructive part in the proof (a $\Rightarrow$ b) of Theorem 3 yields the following decomposition:

$$\begin{bmatrix} I_p & F & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_{(r-q)} \end{bmatrix}^{\mathrm{T}} D^{\mathrm{T}} J_{mr} D \begin{bmatrix} I_p & F & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_{(r-q)} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{E}_{a11} & 0 & 0 \\ \tilde{E}_{a21} & \tilde{E}_{a22} & 0 \\ \tilde{E}_{b11} & \tilde{E}_{b12} & \tilde{E}_{b2} \end{bmatrix}^{\mathrm{T}} J_{pr} \begin{bmatrix} \tilde{E}_{a11} & 0 & 0 \\ \tilde{E}_{a21} & \tilde{E}_{a22} & 0 \\ \tilde{E}_{b11} & \tilde{E}_{b12} & \tilde{E}_{b2} \end{bmatrix}.$$

The proof is concluded by restricting $\tilde{E}_{a11}$, $\tilde{E}_{a22}$ and $\tilde{E}_{b2}$ to be strictly positive definite matrices so as to remove all non-uniqueness in the above selections. □

Note that E in Eq. (9) reduces to the lower triangular matrix E of Theorem 3 when $D_{11}^P = 0$. Furthermore, with E selected as in Theorem 5 (which is always possible whenever the normalised $\mathscr{H}_\infty$ control problem is solvable), the notions of minimum entropy controller and central control (in the sense of this paper) always coincide as in Mustafa and Glover (1988), Glover and Mustafa (1989), Mustafa and Glover (1990), Mustafa et al. (1991), as one would expect and desire.

5. Conclusions

In this paper, we show how to pin down the non-unique factors resulting from the J-lossless factorisation, thereby

[1] This Q is in general not unique, however any single uniquely specifiable minimising Q is suitable.

ensuring that there is only one way in which the admissible controller set can be characterised and in turn guaranteeing that the central controller (corresponding to the centre of this admissible controller set) is (a) uniquely defined, and (b) corresponds to the central and minimum entropy controller frequently discussed in the literature.

In the process of ensuring that the centre of the parametrised set of admissible controllers in the chain-scattering framework corresponds to the central and minimum entropy controller frequently discussed in the literature, we discuss and uncover a number of properties associated with the central controller that are buried when the standard assumptions in the literature are adopted (particularly, when $D_{11}^P = 0$ is assumed). It is shown that this pinning-down of the factors resulting for the J-lossless factorisation must be performed on the basis of an educated selection of a loop-shifting matrix provided that we want the centre of the parametrised set of admissible controllers to posses all the properties and be in fact the same as the central and minimum entropy controller of the literature.

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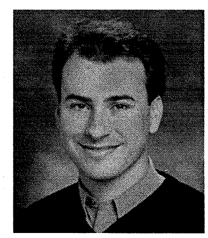

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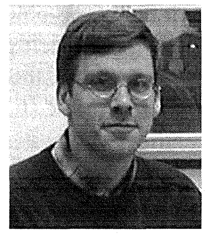

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