Use of integrator in nonlinear $H_\infty$ design for disturbance rejection

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Abstract

The disturbance suppression problem for nonlinear systems is examined in this paper. We review the so-called nonstandard mixed sensitivity problem, which introduces an integrator to a selected weight, as well as the linear classical disturbance suppression problem and the linear $H_\infty$ disturbance suppression problem. We extend this $H_\infty$ problem to the nonlinear case, and present a method to reduce the order of the state feedback Hamilton-Jacobi (HJ) partial differential equation for this nonlinear $H_\infty$ problem by extending the concept of comprehensive stability (Proceedings of the 36th Conference on Decision and Control, 1997, p. 4653; IEEE Trans. Ind. Electron. (1998) 488). Finally, we investigate the structure of the output feedback $H_\infty$ controller for disturbance suppression, and draw the conclusion that, as in the linear case, there must also be an integrator in the controller.

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1. Introduction

This paper is mainly concerned with constant input disturbance rejection under additive norm bounded model uncertainty. Similar problems, with parametric model uncertainty are dealt with in Byrnes, Priscoli, and Isidori (1997). However, as well as robust constant disturbance rejection, we also consider closed-loop performance specifications in terms of $\gamma$-dissipativity basing our design on $H_\infty$ methods. In recent years, $H_\infty$ methods have been employed to handle disturbance suppression problems (Zhou, Doyle, & Glover, 1996; Mita, Hirata, Murata, & Zhang, 1998) for linear systems. The main methodological device is to introduce an integrator in a selected weight function and then formulate the disturbance rejection problem as a mixed sensitivity problem. Here, the mixed sensitivity problem is the problem of simultaneously achieving bounds on weighted versions of the sensitivity and complementary sensitivity functions (see Mita et al., 1998). However, these problems are nonstandard $H_\infty$ problems, because they have an un-stabilisable pole at the origin, which violates the pre-requisite conditions of standard $H_\infty$ control theory. There are several indirect ways to get around this problem, such as by using singular perturbation techniques or changing the system block diagram to absorb the integrator weight into the control loop (Zhou et al., 1996). Mita et al. (1998) use the so-called extended $H_\infty$ theory to give a relatively direct alternative solution of this nonstandard $H_\infty$ problem for linear systems. Furthermore, the integrator weighting leads to order reduction of the Riccati equation by using a so-called quasi-stabilising solution. As for a classical control design, the controller arising from either of the two $H_\infty$ approaches in Mita et al. (1998) and Zhou et al. (1996) normally contains an integrator. Here we extend these ideas to the nonlinear disturbance suppression problem. As in the linear case, for the general nonlinear $H_\infty$ problem it is convenient to regard some problems as standard (Mita, Xin, & Anderson, 1997; Mita et al., 1998; Zhou et al., 1996), the remaining ones then being nonstandard. Many papers and books (Isidori, 1994; Van der Schaft, 1996; Helton & James, 1999) on nonlinear $H_\infty$ control deal exclusively with standard nonlinear $H_\infty$ control problems. In this paper, we consider issues that arise due to the state-feedback $H_\infty$ problem being nonstandard, assuming that we already have access to a state measurement or estimate. We do not discuss the construction of an appropriate state-estimate. For output feedback...
problems, there are two broad approaches for constructing an $H_\infty$ state estimate. In Isidori (1994) and Van der Schaft (1996), a finite dimensional filter is constructed leading to sufficient or necessary conditions for the existence of an output-feedback controller. In contrast Helton and James (1999) exploits information state ideas, leading to an infinite dimensional filter equation, which nevertheless, leads to necessary and sufficient conditions for solving the $H_\infty$ output feedback problem. Each of Isidori (1994), Van der Schaft (1996) and Helton and James (1999), however, deals with the standard $H_\infty$ problem. In this paper, we investigate the constant disturbance rejection problem. Not surprisingly, the $H_\infty$ constant disturbance rejection problem that we consider for the nonlinear case inherits the difficulty of the linear case: the existence of un-stabilisable states makes the problem nonstandard. In this paper, we will present a method which can simplify (by order reduction) the Hamilton Jacobi (HJ) Partial differential equation (PDE) for the nonlinear disturbance rejection problem by using the concept of comprehensive stability, which is extended from the linear case (see Mita et al., 1998). Furthermore, we can show that the controller for output feedback control contains an integrator in a sense defined later, in Section 5.

In the next section, we examine, for linear systems, the mixed sensitivity $H_\infty$ method, and in particular, the so-called extended $H_\infty$ method, which can deal with the robust constant disturbance suppression problem. In Section 3, we set up the disturbance suppression problem for the nonlinear case. Section 4, the main part, gives an order reduction theorem for the state feedback HJ PDE arising from the nonlinear constant disturbance suppression problem. Finally in Section 5, we probe the structure of the output feedback $H_\infty$ controller of the system under consideration, and show that it normally contains an integrator.

2. $H_\infty$ treatment of the classical disturbance suppression problem for linear systems

Let us consider a classical disturbance rejection problem as shown in Fig. 1. This depicts a linear time-invariant single-input single-output (SISO) system. It consists of the interconnection of a plant $P_0(s)$ and controller $C(s)$ forced by a command signal $r$, as well as an input disturbance $d_u$ and an output disturbance $d_y$. The disturbance rejection problem is interpreted as making $e_y(t)$ tend asymptotically to zero, i.e., $\lim_{t \to \infty} e_y(t) = 0$, in the presence of external disturbances. The classical disturbance suppression technique that involves introduction of an integrator demands separate theoretical consideration of stability, and certainly does not deal directly with the robust stability issue. To guarantee robust stability, we need to rely on a theory of robust control, such as $H_\infty$ theory. There are at least two ways (Zhou et al., 1996; Mita et al., 1998) to design an $H_\infty$ controller for the linear disturbance suppression problem. The main methodological device is to introduce an integrator into a selected weight function. In Zhou et al. (1996) an integrator is introduced into one of the output weights $W_c$ (see Fig. 2 left), while in Mita et al. (1998) there is an integrator in one of the input weights $W_d$ (Fig. 2 right). It can be checked, for the linear case, that the two $H_\infty$ problems are duals of each other. So, without loss of generality, we can choose the mixed sensitivity problem described in Fig. 2 (right) as the basis for our discussions. In this diagram, $P_0$ represents the given plant, $1/s$ and $W_d$ are input weights, $W_c$ is an output weight, and $C$ is the controller which needs to be constructed in such a way that it can stabilise the plant $P_0$, and make the infinity norm of the transfer function from $[\hat{w}_1 \hat{w}_2]'$ to $z$ less than some given bound $\gamma$. Note that at zero frequency the integrator ensures that the gain from the integrator output to $z$ will be zero, and this is the mechanism for achieving constant disturbance suppression. Given the plant and the weights, the standard approach is to seek to formulate the problem as an $H_\infty$ problem. However, this problem does not satisfy all the pre-requisite conditions of the standard $H_\infty$ control problem (which includes a stability condition (Green & Limebeer, 1995; Zhou et al., 1996), because of an un-stabilisable mode at the origin. Therefore, this problem is termed nonstandard. More precisely, consider the state-variable realisation of the "generalised plant" with input $\hat{w}_1, \hat{w}_2$ and $u$ and output $z$ and $y$ in Fig. 2 (right). The entire state is not stabilisable from $u$, because the integrator driven by $\hat{w}_1$ is unaffected by $u$. The so-called extended $H_\infty$ controller (Mita et al., 1998) will solve the mixed sensitivity problem described in Fig. 2 (right). The synthesis of the extended $H_\infty$ controller requires a "quasi-stabilising" solution (Mita et al., 1998) of the "X"-Riccati equation. The original $(n+1)$th order Riccati equation can be constructed from the solution of a reduced order $n$th order equation, $n$ being the degree of $P_0$. Similarly we can use extended $H_\infty$ controller design to solve the mixed sensitivity problem of Fig. 2 (left). Not surprisingly, for this dual formulation, it is possible to simplify the controller synthesis by constructing the solution to the original $(n+1)$th order "Y" Riccati equation from the solution of a reduced $n$th order equation.
3. Setting up the disturbance suppression problem formulation in the nonlinear case

We consider the classical disturbance problem shown as in Fig. 1, except that the plant may be nonlinear. In order to give a more explicit description, we suppose that the SISO nonlinear plant, \( P_0 \), is modelled as follows:

\[
P_0 : \begin{cases}
    \dot{x}_0 = A(x_0) + B_1(x_0)w_1 + B_2(x_0)u, \\
y = C_2(x_0) + w_2.
\end{cases}
\]  

We assume that the functions appearing in systems of this paper are smooth with bounded first and second order partial derivatives. Here, \( w = [w_1 \ w_2] \), and \( w_1 \in R \) corresponds to a plant input disturbance, while \( w_2 \in R \) corresponds to a plant output disturbance. The introduction of the disturbance \( w_2 \) can be interpreted as a way of capturing additive norm bounded modelling uncertainty for output feedback \( H_\infty \) control. In order that the constant disturbance rejection problem is solvable, it is necessary that the range space of \( B_2(x_0) \) is within the range space of \( B_1(x_0) \). It is then in principle possible to find an input transformation \( Ge = u \) such that \( B_2(x_0)G = B_1(x_0) \), so that we suppose the matching condition \( B_1(x_0) = B_2(x_0) \) without loss of generality (For the input disturbance rejection problem, this condition is always satisfied). As mentioned in the previous section, for the linear case Zhou et al. (1996), Mita et al. (1998), there are two ways to perform this step. The first one is depicted in Fig. 2 (left), and the second one in Fig. 2 (right). The first way, as stated in the last section, leads to a solution allowing order reduction of the "Y"-Riccati equation (or observer Riccati equation) in the linear case. However, for the nonlinear case, there is no simple and explicit "filter" \( H \) PDE which is equivalent to the "Y" Riccati equation of linear \( H_\infty \) system theory. If we choose the second formulation, it turns out that we can find the order of the control \( H \) PDE for the state feedback problem (which is equivalent to the "X" Riccati equation in the linear case). Therefore, we elect to extend this nonlinear disturbance suppression problem to an \( H_\infty \) problem along the lines of Mita et al. (1998). The framework of this nonlinear \( H_\infty \) problem is shown in Fig. 3. In order to slightly extend the application scope of our method, we choose \( (1/s)G_{w1}(s) \) instead of only \( 1/s \) as the weight of \( \hat{w}_1 \). Here \( G_{w1}(s) \) is a stable and properly proper transfer function and \( G_{w2}(s) \) is a stable and strictly proper transfer function. The state equation of the weight transfer function becomes

\[
\dot{x}_{w1} = \alpha \hat{w}_1,
\]

\[
\dot{x}_{w2} = A_{w2}x_{w2} + B_{w2}\hat{w}_2,
\]

\[
w_1 = w_{11} + w_{12} = x_{w1} + C_{w12}x_{w12}.  
\]  

We assume that \( G_{w2}(s) \) in Fig. 3 is a rational stable proper transfer function with no finite and infinite \( j\omega \)-axis zeros, which can be expressed as

\[
\dot{x}_w = A_wx_w + D_w\hat{w}_2.
\]

The equations of the generalised system are now:

\[
\dot{x}_w = A_wx_w + B_w\hat{w}_1,
\]

\[
y = C_2(x_0) + C_{w2}x_w + D_w\hat{w}_2.  
\]  

The choice of \( z \) is important. We elect to set \( z = e' \), a small difference from the linear case. That is, we split the disturbance \( w_1 \) into two components, \( w_{11} = x_{w11} \) and \( w_{12} = C_{w12}x_{w12} + u \),

\[
z = e' = x_{w11} + u.  
\]  

(4*)

For the linear case, \( z = e = y_w + u \). We also could choose \( z = e \) here, but it is more convenient to provide another choice of \( z \).
4. Simplification of the state feedback HJ PDE for disturbance suppression problem

Here, we extend the concept of the so-called comprehensive stability (Mita et al., 1998) to the nonlinear $H_\infty$ problem. This includes the nonlinear disturbance rejection problem, which contains un-stabilisable states. The constant disturbance rejection problem (as we have formulated it) is a nonstandard $H_\infty$ problem, because $x_{wi}$ is not stabilisable from $u$.

First we introduce the standard nonlinear state feedback $H_\infty$ control problem: See Fig. 4, let the state space model for plant $P$ be

$$
\dot{x} = A(x) + B_1(x)\dot{\nu} + B_2(x)u, \\
z = C_1(x) + D_{12}(x)u, \\
y = x.
$$

(5)

The standard state feedback $H_\infty$ control problem is to find a controller $u = K(x)$ which makes the closed loop $(P, K)$ $\gamma$-dissipative and internally stable, see Helton and James (1999). Internal stability is the condition that $x(t) \to 0$ as $t \to 0$ for all $x(0)$ and $\dot{\nu} \in L_2[0, \infty)$.

**Theorem 1.** Consider the system defined by Eq. (5), and suppose that $\exists \alpha, \beta : \alpha I \geq E_1 = D_{12}^T D_{12} \geq \beta I > 0$ for all $x$. Suppose one can find a strictly positive proper smooth function $V$ of $x$, such that $V(x) > 0$ for $x \neq 0$, $V(0) = 0$ and which (a) satisfies the state feedback Hamilton–Jacobi PDE ($HJ$ PDE)

$$
\nabla_x V(A - B_2 E_1^{-1} D_{12}^T C_1) \\
+ \frac{1}{2} \nabla_x V(\gamma^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T) \nabla_x V^T \\
+ \frac{1}{2} C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0.
$$

(6)

and (b) makes the vector field

$A - B_2 E_1^{-1} D_{12}^T C_1 + (\gamma^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T) \nabla_x V^T$

asymptotically stable. Then the central controller for the state feedback problem, which guarantees $\gamma$-dissipativity and internal stability, is defined as: $K^*(x) = -E_1(x)^{-1}[D_{12}(x)^T C_1(x) + B_2(x)^T \nabla_x V(x)^T]$. Furthermore, even if (b) is not fulfilled, the closed-loop satisfies the dissipation inequality for all $T$, $x(0)$, and $\dot{\nu}(\cdot)$:

$$
V(x(t)) + \frac{1}{2} \int_0^T |z(t)|^2 dt \leq \gamma^2 \frac{1}{2} \int_0^T |\dot{\nu}(t)|^2 dt + V(x(0)).
$$

**Proof.** See Helton and James (1999).

For some mixed sensitivity problems, such as (4), there exist some un-stabilisable states, so it is obvious that no stabilising solution for the HJ PDE exists. In order to get around this obstacle, we extend the concepts of comprehensive stability and essential stability (Mita et al., 1997) to nonlinear systems.

**Definition 2.** The closed-loop system $(P, K)$ in Fig. 4 is essentially stable if the interconnection of the physical plant $P_0$ and controller $K$ is internally stable, or equivalently, if the only noninternally stable modes of $(P, K)$ are those associated with the weighting.

The motivation is that the weighting is not present in any physical sense, while $P_0$ and $K$ are physically present.

**Definition 3.** The closed-loop system $(P, K)$ in Fig. 4 is said to be comprehensively stable if it is essentially stable, and the closed-loop from $\dot{\nu}$ to $z$ is $\gamma$-dissipative. When this is the case, $K$ is called a comprehensively stabilising controller.

**Definition 4.** The system (of Fig. 4) with input $\dot{\nu}$ and output $z$ is said to be zero-detectable if the conditions that $\dot{\nu}(t) = 0$ and $z(t) = 0$ for all $t \geq 0$, are sufficient to imply that $\lim_{t \to \infty} x(t) = 0$.

We present a lemma as follows, which will be needed for the main stability theorem. It comes from a simple extension of La Salle's invariance principle (La Salle & Lefschetz, 1961).

**Definition 5.** Let $x = [x_w^T \ x_z^T]^T$. Define $\Pi$ as the projection from $\mathcal{R}^{\dim(x)}$ to $\mathcal{R}^{\dim(x_z)}$ in the obvious way by $\Pi([x_w^T \ x_z^T]^T) = x_z$.

**Lemma 6.** Let $V(x)$ be a scalar function with continuous partial derivatives. Let $B_r$ be the set defined as $\{x : V(x) < r\}$. Assume that for a fixed but arbitrary $r \in \mathcal{R}$, $\Pi B_r$ is a bounded set and that also within $B_r$, the following conditions hold

- $\dot{V}(x) \leq 0$,
- $V(x) > 0$ for $x_z \neq 0$ and $V(x) = 0$ for $x_z = 0$,
- for every trajectory of $x$ starting from $x(0)$ within $B_r$, there is a bound for $x(t)$ (which may possibly depend on $x(0)$).
Let $\mathcal{N}$ be the set of all points within $B_r$ where $\dot{V}(x) \leq 0$ and $M$ be the largest invariant set within $\mathcal{N}$. Then for every possible $x(0)$ in $B_r$, as $t \to \infty$, $x(t) \to M$ and consequently every associated projection $x_0 = \Pi(x)$ tends to $M = \Pi(M)$.

**Proof.** Since $\dot{V}(x) \leq 0$ then $V(x(t)) \leq V(x(0)) = v$ as $t \to \infty$ so that $x(t) \in B_r$ for $t \geq 0$. Since $V \geq 0$, it follows that $V(x(t))$ has a limit $l$ as $t \to \infty$, where $l \leq v$. Let $\Gamma$ be the (positive) limiting set of $x(t)$. Note that $\Gamma$ is not empty due to the boundedness of the trajectories of $x(t)$ on $B_r$. By the continuity of $\dot{V}(x)$ we conclude that $V(x_T) = l$ for all $x_T \in \Gamma$ and that therefore $\Gamma \subset B_r$ and $\dot{V}(x) \equiv 0$ on $\Gamma$. Since $\Gamma$ is an invariant set it follows that $\Gamma \subset M$. Since $x(t)$ remains bounded within $B_r$, it follows that $x \to M$ as $t \to \infty$ and the theorem conclusion follows.

**Theorem 7.** Consider the system defined by Eq. (5), and suppose that $3a_1 \beta : a_1 \geq E_1 = D_{12}^T D_{12} \geq \beta > 0$ for all $x$. Suppose also that $u = K(x)$ for some $K$ such that $K(0) = 0$. Suppose that the state vector of $P$ is of the form $[x^T_1, x^T_2]$, in which the components $x_1$ are stabilisable from $u$ and the components $x_2$ are associated only with weights and are not necessarily stabilizable. Then the closed-loop system $(P, K)$ will be comprehensively stable, given the following conditions are satisfied:

- There exists a storage function $V$, such that $V(x) > 0$ if $x_1 \neq 0$ and $V(x) = 0$ if $x_1 = 0$, which satisfies the dissipative inequality:

$$\dot{V}(x) \leq \frac{1}{2} [v_1^2 \| \dot{w} \|^2 - \| z \|^2]. \tag{7}$$

- The states $x_1$ are zero-detectable.

**Proof.** From Eq. (7), we can calculate that the closed loop from $\dot{w}$ to $z$ is $\gamma$-dissipative. Now we only need to prove that the state $x_1$ is asymptotically stabilised. Because inequality (7) is satisfied for all $\dot{w}$, for the case when $\dot{w} = 0$; we have $\dot{V}(x) \leq -\| z \|^2$. Now we appeal to Lemma 6. The set of trajectories for which $\dot{V} \equiv 0$ is the set for which $x(t) \equiv 0$. By the theorem hypothesis, $\dot{w} \equiv 0$ and $z = 0$ imply $\lim_{t \to \infty} x_1(t) = 0$.

**Theorem 8.** Consider the system described by Eq. (4). Suppose that the state vector of the plant $P$ is of the form $[x_{w1}, x^T_1] = [x_{w1}, x^T_1, x^T_2, x^T_3]$, where the sub-state $x_1$ is zero-detectable. If there exists a function $\tilde{V}(x)$, such that $\tilde{V}(x) > 0$ if $x_1 \neq 0$ and $V(x) = 0$ if $x_1 = 0$, which satisfies the following HJ PDE:

$$\nabla_x \tilde{V}^T A + \frac{1}{2} \nabla_{x_1} \tilde{V} (y^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T) \nabla_{x_1} V = 0, \tag{8}$$

then system (4) can be comprehensively stabilised by the central controller $K^*(x_1)$, defined as

$$K^*(x_1) = -E_1(x_1)^{-1} (B_1(x_1)^T C_1(x_1) + B_2(x_1)^T \nabla_{x_1} \tilde{V}(x)^T).$$

In the above equations the terms are given by

$$\tilde{A} = \tilde{A}(x_1) = \begin{bmatrix} A_{w1} x_{w1} & \cdots & A_{w1} x_{w1} \\ & \ddots & \vdots \\ & & A_{w1} x_{w1} \end{bmatrix},$$

$$\tilde{B}_1 = \begin{bmatrix} B_{w1} & 0 \\ 0 & B_{w2} \end{bmatrix},$$

$$\tilde{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_1(x) = x_{w1}, \tilde{D}_1 = D_{12}$$

and

$$\tilde{E}_1 = \tilde{D}_1^T \tilde{D}_{12}.$$
able to be described as constant signals plus signals in $L_2$. Because of our assumption that the nonlinear plant does not contain an integrator, the observed output $y$ must be a zero constant signal plus a signal in $L_2$. We now observe that the controller $K$ has an input $y \in L_2$ and an output $u$ which is a nonzero constant signal plus a signal in $L_2$. By Definition 10, the controller contains an integrator.

6. Conclusion

This paper presents a modest extension of nonlinear $H_{\infty}$ theory in order to solve the constant disturbance rejection problem. We have suggested a nonlinear extension of a concept introduced for the corresponding linear problem, that of the "comprehensively stabilising" controller, and have achieved an order reduced $HJ$ PDE for the state feedback problem. Furthermore, we draw the conclusion that the output feedback controller normally must contain an integrator for constant disturbance suppression. This method improves our intuitive understanding of the linear problem.

Appendix A. Proof of Theorem 8

Let $u = \bar{u} + \hat{u}$, and

$$\dot{\bar{u}} = -E_1(x)^{-1}D_{12}(x)^T C_1(x) = x_{w1}.$$  \hfill (A.1)

Then, Eqs. (4) and (4*) together become

$$\begin{align*}
\dot{x}_{w1} &= \alpha \bar{v}_1, \\
\dot{x}_{w2} &= A_{w2} x_{w2} + B_{w2} \dot{w}_2, \\
\dot{x}_0 &= A(x_0) + B_1(x_0) C_{w1} x_{w12} + B_1(x_0) \bar{u}, \\
z &= \bar{u}, \\
y &= C_2(x_0) + C_{w2} x_{w2} + D_{w2} \dot{w}_2.
\end{align*}$$  \hfill (A.2)

For the system defined by Eq. (A.2), set

$$\begin{align*}
\dot{A}(x) &= [0 \ A_{w1} x_{w1} \ A_{w2} x_{w2} \ A(x_0) + B_1(x_0) C_{w1} x_{w12}]^T, \\
\dot{B}_1(x) &= \begin{bmatrix}
\alpha & B_{w1} \\
0 & B_{w2}
\end{bmatrix}^T, \\
\dot{B}_2(x) &= [0 \ 0 \ B_1(x_0)]^T, \\
\dot{D}_{12}(x) &= 1, \text{ and } \dot{E}_1 \triangleq D_{12}^T D_{12}.
\end{align*}$$

Then according to Theorem 1, the HJ PDE for the above system is

$$\begin{align*}
&V_{w12} A_{w12} x_{w12} + V_{w2} A_{w2} x_{w2} + V_{x0}(A(x_0) + B_1(x_0) C_{w1} x_{w12}) \\
&+ \frac{1}{2} \gamma^2 \begin{bmatrix} V_{w11}^T \ V_{w2}^T \ \end{bmatrix}^T \begin{bmatrix}
\alpha^2 & \alpha B_{w1} & 0 & 0 \\
\alpha B_{w1} & B_{w1} B_{w12} & 0 & 0 \\
0 & 0 & B_{w2} B_{w2} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} V_{w11} \\
V_{w2} \\
V_{x0}
\end{bmatrix} \\
&- \frac{1}{2} \begin{bmatrix} V_{w11}^T \ V_{w2}^T \ \end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} V_{w11} \\
V_{w2} \\
V_{x0}
\end{bmatrix} \begin{bmatrix} V_{w11}^T \\
V_{w2}^T \\
V_{x0}^T
\end{bmatrix}
= 0.
\end{align*}$$  \hfill (A.3)

Now with $V$ the solution of Eq. (8), we can verify that $\bar{V}(x_{w11}, x_{w2}, x_{w12}, x_0) = \bar{V}(x_{w11}, x_{w2}, x_0)$ satisfies (A.3). For with this identification, $\bar{V}_{x11} = 0$, and Eq. (A.3) becomes

$$\begin{align*}
&\begin{bmatrix} \bar{p}_{w11}^T \\
\bar{p}_{w2}^T \end{bmatrix} \begin{bmatrix}
A_{w12} x_{w12} \\
A_{w2} x_{w2}
\end{bmatrix} + \begin{bmatrix} B_{w12} \bar{p}_{w12}^T & 0 & 0 \\
0 & B_{w2} B_{w2} & 0
\end{bmatrix} \begin{bmatrix} \bar{p}_{w11}^T \\
\bar{p}_{w2}^T \end{bmatrix} \\
&\quad - \frac{1}{2} \begin{bmatrix} \bar{p}_{w11}^T \\
\bar{p}_{w2}^T \end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix} \bar{p}_{w11}^T \\
\bar{p}_{w2}^T \\
\bar{p}_{x0}
\end{bmatrix} \\
&= 0.
\end{align*}$$  \hfill (A.4)

This is identical to Eq. (8), which is true by hypothesis. Since the solution $\bar{V}(x)$ has the property that $\bar{V}(x) > 0$ if $x \neq 0$, $\bar{V}(x) = 0$ if $x = 0$, then from Theorem 1 we conclude that equation (A.3) has a solution that makes the closed-loop $(P, K^*)$ $\gamma$-dissipative without necessarily satisfying the closed-loop asymptotic stability condition of Theorem 1. Because $x_3$ is zero-detectable by hypothesis, then from the $\gamma$-dissipativity property that we have demonstrated immediately previously, and from application of Theorem 7, we conclude that the closed-loop $(P, K)$ is comprehensively stabilised.
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