Optimizing FIR Approximation for Discrete-Time IIR Filters

Yutaka Yamamoto, *Fellow, IEEE*, Brian D. O. Anderson, *Fellow, IEEE*, Masaaki Nagahara, and Yoko Koyanagi

***Abstract*—Finite-impulse response (FIR) filters are often preferred to infinite-impulse response (IIR) filters because of their various advantages in respect of stability, phase characteristic, implementation, etc. This letter proposes a new method to approximate an IIR filter by an FIR filter, which directly yields an optimal approximation with respect to the H^∞ error norm. We show that this design problem can be reduced to a linear matrix inequality. We will also make a comparison via a numerical example with an existing method, known as the Nehari shuffle.**

***Index Terms*—Finite-impulse response (FIR) filter approximation, H^∞ optimization, linear matrix inequality.**

I. Introduction

FINITE-IMPULSE response (FIR) filters are often preferred to infinite-impulse response (IIR) filters, which have infinitely many nonzero Markov parameters, for the following reasons [7].

- FIR filters are intrinsically stable; the stability issue is a nonissue.
- They can easily realize various features that are not possible or are difficult to achieve with IIR filters, e.g., linear phase property.
- They can be free from certain problems in implementation, e.g., limit cycles, attributed to quantization and the existence of a feedback loop in IIR filters.

On the other hand, a design process may have to start with an IIR filter for a variety of reasons. For example, we have a large number of continuous-time filters available, and a digital filter may be obtained by discretizing one of them. It is then desired that such an IIR filter be approximated by an FIR filter. The following problem is thus very natural and of importance.

Problem 1: Given an IIR filter $K(z)$ and a positive integer N, find an optimal FIR approximant $K_f(z)$ that has order N and approximates $K(z)$ with respect to a certain performance measure.

There is a very elegant method called the *Nehari shuffle*, proposed by Kootsookos *et al.* [3], [4]. An advantage is that this procedure gives rise to certain *a priori* and *a posteriori* error bounds. On the other hand, it does not necessarily give an optimal approximation with respect to the H^∞ norm, although it is effectively guaranteed to outperform impulse response truncation as an approximation method.

Manuscript received July 3, 2002; revised October 26, 2002. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Xi Zhang.

Y. Yamamoto, M. Nagahara, and Y. Koyanagi are with the Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan.

B. D. O. Anderson is with the Research School of Information Sciences and Engineering, Australian National University, Canberra, ACT 0200, Australia.

Digital Object Identifier 10.1109/LSP.2003.815615

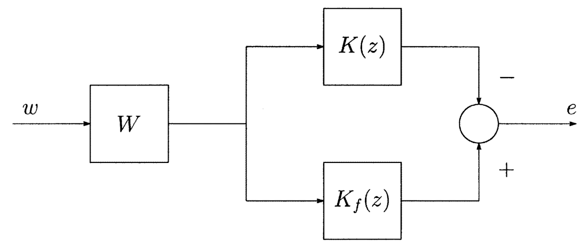

Fig. 1. Error system T_{ew}.

We here propose a method that directly deals with (sub)optimal approximants with respect to the H^∞ error norm. The following are shown.

- The design problem is reducible to a linear matrix inequality (LMI) [1].
- The obtained filter can be made close to being optimal by an iterative procedure.

A comparison with the Nehari shuffle is made for the Chebyshev filter of order eight, which has been studied in detail in [4].

II. FIR Approximation Problem

Consider the block diagram (Fig. 1). $K(z)$ is a given (rational and stable) IIR filter; $W(z)$ is a proper and rational weighting function; and $K_f(z)$ is an FIR filter of a prespecified order N. Denote by $T_{ew}(z)$ the transfer function from the external signal w to the error e in Fig. 1. $W(z)$ determines a weighting in the frequency domain, and the objective here is to find $K_f(z)$ that makes the H^∞ error norm less than a prespecified bound $\gamma > 0$, i.e.,

$$\|T_{ew}\|_\infty := \sup_{w \in l^2} \frac{\|e\|_2}{\|w\|_2} < \gamma.$$

By successively choosing γ smaller, one can approach the optimal filter.

Introduce state–space realizations

$$W(z) := C_W(zI - A_W)^{-1}B_W + D_W$$
$$K(z) := C_K(zI - A_K)^{-1}B_K + D_K$$

and put

$$\begin{aligned} K_f(z) &:= \sum_{k=0}^{N} a_k z^{-k} \\ &= C_f(\alpha)(zI - A_f)^{-1}B_f + D_f(\alpha) \\ C_f(\alpha) &= [a_N, a_{N-1}, \dots, a_1] \\ D_f(\alpha) &= a_0 \end{aligned}$$

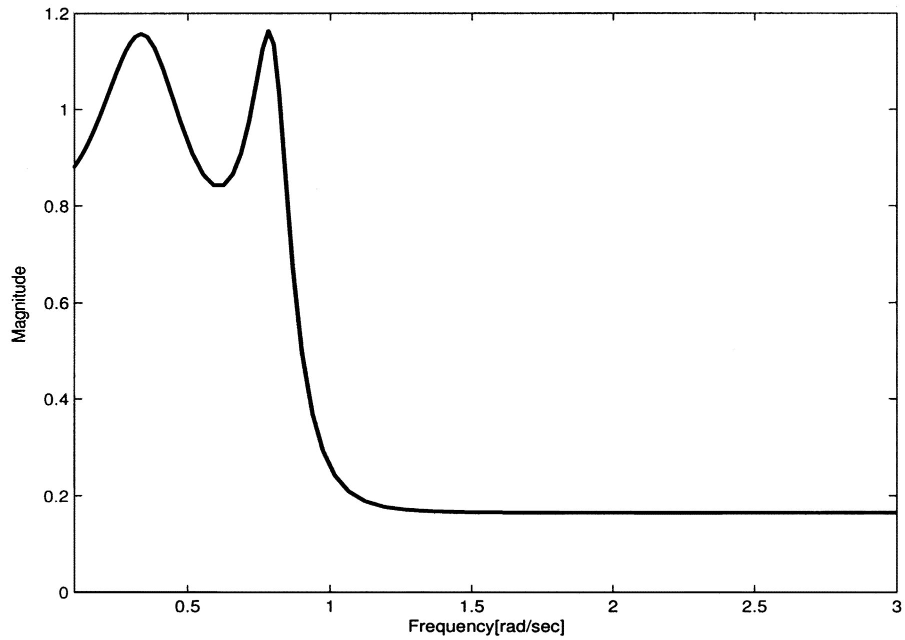

Fig. 2. Inverse of weighting function W.

where $\alpha = [a_N, a_{N-1}, \ldots, a_0]$ denotes the vector of Markov parameters of the filter $K_f(z)$ to be designed. The matrices A_f and B_f are defined as follows:

$$A_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \quad B_f = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and they contain just zeros and ones.

A realization of T_{ew} is given as follows:

$$\begin{aligned} T_{ew}(z) &= C(\alpha)(zI - A)^{-1}B + D(\alpha). \\ A &= \begin{bmatrix} A_W & 0 & 0 \\ B_K C_W & A_K & 0 \\ B_f C_W & 0 & A_f \end{bmatrix} \\ B &= \begin{bmatrix} B_W \\ B_K D_W \\ B_f D_W \end{bmatrix} \\ C(\alpha) &= [\,(D_f(\alpha) - D_K)C_W \quad -C_K \quad C_f(\alpha)\,] \\ D(\alpha) &= [\,(D_K + D_f(\alpha))D_W\,]. \end{aligned}$$

The important point is that the design parameter α appears only in the C and D matrices linearly, and the underlying structure is of the so-called one-block H^∞-optmization type. Hence, the overall transfer operator is linear in α, and the design problem of choosing α to minimize the H_∞ norm can be expected to become a linear matrix inequality (LMI). In fact, the bounded real lemma [1] readily yields the following.

Theorem 1: $\|T_{ew}\|_\infty < \gamma$ if and only if there exists $P > 0$ such that

$$\begin{bmatrix} A^T P A - P & A^T P B & C(\alpha)^T \\ B^T P A & -\gamma I + B^T P B & D(\alpha)^T \\ C(\alpha) & D(\alpha) & -\gamma I \end{bmatrix} < 0. \quad (1)$$

Proof: By the bounded real lemma [1], $\|T_{ew}\| < \gamma$ is equivalent to the condition that there exists a matrix $P > 0$ such that

$$Q^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} Q < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} \quad (2)$$

where

$$Q = \begin{bmatrix} A_W & B_W \\ C_W(\alpha) & D_W(\alpha) \end{bmatrix}.$$

Then, the inequality is converted to (2) by using the Schur complement [1]. ■

Theorem 1 gives an LMI characterization for the existence of an FIR filter $K_f(z)$ such that $\|T_{ew}\|_\infty$ is less than γ. Whether (1) is satisfied can easily be checked by standard MATLAB (particularly, LMI toolbox) routines [2] as follows. Let x be the vector consisting of all variables in α, P and γ in (1). The matrix in (1) is linear with respect to these variables and, hence, can be rewritten in the form $M(x) = \sum_i A_i x_i$ where A_i is a symmetric constant matrix, and x_i is the ith entry of x. The matrix $M(x)$ is easily obtained with the MATLAB function `lmiedit`. Let c be a vector such that $c^T x = \gamma$; this can be obtained by the function `defcx`. Whether γ satisfies (1) can be checked easily by function `feasp`. Minimizing $\gamma = c^T x$ subject to $M(x) < 0$ by using function `mincx` (which also checks feasibility, so `feasp` is not needed), we can approach the optimal filter coefficients α.

III. Numerical Example

A. *Comparison of H^∞ Design via LMI and the Nehari Shuffle*

Take the following Chebyshev filter of order eight:

$$K(z) = 10^{-3} \times \frac{0.047\,05 z^8 + 0.3764 z^7 + 1.317 z^6}{z^8 - 4.953 z^7 + 11.71 z^6}$$

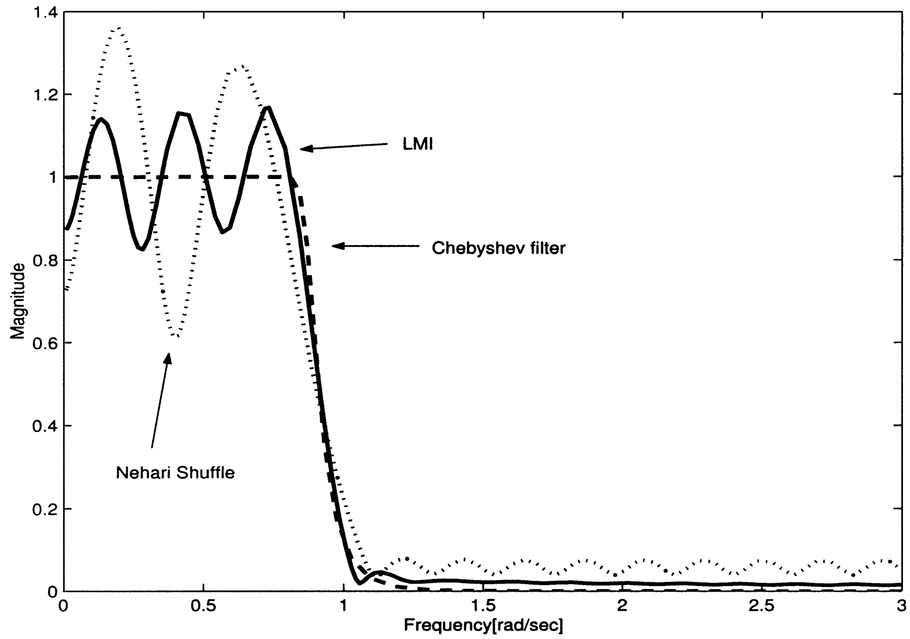

Fig. 3. Gain responses of FIR approximants K_f.

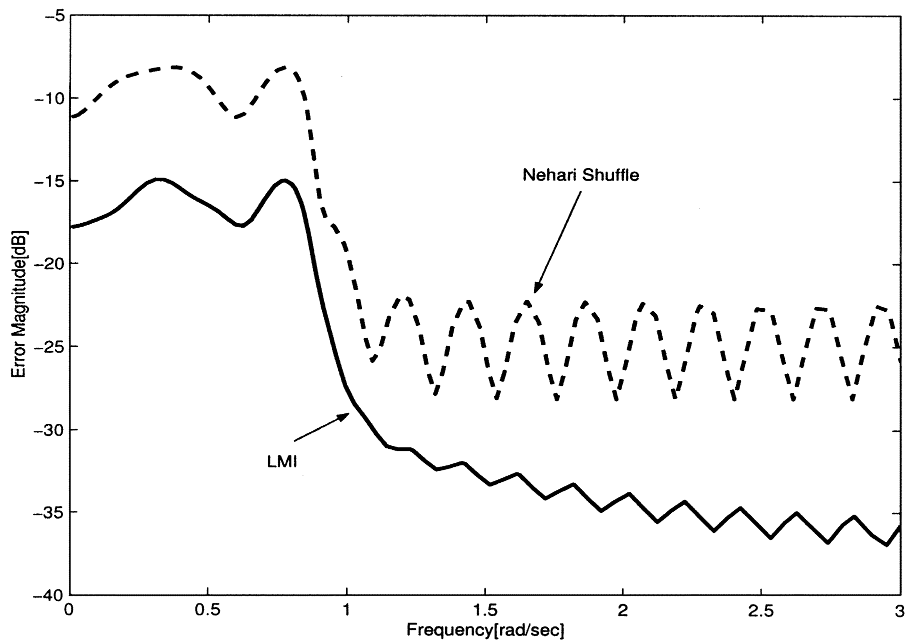

Fig. 4. Gain responses of error system $K_f - K$.

$$\frac{+2.635z^5 + 3.294z^4 + 2.635z^3 + 1.317z^2}{-16.95z^5 + 16.29z^4 - 10.58z^3 + 4.552z^2} \frac{+0.3764z + 0.047\,05}{-1.161z + 0.1369}.$$

as a target filter to be approximated. This has been studied extensively by Kootsookos and Bitmead [4] for the Nehari shuffle, and is suitable for comparison with the present method. For simplicity, we confine ourselves to approximations by FIR filters with 32 tap coefficients (of order 31).

The design depends crucially on the choice of the weight $W(z)$. A natural choice [5] would be to take $W(z)$ to be equal to $K^{-1}(z)$ (or some variant of it having the same gain on the unit circle, since K is not minimum phase). This is a relative error approximation, where (approximately) decibel and phase errors are weighted uniformly with frequency. Since the error criterion in Fig. 1 is taken with respect to the H^∞ norm, it approximates equal amplitude at all frequencies, and this will have the effect of attenuating the stopband error with the weight of $K^{-1}(z)$ (which is very large), while maintaining a reasonable passband characteristic. Unfortunately, however, due to the very small gain of $K(z)$ in the stopband, this will make the solution of the approximation problem (Fig. 1) numerically difficult. Neither the Nehari shuffle nor the LMI method gave a satisfactory result in this case. Hence, one should sacrifice the stopband attenuation to obtain a reasonable $W(z)$. There is also a tradeoff, empirically observed, between the stopband attenuation and the passband ripples.

Kootsookos and Bitmead [4], thus, employed the weight as depicted in Fig. 2.

To be precise, the frequency response shown here is the inverse of the para-Hermitian conjugate of the weight function. The reason for taking the para-Hermitian conjugate is that the Nehari shuffle makes use of causal approximation of an anticausal transfer function, so that we must reciprocate the poles and zeros. Then, by taking the inverse, the weight attenuates the stopband by the inverse of its gain and approximately shapes the passband as it is in the passband. On the other hand, for the FIR approximation as in Fig. 1, we simply take the inverse of this weight, since we do not need to make the weight antistable.

The gain responses of obtained FIR filters based on the Nehari shuffle and Theorem 1 are given in Fig. 3. We see that the gain of the H^∞ approximant shows smaller passband ripples and better stopband attenuation than those by the Nehari shuffle.

Fig. 4 shows the error magnitude response. The FIR filter designed by the LMI method has the advantage of 5–7-dB smaller error over the one obtained by the Nehari shuffle.

IV. Conclusion

We have given an LMI solution to the optimal H^∞ approximation of IIR filters via FIR filters. A comparison with the Nehari shuffle is made with a numerical example, and it is observed that the LMI solution generally performs better. For an application to the sampled-data setting, see also [6].

Acknowledgment

The authors wish to thank P. Kootsookos and R. Bitmead for providing us with detailed data concerning the weighting function in Fig. 2, and particularly the MATLAB routines generating it.

References

[1] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA: SIAM, 1994.

[2] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. Natick, MA: The MathWorks, 1995.

[3] P. J. Kootsookos, R. B. Bitmead, and M. Green, "The Nehari shuffle: FIR(q) filter design with guaranteed error bounds," *IEEE Trans. Signal Processing*, vol. 40, pp. 1876–1883, July 1992.

[4] P. J. Kootsookos and R. B. Bitmead, "The Nehari shuffle and minimax FIR filter design," in *Control and Dynamic Systems*. Orlando, FL: Academic, 1994, vol. 64, pp. 239–298.

[5] G. Obinata and B. D. O. Anderson, *Model Reduction for Control System Design*. Berlin, Germany: Springer-Verlag, 2001.

[6] Y. Yamamoto, A. G. Madievski, and B. D. O. Anderson, "Approximation of frequency response for sampled-data control systems," *Automatica*, vol. 35, pp. 729–734, 1999.

[7] G. Zelniker and F. J. Taylor, *Advanced Digital Signal Processing*. New York: Marcel-Dekker, 1994.