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Model validation for control and controller validation in a prediction error identification framework—Part I: theory☆

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Abstract

We propose a model validation procedure that consists of a prediction error identification experiment with a full order model. It delivers a parametric uncertainty ellipsoid and a corresponding set of parameterized transfer functions, which we call prediction error (PE) uncertainty set. Such uncertainty set differs from the classical uncertainty descriptions used in robust control analysis and design. We develop a robust control analysis theory for such uncertainty sets, which covers two distinct aspects: (1) *Controller validation.* We present necessary and sufficient conditions for a specific controller to stabilize—or to achieve a given level of performance with—all systems in such PE uncertainty set. (2) *Model validation for robust control.* We present a measure for the size of such PE uncertainty set that is directly connected to the size of a set controllers that stabilize all systems in the model uncertainty set. This allows us to establish that one uncertainty set is better tuned for robust control design than another, leading to control-oriented validation objectives.

Keywords: System identification; Identification for robust control; Model validation; Controller validation

1. Introduction

In this paper, we propose a new validation framework that connects prediction error (PE) identification and robustness theory. This framework consists of a new method to design an uncertainty region using the tools of PE identification, coupled with robustness tools that are adapted to this uncertainty region. These robustness tools pertain both to the robustness analysis of a specific controller vis-à-vis all models in the model uncertainty region (*controller validation*), and to the quality assessment (for model-based control design) of the model uncertainty region (*model validation for control*). Most of the technical results of this paper have already been published in a succession of papers in which we have addressed a sequence of sub-problems. The interest of the present paper is that it connects all these disparate technical results into a coherent framework from measured data to robust controller, via an uncertainty region and a model deduced from PE identification. In a companion paper, we illustrate the methodology and the theoretical results of this paper with two realistic simulation examples.

1.1. Model validation

There are many different ways of understanding the concept of model validation, and many different frameworks under which model validation questions have been formulated. For the sake of clarity, we first define the terminology adopted in this paper. We shall say that *model validation* consists in constructing a model set $\mathscr{D}$ that contains the true system G_0, perhaps at a certain probability level. This

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model set, typically called *uncertainty set*, is constructed from a combination of data and prior assumptions. In contrast, we shall use the term *identification* for the estimation of a nominal model G_{mod}. Our reasons for distinguishing these two terms will become clear in the sequel. The model G_{mod} will be called validated if it belongs to the validated set $\mathscr{D}$.

As stated above, there exist many other frameworks and definitions of the terms validation and identification. For example:

- it is very often the case that a single procedure delivers both a nominal model and an uncertainty set;
- there are procedures, such as set membership identification, where one estimates a model uncertainty set without estimating a nominal model;
- there are frameworks that lead to the *invalidation* of models rather than the validation of model sets.

The history of model validation in PE identification is as old as PE identification itself. A reputable engineer should never deliver a product, whether it be a measurement device or a model, without a statement about its quality. However, the information about the quality of a model resulting from PE identification was classically presented via a battery of model validation tests, such as the whiteness of the residuals, the cross-correlation between inputs and residuals, parameter and transfer function covariance formulae, implicit descriptions of the bias distribution via integral formulae, etc. Thus, despite its enormous practical successes, PE identification was not delivering the classical uncertainty descriptions upon which mainstream robustness theory was built all through the 1980s, namely frequency domain uncertainty descriptions.

As a consequence, a huge gap appeared at the end of the 1980s between Robustness Theory and PE identification as was evidenced in the 1992 Santa Barbara Workshop (Smith & Dahleh, 1994). This gap drove the control community to develop new techniques, different from PE identification, in order to obtain, from measured data, an uncertainty region containing the true system. Several directions have been pursued: *set membership identification* (Giarré, Milanese, & Taragna, 1997; Giarré & Milanese, 1997), *model invalidation* (Poolla, Khargonekar, Tikku, Krause, & Nagpal, 1994; Kosut, 1995; Chen, 1997; Boulet & Francis, 1998), H_∞ and *worst-case identification* (Helmicki, Jacobson, & Nett, 1991; Gu & Khargonekar, 1992; Mäkilä & Partington, 1999; Mäkilä, Partington, & Gustafsson, 1995). These new techniques aimed at producing one of the standard linear fractional frequency domain uncertainty regions that are used in mainstream Robust Control Theory (such as additive, multiplicative, coprime factor uncertainty regions). The drawbacks of these techniques are that they are based on prior assumptions about the unknown system and the noise that are difficult to ascertain. In addition, because they are based on worst-case assumptions rather than on the idea of averaging out the noise, they typically lead to conservative uncertainty sets (Hjalmarsson, 1994).

Thus, attempts were made to construct frequency domain uncertainty regions around nominal models identified using PE identification methods. In the case where the chosen model structure is able to represent the true system, the only error in the estimated transfer function is the *variance error* (or noise-induced error), for which reliable formulae exist (Ljung, 1999). An interesting attempt to extend these formulae to the case where undermodeling is present was proposed in Hjalmarsson and Ljung (1992). The main difficulty in computing the total (mean square) error around a nominal transfer function, estimated by PE identification, has always been the estimation of the *bias error*. A first attempt at computing explicit expressions for the bias error was made by Goodwin and collaborators, using the stochastic embedding (SE) approach (Goodwin, Gevers, & Ninness, 1992). The basic idea is to treat the model error (i.e. the bias) as a realization of a stochastic process whose variance is parameterized by a few parameters, and then to estimate these parameters from data; this allows one to compute the size of the unmodeled dynamics, in a mean square sense. The only prior knowledge is the parametric structure of the variance of the unmodeled dynamics. However, the methodology is limited to models that are linear in the parameters.

Another approach to estimate the model error of a nominal restricted complexity model in a PE framework is the model error model (MEM) approach proposed by Ljung (Ljung, 1997); see also Ljung (1998, 2000) and Reinelt, Garulli, Ljung, Braslavsky, and Vicino (1999). The key idea is to estimate an unbiased model of the error between the nominal model and the true system by a simple step of PE identification with full order model structure, using validation data. The mean square error on this model error estimate is then a variance error only (by virtue of its full order structure), for which standard formulae exist, as already stated. In Ljung (1997, 1998) this error was computed in the frequency domain from the ellipsoidal confidence region on the parameter vector of the identified MEM, using a first-order approximation. This results in an ellipse at each frequency in the Nyquist plane; these ellipses can be collected together to make up a frequency domain uncertainty region, say $\mathscr{L}$. However, the mapping from the ellipsoidal uncertainty region in parameter space to this frequency domain uncertainty region $\mathscr{L}$ is rather subtle, even in the case of linearly parameterized model structures, and so is the computation of the corresponding probability levels. For a thorough analysis, see Bombois, Anderson, and Gevers (2001a).

The model validation approach that we develop in this paper is inspired by the MEM approach, but it uses the obvious (and much simpler) alternative of identifying a full order model directly for the full system, rather than for the model error between the full system and a low-order estimate of this system. Hence, the MSE is again a variance error only.

The *mixed stochastic-deterministic* methods for the construction of uncertainty regions presented in Hakvoort and

Van den Hof (1997), de Vries and Van den Hof (1995) and Venkatesh and Dahleh (1997) are based on a mixture of stochastic assumptions on the noise, and deterministic assumptions on the decay rate of the tail of the system's response. Just as for the SE approach, the authors show that they can obtain, from data, an estimate of the noise variance and of the prior bounds on this decay rate. Also common with the SE approach is a restriction of the method to models that are linear in the parameters, i.e. fixed denominator models.

All the methods described above, which compute uncertainty sets on the basis of stochastic assumptions on the noise, do of course lead to stochastic descriptions of the uncertainty sets, i.e. the true system belongs to the validated set $\mathscr{D}$ with probability α; the level α is entirely a choice of the designer.

1.2. Controller validation

Once an uncertainty set $\mathscr{D}$ of models has been constructed by a procedure of identification or model validation, one can address controller validation questions. Namely, for such set $\mathscr{D}$, and for a tentative controller $C(z)$ one can ask one or both of the following questions:

- does the controller $C(z)$ stabilize all models in the model set $\mathscr{D}$? This is called *controller validation for stability*.
- does the controller $C(z)$ achieve a given level of closed loop performance with all models in the model set $\mathscr{D}$? This is called *controller validation for performance*.

In this paper, we offer a solution to both questions in the context of uncertainty sets obtained by PE methods. Note that in the approach adopted in the present paper, we pose the controller validation questions with respect to the uncertainty set $\mathscr{D}$ with the knowledge that the true system belongs to $\mathscr{D}$ with probability α. This will lead us to ascertain that a controller $C(z)$ that stabilizes all models in $\mathscr{D}$ stabilizes the true system with probability at least equal to α. A more precise and less conservative estimate of the probability that $C(z)$ stabilizes the true system can be obtained using the theory of randomized algorithms: see Campi, Lecchini, and Savaresi (2000, 2002).

1.3. Model validation for control

The assessment of the quality of a model cannot be decoupled from the purpose for which the model is to be used. The research on identification for control has, in the last 10 years, focused on the design of identification criteria that delivered a *control-oriented nominal model*. Similarly, the validation experiment must be designed in such a way as to deliver uncertainty sets that are tuned for robust control design. Thus, one must think in terms of "control-oriented validation design". In this paper, we highlight the connection between validated uncertainty sets obtained by PE methods and sets of stabilizing controllers.

1.4. Contribution of our work

We develop a framework that connects Robustness Theory and PE identification with full order model structures. Our results can be extended to PE identification with biased (i.e. low order) model structures using the stochastic embedding framework and linearly parameterized model structures (Bombois, Gevers, & Scorletti, 2000a). This extension will be briefly discussed in this paper.

The starting point for this framework is the observation that if a full order rational model structure is used, then a straightforward step of PE identification delivers an ellipsoidal confidence region in parameter space, constructed from the estimated parameter covariance matrix. This ellipsoid can be mapped without any approximation to an uncertainty set $\mathscr{D}$ in transfer function space to which the unknown system belongs with a prescribed probability.

Our first contribution is *to derive a general expression for the transfer function uncertainty set $\mathscr{D}$ obtained by a step of PE validation*, i.e. identification with a full order model structure. We call such set the generic *PE model uncertainty set*, because it is the uncertainty set that results from either open-loop or closed-loop identification in a PE framework. This generic PE uncertainty set takes the form of a set of parameterized transfer functions whose (real) parameter vector is constrained to lie in an ellipsoid. The center of this uncertainty region is the full order identified model. Since our validation procedure is based on signal statistics, this implies that a statement like "On the basis of the data I have collected, the true system G_0 lies in this set" really means "On the basis of the data I have collected, and with probability 95%, say, G_0 lies in this set."

Our second contribution is *to furnish robust stability and robust performance analysis tools that are adapted to this PE uncertainty set $\mathscr{D}$*, i.e. without embedding it in a classical, but necessarily larger, uncertainty set as we initially did in Bombois, Gevers, and Scorletti (1999). Even though earlier robustness results have been derived for parametric uncertainty sets (Fan, Tits, & Doyle, 1991; Rantzer, 1992; Bhattacharyya, Chapellat, & Keel, 1995), *they did not cover uncertainty sets like our generic PE uncertainty region $\mathscr{D}$*. However, they have helped us to develop new robustness tools adapted to the uncertainty region $\mathscr{D}$. Our main robust stability result is a necessary and sufficient condition for the stabilization of all plants in $\mathscr{D}$ by a given controller. Our main robust performance result is to show that the worst-case performance achieved by a given controller over all plants in the PE uncertainty set $\mathscr{D}$ is the result of a linear matrix inequality (LMI)-based optimization problem.

Our third contribution is on *model validation for robust control* in a PE framework. This is a design problem, where our contribution is *to characterize what quality a validated*

PE uncertainty set $\mathscr{D}$ must possess for it to be tuned for robustly stable control design. We define a "measure of size" of a validated PE uncertainty set $\mathscr{D}$ that is connected to the size of a set of model-based controllers that stabilize all models in the model set $\mathscr{D}$. This measure of size, called the *worst-case ν-gap* between the model and the validated set, $\delta_{WC}(G_{mod}, \mathscr{D})$, was initially introduced in Gevers, Bombois, Codrons, De Bruyne, and Scorletti (1999a) (see also Bombois et al., 1999). It is an extension of the ν-gap, a distance measure between two transfer functions introduced in Vinnicombe (1993). We show that this worst-case ν-gap can be computed using an LMI-based optimization problem at each frequency. We also show in Section 6 that the smaller the worst-case $\delta_{WC}(G_{mod}, \mathscr{D})$ gap between the nominal model G_{mod} used for control design and the validated PE uncertainty set $\mathscr{D}$, the larger is the set of G_{mod}-based controllers that are guaranteed to stabilize all systems in $\mathscr{D}$. Our result therefore establishes a link between identification experiment design and controller stability robustness: an uncertainty set is "tuned for robust control design" if its worst case ν-gap is small. In Hildebrand and Gevers (2002) a solution has been obtained to the problem of optimal experiment design with respect to the worst-case ν-gap.

In a companion paper (Gevers, Bombois, Codrons, Scorletti, & Anderson, 2003), we give two realistic simulation examples of our methodology. In these examples, the objective is to design a controller satisfying a number of specifications with the unknown true system. Several PE identification experiments are performed on the true system, under different experimental conditions, leading in each case to a model G_{mod} and an uncertainty region $\mathscr{D}$ containing G_0 at a certain probability level. The worst-case ν-gap is then used to assess the quality of the pair $\{G_{mod}, \mathscr{D}\}$ for robustly stable control design. When this test is judged to be satisfactory, the model G_{mod} is used to design a controller satisfying the specifications with the model. The controller validation results are then used to verify if these specifications are also satisfied with all systems in $\mathscr{D}$, and therefore also with the true system.

1.5. Structure of paper

The paper is organized as follows. In Section 2 we present our prediction error validation procedure, both in open and in closed loop. Section 3 presents necessary and sufficient conditions for a given controller to robustly stabilize all models in an uncertainty set validated by prediction error methods, while Section 4 shows how to compute the worst-case performance of this controller with respect to all models in that uncertainty set. In Section 5 we define the worst-case ν-gap and the worst-case chordal distance function, two control-oriented measures of quality of a validated set. In Section 6 we build upon these quality measures to introduce a procedure for control-oriented model validation. Conclusions are presented in Section 7.

2. Uncertainty regions obtained by PE identification

In this section we show that we can design an uncertainty region containing the true system G_0, at a certain probability level α, using a PE identification experiment with a full order model structure, and this without adding any further assumptions on the true system G_0 than the classical assumptions required by PE identification. In order to remain concise, we consider here only the case of open-loop identification, but we have shown in Bombois (2000) and Bombois et al. (1999) that an uncertainty region containing G_0 with probability α can also be deduced from different types of closed-loop identification.

All through this paper we shall consider that input–output data y and u are generated from a single-input single-output Linear Time Invariant unknown "*true system*":

$$\mathscr{S}: y(t) = G_0(z)u(t) + v(t), \tag{1}$$

where $G_0(z)$ is a discrete-time rational transfer function having the following general form:

$$G_0(z) = G(z,\theta_0) = \frac{z^{-d}(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{Z_2(z)\theta_0}{1 + Z_1(z)\theta_0}, \tag{2}$$

where d is the delay; $\theta_0^{\mathrm{T}} = [a_1 \ \dots \ a_n \ b_0 \ \dots \ b_m] \in \mathbf{R}^{q\times 1}$, $(q \triangleq (n+m+1))$; $Z_1(z) = [z^{-1}\ z^{-2}\ \dots\ z^{-n}\ 0\ \dots\ 0]$ is a row vector of size q; $Z_2(z) = z^{-d}[0\ \dots\ 0\ 1\ z^{-1}\ z^{-2}\ \dots\ z^{-m}]$ is a row vector of size q. We will further assume that $v(t)$ is additive wide sense stationary noise that can be described as the output of a white noise driven filter. Observe that PE identification theory requires no additional assumptions on the noise; in particular $v(t)$ need not be Gaussian: see Ljung (1999).

2.1. Open-loop PE identification

A full order model structure for $G_0(z)$ is given by

$$\mathscr{M}_{ol} = \left\{ G(z,\theta) \mid G(z,\theta) = \frac{Z_2(z)\theta}{1 + Z_1(z)\theta} \right\}, \tag{3}$$

where $\theta \in \mathbf{R}^{q\times 1}$. From N input and output data obtained on the true system G_0, we can compute a model $G(\hat{\theta}) \in \mathscr{M}_{ol}$ and an estimate of the covariance matrix P_θ of $\hat{\theta}$, using classical PE identification. The true parameter vector θ_0 lies then with probability $\alpha(q,\chi) = Pr(\chi^2(q) < \chi)$ in the ellipsoidal uncertainty region (Ljung, 1999)

$$U_{ol} = \{\theta \mid (\theta - \hat{\theta})^{\mathrm{T}} P_\theta^{-1} (\theta - \hat{\theta}) < \chi\} \tag{4}$$

with $\chi^2(q)$ the chi-square probability distribution with q parameters.[1] This parametric uncertainty region U_{ol} defines a corresponding uncertainty region in the space of transfer functions which we denote $\mathscr{D}_{ol}$:

$$\mathscr{D}_{ol} = \left\{ G(z,\theta) \mid G(z,\theta) = \frac{Z_2(z)\theta}{1+Z_1(z)\theta} \text{ and } \theta \in U_{ol} \right\}. \quad (5)$$

Property of $\mathscr{D}_{ol}$. $G_0 \in \mathscr{D}_{ol}$ *with probability* $\alpha(q,\chi)$.

Role of the experimental conditions. The validated model set $\mathscr{D}_{ol}$ depends very much on the experimental conditions under which the validation has been performed. This is perhaps not so apparent in definition (5) of $\mathscr{D}_{ol}$ via the parameter covariance matrix P_θ. However, let us recall that a reasonable approximation for the covariance of the transfer function estimate $G(z,\hat{\theta})$ is given, for sufficiently large q and N, by (Ljung, 1999):

$$cov(G(e^{j\omega},\hat{\theta})) \approx \frac{q}{N}\frac{\phi_v(\omega)}{\phi_u(\omega)}. \quad (6)$$

This shows the role of the signal spectra $\phi_u(\omega)$ and $\phi_v(\omega)$, as well as the number of data, in shaping the uncertainty set $\mathscr{D}_{ol}$. Clearly, with a very small input signal energy and a small number of data, the validated region $\mathscr{D}$ would be very large, and hence almost any model would be validated by such experiment. However, such uncertainty region would be useless. A validation experiment is useful for control design if the resulting uncertainty set is small and if its distribution in the frequency domain is "control-oriented"; this last concept will be made precise in Sections 5 and 6.

2.2. The generic PE uncertainty set

A PE identification experiment with a full order model can similarly be performed on closed-loop data in order to design a corresponding uncertainty region based on a parameter covariance estimate. Alternatively, the MEM approach can be used to estimate a full order model of the model error (i.e. the unmodeled dynamics) using open-loop data (Ljung, 1997, 1998) or closed-loop data (Gevers, Codrons, & De Bruyne, 1999b), again leading to a parametric confidence ellipsoid and a corresponding transfer function uncertainty set. Whether the identification procedure is performed in open loop or in closed loop, whether it directly estimates a full order model of the true system or a full order model of the model error, it can be shown that the resulting uncertainty sets $\mathscr{D}$ can all be described in the generic form defined in the following proposition. For ease of reference, we call this uncertainty set the *generic PE uncertainty set.*

Proposition 1. *Consider $G_0(z)$, the true system defined in (2). The uncertainty regions $\mathscr{D}$ resulting from prediction error identification, and which contain the true system G_0 at a prescribed probability level, can all be described in the following generic form:*

$$\mathscr{D} = \left\{ G(z,\delta) \mid G(z,\delta) = \frac{e(z)+Z_N(z)\delta}{1+Z_D(z)\delta} \text{ and } \delta \in U = \{\delta \mid (\delta-\hat{\delta})^T R(\delta-\hat{\delta}) < 1\} \right\}, \quad (7)$$

where $\delta \in \mathbf{R}^{k\times 1}$ is a real parameter vector; $\hat{\delta}$ is the parameter estimate resulting from the identification step; $R \in \mathbf{R}^{k\times k}$ is a symmetric positive definite matrix that is proportional to the inverse of the covariance matrix of $\hat{\delta}$; $Z_N(z)$ and $Z_D(z)$ are row vectors of size k of known transfer functions; and $e(z)$ is a known transfer function.

Proof. See Bombois (2000). Note that $\mathscr{D}_{ol}$ in (5) has structure (7) with $e(z)=0$, $\theta=\delta$ and $R=(\chi P_\theta)^{-1}$.

Proposition 1 defines the general structure of the uncertainty region $\mathscr{D}$ which results from a PE identification experiment with unbiased model structures. Let us point out the following characteristics of this uncertainty region.

- The uncertainty region $\mathscr{D}$ is simply the result of a PE identification experiment with full order model structure. Such PE identification step, performed with the purpose of constructing an uncertainty region, will be called *a validation experiment* in the sequel.
- The true system G_0 lies in $\mathscr{D}$ with a probability level that is entirely fixed by the designer.
- The uncertainty region $\mathscr{D}$ is "centered" at $G(z,\hat{\delta})$, which is a full order model of the true system G_0 deduced from the identified parameter vector $\hat{\delta}$. For control design, one can of course use $G(z,\hat{\delta})$. However, it is a high order model, and will therefore lead to a high-order controller. Hence, it is often the case that one will choose a low-order approximation G_{mod} of $G(z,\hat{\delta})$ as the *model for control design.*
- Sometimes a low-order model G_{mod} has been obtained a priori, by modeling or by a separate identification experiment. The PE validation step is then performed for the purpose of constructing an uncertainty set $\mathscr{D}$, that is known to contain the true G_0 at a certain probability level. The prior model G_{mod} is then used for control design if it is validated, i.e. if $G_{mod} \in \mathscr{D}$.
- Different identification experiments (i.e. open-loop or closed-loop identification, different measured data, etc.) lead to different identified parameter vectors, different covariance matrices, and therefore also different uncertainty sets $\mathscr{D}^{(i)}$.

2.3. Extension to PE identification with reduced order model structures

Expression (7) defines the structure of the uncertainty region obtained from a PE identification experiment with full

[1] This result holds even if $v(t)$ is not Gaussian or is colored; this is a consequence of the central limit theorem. In addition, the probability level α can be chosen as close to 1 as desired.

order model structure, i.e. without undermodeling. We have shown in Bombois et al. (2000a) that the stochastic embedding tools presented in Goodwin et al. (1992) allow one to design uncertainty regions $\mathscr{L}$ using PE identification with low order (and hence biased) model structures. The uncertainty region $\mathscr{L}$ is then a ratio of transfer functions parameterized by a transfer vector whose frequency response is real and constrained to lie in an ellipse at each frequency. The structure of the uncertainty region $\mathscr{L}$ is thus quite similar to that of the uncertainty region $\mathscr{D}$. This approach therefore allows us to also handle infinite dimensional unmodeled dynamics in a PE framework.

In the sequel, we develop two sets of results for PE uncertainty regions $\mathscr{D}$. The first are *controller validation results*: they allow us to verify whether a controller $C(z)$ stabilizes and achieves a prescribed level of performance with all systems in such uncertainty region $\mathscr{D}$ and therefore also with the true system G_0. The other set of results pertain to *model validation for control*, and are based on a measure of size of the region $\mathscr{D}$ that is connected to the size of stabilizing controller sets. These are presented in Sections 5 and 6.

3. Necessary and sufficient conditions for stabilization of a PE model set

In this section we establish *necessary and sufficient conditions* for some given controller $C(z)$ to stabilize all models in a generic PE uncertainty set defined by (7) (*controller validation for stability*). A controller that stabilizes all models in a PE model set will be called *validated for stability*.

For the standard uncertainty sets that are used in robust control theory, such as additive, multiplicative, coprime factor uncertainty sets, necessary and sufficient conditions are usually obtained by rewriting the closed-loop connections of the controller with all plants in the uncertainty region as a set of loops that connect a known fixed dynamic matrix $M(z)$, that includes this controller, to an uncertainty part $\Delta(z)$ of known structure that belongs to the prescribed uncertainty domain: the so-called LFT framework (Fan et al., 1991; Doyle, 1982; Zhou, Doyle, & Glover, 1995; Packard & Doyle, 1993; Hinrichsen & Pritchard, 1988). For our PE uncertainty sets $\mathscr{D}$, a necessary and sufficient condition for robust stability is obtained by showing that the set of feedback connections of C with all models in $\mathscr{D}$ can be reformulated, using an LFT framework, as a set of feedback connections $[M_{\mathscr{D}}(z)\ \phi]$, where $M_{\mathscr{D}}(z)$ is fixed and contains the controller information, and where the uncertainty part ϕ is a real vector, linearly related to the parameter vector δ that defines $\mathscr{D}$: see (7). We have then shown that the (real) stability radius linked with the set of loops $[M_{\mathscr{D}}(z)\ \phi]$ can be computed exactly and efficiently, using results of Hinrichsen and Pritchard (1988) and Rantzer (1992). The full analysis of our results, together with the proofs, can be found in Bombois, Gevers, Scorletti, & Anderson, (2001b). They hold only for single-input single-output systems. Here we recall the main robust stability result of Bombois et al. (2001b).

Theorem 1. *Consider a generic PE uncertainty set of the form* (7) *and a controller* $C(z)=X(z)/Y(z)$ *that stabilizes the center of that set,* $G(z,\hat{\delta})$. *Then all models in* $\mathscr{D}$ *are stabilized by* $C(z)$ *if and only if*

$$\max_{\omega} \mu(M_{\mathscr{D}}(e^{j\omega})) \leqslant 1, \tag{8}$$

where

- $M_{\mathscr{D}}(z)$ *is defined as*

$$M_{\mathscr{D}}(z)=\frac{-(Z_D+X(Z_N-eZ_D)/(Y+eX))T^{-1}}{1+(Z_D+X(Z_N-eZ_D)/(Y+eX))\hat{\delta}}. \tag{9}$$

- T *is a square root of the matrix* R *defining* $\mathscr{D}: R=T^{\mathrm{T}}T$.
- $\phi=T(\delta-\hat{\delta})$, *whereby* $\delta\in U \Leftrightarrow |\phi|_2<1$.
- $\mu(M_{\mathscr{D}}(e^{j\omega}))$ *is called the stability radius of the loop* $[M_{\mathscr{D}}(z)\ \phi]$. *For a real vector* ϕ *it is computed as follows:*

$$\mu(M(e^{j\omega}))=\sqrt{|\mathrm{Re}(M)|_2^2-\frac{(\mathrm{Re}(M)\mathrm{Im}(M)^{\mathrm{T}})^2}{|\mathrm{Im}(M)|_2^2}}$$

if $\mathrm{Im}(M)\neq 0$

$$\mu(M(e^{j\omega}))=|M|_2 \quad \textit{if } \mathrm{Im}(M)=0.$$

Proof. See Bombois et al. (2001b). □

We note that checking the stability of a controller $C(z)$ with all models in the PE uncertainty set $\mathscr{D}$ requires that a frequency domain point-by-point inequality be satisfied. The same will hold for most of the robust stability results presented later in this paper. Thus, such test belongs to the same family of methods as the Nyquist stability test. It might well be possible to replace such pointwise stability test by one based directly on $C(z)$, the structure of $\mathscr{D}$ and the normalized covariance matrix R. This is the subject of future investigations.

The necessary and sufficient conditions for controller validation depend critically on the uncertainty set $\mathscr{D}$. A controller that is not validated for stability with a validated PE uncertainty set $\mathscr{D}^{(1)}$ may well be validated for stability with another validated PE uncertainty set $\mathscr{D}^{(2)}$, obtained using different experimental conditions. Of course, this observation applies to all robust control methodologies, i.e. the validation of a controller always depends on the model uncertainty set. What distinguishes our approach from most others is that, in mainstream robust control theory or in model invalidation theory, the uncertainty sets used for control analysis and/or design are either assumed a priori or obtained by overbounding uncertainty sets estimated from data and assumptions. Here we work directly with uncertainty sets identified from data without overbounding. Our analysis in Section 5 will give us at least some handle on how we can shape these uncertainty sets towards robust control design.

4. Worst case performance over a PE model set

In this section, we show that we can compute the worst-case performance achieved by some controller $C(z)$ over all models in a PE uncertainty region $\mathscr{D}$. This worst-case performance is of course an upper bound for the closed-loop performance achieved by this controller with the true system. We shall say that a controller is *validated for performance* if the worst-case performance over all models in $\mathscr{D}$ remains below a prespecified threshold. There is no unique way of defining the performance of a closed-loop system. However, most commonly used performance criteria are derived from some norm of a frequency weighted version of the transfer matrix $T(G,C)$ of the closed-loop system $[G\ C]$ defined by

$$T(G,C)=\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}=\begin{pmatrix} \dfrac{GC}{1+GC} & \dfrac{G}{1+GC} \\ \dfrac{C}{1+GC} & \dfrac{1}{1+GC}\end{pmatrix}. \tag{10}$$

Thus we shall start from the following very general definition.

Definition 1. The performance of a closed-loop system $[G\ C]$ is defined as the following frequency function:

$$J(G,C,W_l,W_r,\omega)$$
$$=\sigma_1\left(\overbrace{\begin{pmatrix} W_{l1} & 0 \\ 0 & W_{l2}\end{pmatrix}}^{W_l} T(G(e^{j\omega}),C(e^{j\omega}))\overbrace{\begin{pmatrix} W_{r1} & 0 \\ 0 & W_{r2}\end{pmatrix}}^{W_r}\right), \tag{11}$$

where $W_{l1}(e^{j\omega}), W_{l2}(e^{j\omega})$ and $W_{r1}(e^{j\omega}), W_{r2}(e^{j\omega})$ are frequency weights that allow one to define specific performance levels, and where $\sigma_1(A)$ denotes the largest singular value of A.

The frequency function J defines a template. Any function that is derived from J can of course also be handled, such as $\|W_lT(G,C)W_r\|_\infty$, as used in de Callafon and Van den Hof (1997), for example. Observe also that the choice of a diagonal structure for W_l and W_r is no loss of generality, since the four transfer functions in $T(G,C)$ can all be weighted differently. For example, a common choice for the performance measure of a closed-loop system is the shape of the modulus of the frequency response of one or several of the four transfer functions defined in (10): see Zames (1981). The worst case performance over a validated PE set is now defined as follows.

Definition 2. Consider a validated PE uncertainty region $\mathscr{D}$ given by (7) and a controller $C(z)$ that is validated for stability with respect to $\mathscr{D}$. The worst-case performance achieved by this controller at a frequency ω over all models in $\mathscr{D}$ is defined as

$$J_{WC}(\mathscr{D},C,W_l,W_r,\omega)$$
$$=\max_{G(z,\delta)\in\mathscr{D}}\sigma_1\left(\overbrace{\begin{pmatrix} W_{l1} & 0 \\ 0 & W_{l2}\end{pmatrix}}^{W_l} T(G(e^{j\omega},\delta), C(e^{j\omega}))\overbrace{\begin{pmatrix} W_{r1} & 0 \\ 0 & W_{r2}\end{pmatrix}}^{W_r}\right). \tag{12}$$

The following theorem, proved in Bombois et al. (2001b), shows how to compute the criterion $J_{WC}(\mathscr{D},C,W_l,W_r,\omega)$ at the frequency ω as the solution of an optimization problem involving LMI constraints (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). A crucial feature that makes this computation possible is the rank one property of the matrix $T(G,C)$; such property does not hold in general for MIMO plants.

Theorem 2. *Consider a PE uncertainty region $\mathscr{D}$ defined by* (7) *and a robustly stabilizing controller $C=X/Y$. Then, at frequency ω, the criterion function $J_{WC}(\mathscr{D},C,W_l,W_r,\omega)$ is obtained as*

$$J_{WC}(\mathscr{D},C,W_l,W_r,\omega)=\sqrt{\gamma_{opt}(\omega)}, \tag{13}$$

where $\gamma_{opt}(\omega)$ is the optimal value of γ for the following standard convex optimization problem involving LMI constraints:

minimize γ

over $\gamma,\ \tau$

subject to $\tau\geqslant 0$ *and*

$$\begin{pmatrix} \mathrm{Re}(a_{11}) & \mathrm{Re}(a_{12}) \\ \mathrm{Re}(a_{12}^*) & \mathrm{Re}(a_{22})\end{pmatrix}-\tau\begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^{\mathrm{T}} & \hat{\delta}^{\mathrm{T}}R\hat{\delta}-1\end{pmatrix}<0, \tag{14}$$

where

- $a_{11}=(Z_N^*W_{l1}^*W_{l1}Z_N+Z_D^*W_{l2}^*W_{l2}Z_D)-\gamma(QZ_1^*Z_1)$,
- $a_{12}=Z_N^*W_{l1}^*W_{l1}e+W_{l2}^*W_{l2}Z_D^*-\gamma(QZ_1^*(Y+eX))$,
- $a_{22}=e^*W_{l1}^*W_{l1}e+W_{l2}^*W_{l2}-\gamma(Q(Y+eX)^*(Y+eX))$,
- $Q=1/(X^*W_{r1}^*W_{r1}X+Y^*W_{r2}^*W_{r2}Y)$,
- $Z_1=XZ_N+YZ_D$.

Proof. See Bombois et al. (2001b). □

5. A robust stability oriented quality measure for $\mathscr{D}$

In this section, we introduce control-oriented quality measures for the generic PE uncertainty set (7) obtained by a validation experiment, namely the *worst-case chordal distance* and the *worst-case ν-gap* between a model G_{mod} [2] and the set $\mathscr{D}$. The worst-case chordal distance is a frequency function, while the worst-case ν-gap is a global measure. These measures are control-oriented because they are related to sets of stabilizing controllers. The worst-case ν-gap is an extension of the ν-gap, introduced in Vinnicombe (1993), which is a measure of distance between two transfer functions. For the sake of completeness, we first briefly recall the definition of ν-gap for scalar transfer functions and the definition of generalized stability margin linked to this metric.

Definition 3. The ν-gap metric between two transfer functions G_1 and G_2, introduced in Vinnicombe (1993) and denoted δ_ν, is defined as

$$\delta_\nu(G_1,G_2) = \begin{cases} \max_\omega \kappa(G_1(e^{j\omega}),G_2(e^{j\omega})) & \text{if } W(G_1,G_2)=0, \\ 1 & \text{otherwise,} \end{cases} \tag{15}$$

where

$$\kappa(G_1(e^{j\omega}),G_2(e^{j\omega})) \triangleq \frac{|G_1(e^{j\omega})-G_2(e^{j\omega})|}{\sqrt{1+|G_1(e^{j\omega})|^2}\sqrt{1+|G_2(e^{j\omega})|^2}} \tag{16}$$

and where

$$W(G_1,G_2) = wno(1+G_1^*G_2)+\eta(G_2)-\tilde{\eta}(G_1).$$

Here $G^*(e^{j\omega}) = G(e^{-j\omega})$, $\eta(G)$ (resp. $\tilde{\eta}(G)$) denotes the number of poles of G in the complement of the closed (resp. open) unit disc, while $wno(G)$ denotes the winding number about the origin of $G(z)$ as z follows the unit circle indented into the exterior of the unit disc around any unit circle pole and zero of $G(z)$.

If the winding number condition $W(G_1,G_2)=0$ is satisfied, then the ν-gap between two plants has a simple frequency domain interpretation (in the SISO case). Indeed, the quantity $\kappa(G_1(e^{j\omega}),G_2(e^{j\omega}))$ is the chordal distance between the projections of $G_1(e^{j\omega})$ and $G_2(e^{j\omega})$ onto the Riemann sphere of unit diameter with South Pole at the origin of the complex plane (Vinnicombe, 1993). The distance $\delta_\nu(G_1,G_2)$ between G_1 and G_2 is therefore, according to (15), the supremum of these chordal distances over all frequencies. Observe that $0 \leqslant \delta_\nu(G_1,G_2) \leqslant 1$.

Consider now a closed-loop system made up of the feedback interconnection of a system G and a controller C: see Fig. 1.

[2] G_{mod} is the model used for control design; typically the center of $\mathscr{D}$ or a low-order approximation of this center contained in $\mathscr{D}$.

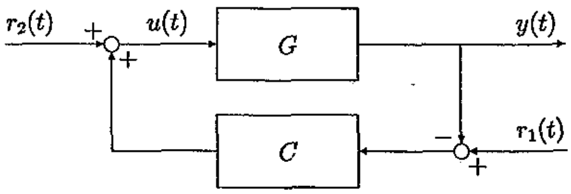

Fig. 1. Closed-loop system.

The closed-loop transfer function matrix between $[r_1\ r_2]^T$ and $[y\ u]^T$ is the matrix $T(G,C)$ defined in (10).

Definition 4. The *generalized stability margin* of the closed-loop system $[G\ C]$ is defined as (Vinnicombe, 1993)

$$b_{GC} = \begin{cases} \|T(G,C)\|_\infty^{-1} & \\ \quad = \min_\omega \kappa(G(e^{j\omega}), -\frac{1}{C(e^{j\omega})}) & \text{if } [G\ C] \text{ is stable} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the generalized stability margin of a closed-loop system $[G\ C]$ is measured by the least chordal distance between the projections on the Riemann sphere of G and of the inverse of $-C$. It is also important to note that, for a given plant G, and whatever the linear controller, the generalized stability margin has a maximum value $b_{opt}(G)$ (see e.g. Zhou & Doyle, 1998) given by

$$b_{opt}(G) = \max_C b_{GC} = \sqrt{1-\|[N\ M]\|_H^2}, \tag{17}$$

where $\|A\|_H$ is the Hankel norm of the operator A (see e.g. Zhou et al. (1995)) and $\{N,M\}$ is the normalized coprime factorization of G.

5.1. Robust stability and the ν-gap

The main interest of the ν-gap metric is its use in a range of robust stability results. One of these results relates the size of the set of robustly stabilizing controllers of a ν-gap uncertainty set (i.e. an uncertainty set defined with the ν-gap) to the size of this uncertainty set (Vinnicombe, 2000).

Proposition 1. *Let us consider the uncertainty set $\mathscr{G}_\gamma$, centered at a model G_{mod}, and defined by*

$$\mathscr{G}_\gamma = \{G \mid \kappa(G_{mod}(e^{j\omega}),G(e^{j\omega})) \leqslant \gamma(\omega) \ \ \forall\omega \text{ and } \delta_\nu(G_{mod},G)<1\},$$

with $0 \leqslant \gamma(\omega) \leqslant 1\ \forall\omega$. Then, a controller C stabilizing G_{mod} stabilizes all plants in the uncertainty region $\mathscr{G}_\gamma$ if and only if it lies in the controller set

$$\mathscr{C}_\gamma = \left\{C(z) \mid \kappa\left(G_{mod}(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right) > \gamma(\omega)\ \forall\omega\right\}.$$

A simpler min–max version of this proposition is as follows.

Proposition 2 (Vinnicombe, 2000). *Let us consider the* ν*-gap uncertainty set* $\mathscr{G}_\beta$ *of size* β*, centered at a model* G_{mod}:

$$\mathscr{G}_\beta = \{G \mid \delta_\nu(G_{\mathrm{mod}}, G) \leqslant \beta\}.$$

Then, a controller C stabilizing G_{mod} *stabilizes all plants in the uncertainty region* $\mathscr{G}_\beta$ *if and only if it lies in the controller set*

$$\mathscr{C}_\beta = \{C(z) \mid b_{G_{\mathrm{mod}}C} > \beta\}.$$

The size β of a ν-gap uncertainty set $\mathscr{G}_\beta$ is thus directly connected to the size of the set of all controllers that robustly stabilize $\mathscr{G}_\beta$. Moreover, the smaller is this size β, the larger is the set of controllers that robustly stabilize $\mathscr{G}_\beta$.

5.2. The worst-case distance between a model and a validated PE model set

We now build on these robust stability results in order to connect validated PE uncertainty sets to sets of robustly stabilizing controllers. Proposition 2 shows that the G_{mod}-based controller set that is guaranteed to robustly stabilize all systems in $\mathscr{D}$ is large if the largest ν-gap between G_{mod} and any plant in $\mathscr{D}$ remains small. We therefore extend the definitions of distance between two models to definitions of *worst-case distance* between a model and all models in a PE model set.

Definition 5. Consider a PE uncertainty set $\mathscr{D}$ of the form (7) and a model G_{mod}. The *worst case chordal distance* at frequency ω between G_{mod} and $\mathscr{D}$ is defined as

$$\kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(e^{j\omega}), \mathscr{D}) = \sup_{G_{\mathscr{G}} \in \mathscr{D}} \kappa(G_{\mathrm{mod}}(e^{j\omega}), G_{\mathscr{D}}(e^{j\omega})).$$

Definition 6. Consider a PE uncertainty set $\mathscr{D}$ of the form (7) and a model G_{mod}. The *worst case* ν*-gap* between G_{mod} and $\mathscr{D}$ is defined as

$$\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D}) = \sup_{G_{\mathscr{G}} \in \mathscr{D}} \delta_\nu(G_{\mathrm{mod}}, G_{\mathscr{D}}).$$

The worst-case ν-gap can alternatively be defined as the supremum over all frequencies of the worst-case chordal distance. This is shown in the following lemma, which is an extension of a proposition presented in Vinnicombe (2000).

Lemma 1. *If* $W(G_{\mathrm{mod}}, G_{\mathscr{D}}) = 0$ *for one plant* $G_{\mathscr{D}} \in \mathscr{D}$*, then the worst-case* ν*-gap* $\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D})$ *defined in Definition 6 can also be expressed in the following way using the worst-case chordal distance*:

$$\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D}) = \sup_{\omega} \kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(e^{j\omega}), \mathscr{D}) \tag{18}$$

where $\kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(e^{j\omega}), \mathscr{D})$ *is defined in Definition 5.*

Proof. See Bombois et al. (2000b). □

5.3. Robustly stabilizing controller sets

Having extended the concept of chordal distance and of ν-gap between plants to that of worst-case chordal distance and worst-case ν-gap between a model and a validated PE uncertainty set $\mathscr{D}$, we can now also extend the stability results of Propositions 1 and 2 to the context of our validated PE sets.

Theorem 3. *Consider a PE uncertainty region* $\mathscr{D}$ *having structure* (7)*, and a model* G_{mod}*, with* $\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D}) < 1$*. Then the set of controllers defined by*

$$\mathscr{C}_\kappa(G_{\mathrm{mod}}, \mathscr{D}) = \left\{ C \mid \kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(e^{j\omega}), \mathscr{D}) < \kappa\left(G_{\mathrm{mod}}(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right) \forall\omega \right\} \tag{19}$$

are guaranteed to stabilize all plants in the uncertainty region $\mathscr{D}$.

Proof. Immediate consequence of Proposition 1 and of the definition of worst-case chordal distance. □

A more compact but more conservative version of this robust stability theorem is as follows.

Theorem 4 (Min–Max version). *Consider a PE uncertainty set* $\mathscr{D}$ *having structure* (7)*, and a model* G_{mod}*, with* $\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D}) < 1$*. Then the set of controllers defined by*[3]

$$\mathscr{C}_\delta(G_{\mathrm{mod}}, \mathscr{D}) = \{C \mid \delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathscr{D}) < b_{G_{\mathrm{mod}}C}\} \tag{20}$$

are guaranteed to stabilize all plants in the uncertainty region $\mathscr{D}$.

Proof. Immediate consequence of Proposition 2. □

5.4. Computation of the worst case chordal distance and worst case ν-gap

We have shown in Bombois et al. (2000b) that the worst-case chordal distance between a model G_{mod} and all systems in $\mathscr{D}$ can be computed at every frequency as the solution of an optimization problem involving LMI constraints. Several algorithms have been devised for solving these problems (Boyd et al., 1994). The worst-case ν-gap can then be computed from Lemma 1 using a gridding procedure.

Theorem 5. *Consider a model* G_{mod} *and a generic PE uncertainty set* $\mathscr{D}$ *defined by* (7). *Then* $\kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(e^{j\omega}), \mathscr{D}) = \sqrt{\gamma_{\mathrm{opt}}(\omega)}$*, where* $\gamma_{\mathrm{opt}}(\omega)$ *is the optimal value of* $\gamma(\omega)$ *in the following standard convex optimization problem*

[3] Observe that this set $\mathscr{C}_\delta(G_{\mathrm{mod}}, \mathscr{D})$ is strictly included in the set $\mathscr{C}_\kappa(G_{\mathrm{mod}}, \mathscr{D})$ defined above.

involving LMI constraints:

minimize γ

over $\gamma,\ \tau$

subject to $\tau \geqslant 0$ *and*

$$\begin{pmatrix} \mathrm{Re}(a_{11}) & \mathrm{Re}(a_{12}) \\ \mathrm{Re}(a_{12}^*) & \mathrm{Re}(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^{\mathrm{T}} & \hat{\delta}^{\mathrm{T}} R \hat{\delta} - 1 \end{pmatrix} \leqslant 0 \tag{21}$$

with $a_{11} = (Z_N^* Z_N - Z_N^* x Z_D - Z_D^* x^* Z_N + Z_D^* x^* x Z_D) - \gamma(Z_N^* Q Z_N + Z_D^* Q Z_D)$, $a_{12} = Z_N^* e - Z_N^* x - Z_D^* e x^* + Z_D^* x x^* - \gamma(Z_N^* e Q + Z_D^* Q)$, $a_{22} = ee^* - e^* x - e x^* + x x^* - \gamma(ee^* Q + Q)$, $Q = 1 + x^* x$ *and* $x = G_{\mathrm{mod}}(\mathrm{e}^{\mathrm{j}\omega})$.

The worst case ν*-gap is then obtained as*

$$\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathcal{D}) = \max_{\omega} \kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(\mathrm{e}^{\mathrm{j}\omega}), \mathcal{D}).$$

Proof. See Bombois et al. (2000b). □

6. Model validation for control

The results of Section 5.3 constitute the basis for connecting the quality of a PE validation experiment with the size of sets of stabilizing controllers. Roughly speaking, they tell us that the smaller the size of the validated set, as measured by the worst-case ν-gap (a single number) or the worst-case chordal distance (a frequency function) between G_{mod} and all members of that set, the larger the set of stabilizing controllers. The results of Theorems 3 and 4 are valid with respect to any model G_{mod}, whether it be $G(z,\hat{\delta})$ or any other model. An important aspect of these robust stability results is that all quantities are available for computation. Observe that the left-hand side of the inequality in (19) is not a function of C while the right-hand side is not a function of $\mathcal{D}$. Thus, $\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathcal{D})$ and $\kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(\mathrm{e}^{\mathrm{j}\omega}), \mathcal{D})$ can be taken as robust stability oriented measures of quality for a validated set, in that the smaller these measures, the larger are the sets of model-based controllers that are guaranteed to stabilize all models in the corresponding uncertainty set.

6.1. Comparing validated PE uncertainty sets

The uncertainty set $\mathcal{D}$ that results from a PE identification experiment is very much dependent on the experimental conditions (open-loop or closed-loop identification, choice of input signal spectrum, number of data, etc.). Different validation experiments lead to different PE uncertainty sets. The results of the previous section and the observations above, allow us to compare the quality of these uncertainty sets in terms of robust stabilization.

Theorem 6. *Consider two different validated PE sets* $\mathcal{D}^{(1)}$ *and* $\mathcal{D}^{(2)}$*, obtained from two different validation experiments. Let* G_{mod} *be a model that belongs to both sets and that is used for control design. Then*

$$\kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(\mathrm{e}^{\mathrm{j}\omega}), \mathcal{D}^{(1)}) < \kappa_{\mathrm{WC}}(G_{\mathrm{mod}}(\mathrm{e}^{\mathrm{j}\omega}), \mathcal{D}^{(2)})\ \forall\omega$$
$$\Rightarrow \mathscr{C}_{\kappa}(G_{\mathrm{mod}}, \mathcal{D}^{(2)}) \subset \mathscr{C}_{\kappa}(G_{\mathrm{mod}}, \mathcal{D}^{(1)}). \tag{22}$$

Proof. Immediate consequence of Theorem 3. □

Theorem 7. *Consider two different validated PE sets* $\mathcal{D}^{(1)}$ *and* $\mathcal{D}^{(2)}$*, obtained from two different validation experiments, both containing the model* G_{mod}*. Then*

$$\delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathcal{D}^{(1)}) < \delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathcal{D}^{(2)})$$
$$\Rightarrow \mathscr{C}_{\delta}(G_{\mathrm{mod}}, \mathcal{D}^{(2)}) \subset \mathscr{C}_{\delta}(G_{\mathrm{mod}}, \mathcal{D}^{(1)}). \tag{23}$$

Proof. Immediate consequence of Theorem 4. □

Corollary 1. *Among k validated uncertainty regions* $\mathcal{D}^{(i)}$ *obtained from k validation experiments for a model* G_{mod}*, the uncertainty region* $\mathcal{D}^*$ *that generates the largest set* $\mathscr{C}_{\delta}(G_{\mathrm{mod}}, \mathcal{D}^{(i)})$, $i = 1, \ldots, k$, *of* G_{mod}*-based robustly stabilizing controllers is*

$$\mathcal{D}^* = \arg\min_{\mathcal{D}^{(i)}} \delta_{\mathrm{WC}}(G_{\mathrm{mod}}, \mathcal{D}^{(i)}).$$

If $\bar{\mathcal{D}}$ *denotes the intersection of the sets* $\mathcal{D}^{(i)}$*, then the set* $\mathscr{C}_{\delta}(G_{\mathrm{mod}}, \bar{\mathcal{D}})$ *contains all controller sets* $\mathscr{C}_{\delta}(G_{\mathrm{mod}}, \mathcal{D}^{(i)})$, $i = 1, \ldots, k$, *including* $\mathscr{C}_{\delta}(G_{\mathrm{mod}}, \mathcal{D}^*)$.

6.2. Practical use of the worst-case ν-gap

As said above, the worst-case ν-gap is a nice and compact measure of how well the uncertainty region $\mathcal{D}$ is tuned for control stability robustness. In order to present a practical use of this measure, let us consider the following situations.

Suppose first that we have performed one validation experiment leading to one uncertainty region $\mathcal{D}$, and no model has yet been chosen for control design. Before we perform any control design, we want to check whether this uncertainty region is suited for the design of robustly stabilizing controllers. One way to evaluate the quality of $\mathcal{D}$ for robust control design is to check whether

$$\delta_{\mathrm{WC}}(G(z,\hat{\delta}), \mathcal{D}) \ll b_{\mathrm{opt}}(G(z,\hat{\delta})), \tag{24}$$

where $b_{\mathrm{opt}}(G(z,\hat{\delta}))$ was defined in (17). Note that both quantities in (24) can be computed. If (24) holds, then the set of controllers $\mathscr{C}_{\delta}(G(z,\hat{\delta}), \mathcal{D})$ that robustly stabilize $\mathcal{D}$ is large, and $\mathcal{D}$ is thus a suitable uncertainty set for the design of a robust controller. If (24) does not hold, it might be advisable to perform a new validation experiment that would lead to a smaller $\delta_{\mathrm{WC}}(G(z,\hat{\delta}), \mathcal{D})$, thereby allowing for a larger choice of robustly stabilizing controllers. The choice of experimental conditions that lead to a very small $\delta_{\mathrm{WC}}(G(z,\hat{\delta}), \mathcal{D})$ is the object of present research; some preliminary considerations are given in the next subsection.

Suppose now that we have obtained an uncertainty set $\mathscr{D}$ for which (24) holds. One can then choose for control design either the full order model $G(z,\hat{\delta})$ estimated during the validation procedure or a low-order approximation G_{mod} of $G(z,\hat{\delta})$. One should then check whether

$$\delta_{\text{WC}}(G_{\text{mod}},\mathscr{D}) \ll b_{\text{opt}}(G_{\text{mod}}). \quad (25)$$

Observe that the following inequality holds:

$$\delta_{\text{WC}}(G_{\text{mod}},\mathscr{D}) \leqslant \delta_{\text{WC}}(G(z,\hat{\delta}),\mathscr{D}) + \delta_{\nu}(G_{\text{mod}},G(z,\hat{\delta})). \quad (26)$$

If (25) holds, then by Theorem 4 the set of G_{mod}-based controllers that robustly stabilize $\mathscr{D}$ is large, and the nominal model G_{mod}, jointly with the uncertainty set $\mathscr{D}$, are suitable for robust control design. If (25) does not hold, it then behooves us to replace the initial model G_{mod} by another model that is closer to the center of $\mathscr{D}$.

Finally, note that the discussion in this subsection has advocated the use of the worst-case ν-gap (and its comparison with the corresponding b_{opt}) to evaluate the quality of an uncertainty region for robust control design. One advantage is that this worst-case ν-gap is just one number. However, at the cost of a more complicated but less conservative analysis, one could also use the worst-case chordal distance between $G(e^{j\omega},\hat{\delta})$ and $\mathscr{D}$ to evaluate the quality of an uncertainty set $\mathscr{D}$, on the basis of Theorem 6 rather than Theorem 7.

6.3. *Control-oriented design of the validation experiment*

It has been argued in Date and Vinnicombe (1999) that, in identification for control, it makes sense to pose the identification problem for the *nominal model* G_{mod} as one of finding the model that minimizes $\delta_{\nu}(G_0,G_{\text{mod}})$; a suboptimal solution to this problem has been proposed in that paper. We have argued in the present paper that it makes sense (from a robust stability point of view) to design the validation experiment such as to minimize $\delta_{\text{WC}}(G(z,\hat{\delta}),\mathscr{D})$. This optimal experiment design problem has been addressed in Hildebrand and Gevers (2002) where optimal inputs have been constructed that minimize this measure.

7. Conclusions

We have developed a PE model validation procedure that leads to uncertainty sets that have a probability level attached to them, the probability level being at the designer's choice. Our model validation procedure is data driven, and does therefore not rely on such typical prior assumptions as a prior bound on the noise, or a bound on the exponential decay of the impulse response of the true system. Our PE model set validation procedure is nothing but an identification experiment with a full order model structure. The validated uncertainty sets are described in a generic form, which we have called the generic PE model uncertainty set. Our validation procedure has been extended in Bombois et al. (2000a) to reduced order model sets, using a stochastic embedding approach.

We have developed robust analysis tools that are compatible with such PE uncertainty sets, both for the full order case and for the reduced order case. These tools allow one to compute necessary and sufficient conditions for the validation of a controller for stability over all models in a PE uncertainty set. They also allow one to compute the worst-case performance achieved by some controller over all models in such set. These results can be characterized as verification tools, i.e. for all models in a validated PE set, they allow one to verify whether some tentative controller meets the specifications.

Another part of our results pertain to *validation design for robust control*. In order to establish a connection all the way from the design of the validation step to the specification of a set of robustly stabilizing controllers, we have defined a measure of the size of the PE validated sets, and shown that it is connected to the size of the controller sets that are guaranteed to robustly stabilize all models in these PE validated sets.

In the companion paper (Gevers et al., 2003) we shall illustrate the use of both the controller validation tools and the validation for control tools developed in this paper on two realistic examples. We end up with a note of caution. The results of our paper have been presented in a SISO context. While many of the new *concepts* on model and controller validation carry over to the MIMO case, the extension of a number of our technical and computational results is by no means trivial.

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