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Brief Paper

Use of integrator in nonlinear H_∞ design for disturbance rejection ☆

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Abstract

The disturbance suppression problem for nonlinear systems is examined in this paper. We review the so-called nonstandard mixed sensitivity problem, which introduces an integrator to a selected weight, as well as the linear classical disturbance suppression problem and the linear H_∞ disturbance suppression problem. We extend this H_∞ problem to the nonlinear case, and present a method to reduce the order of the state feedback Hamilton–Jacobi (HJ) partial differential equation for this nonlinear H_∞ problem by extending the concept of comprehensive stability (Proceedings of the 36th Conference on Decision and Control, 1997, p. 4653; IEEE Trans. Ind. Electron. (1998) 488). Finally, we investigate the structure of the output feedback H_∞ controller for disturbance suppression, and draw the conclusion that, as in the linear case, there must also be an integrator in the controller.

Keywords: Nonlinear control; H_∞ control; Disturbance rejection

1. Introduction

This paper is mainly concerned with constant input disturbance rejection under additive norm bounded model uncertainty. Similar problems, with parametric model uncertainty are dealt with in Byrnes, Priscoli, and Isidori (1997). However, as well as robust constant disturbance rejection, we also consider closed-loop performance specifications in terms of γ-dissipativity basing our design on H_∞ methods. In recent years, H_∞ methods have been employed to handle disturbance suppression problems (Zhou, Doyle, & Glover, 1996; Mita, Hirata, Murata, & Zhang, 1998) for linear systems. The main methodological device is to introduce an integrator in a selected weight function and then formulate the disturbance rejection problem as a mixed sensitivity problem. Here, the mixed sensitivity problem is the problem of simultaneously achieving bounds on weighted versions of the sensitivity and complementary sensitivity functions (see Mita et al., 1998). However, these problems are nonstandard H_∞ problems, because they have an un-stabilisable pole at the origin, which violates the pre-requisite conditions of standard H_∞ control theory. There are several indirect ways to get around this problem, such as by using singular perturbation techniques or changing the system block diagram to absorb the integrator weight into the control loop (Zhou et al., 1996). Mita et al. (1998) use the so-called extended H_∞ theory to give a relatively direct alternative solution of this nonstandard H_∞ problem for linear systems. Furthermore, the integrator weighting leads to order reduction of the Riccati equation by using a so-called quasi-stabilising solution. As for a classical control design, the controller arising from either of the two H_∞ approaches in Mita et al. (1998) and Zhou et al. (1996) normally contains an integrator. Here we extend these ideas to the nonlinear disturbance suppression problem. As in the linear case, for the general nonlinear H_∞ problem it is convenient to regard some problems as standard (Mita, Xin, & Anderson, 1997; Mita et al., 1998; Zhou et al., 1996), the remaining ones then being nonstandard. Many papers and books (Isidori, 1994; Van der Schaft, 1996; Helton & James, 1999) on nonlinear H_∞ control deal exclusively with standard nonlinear H_∞ control problems. In this paper, we consider issues that arise due to the state-feedback H_∞ problem being nonstandard, assuming that we already have access to a state measurement or estimate. We do not discuss the construction of an appropriate state-estimate. For output feedback

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problems, there are two broad approaches for constructing an H_∞ state estimate. In Isidori (1994) and Van der Schaft (1996), a finite dimensional filter is constructed leading to sufficient or necessary conditions for the existence of an output-feedback controller. In contrast Helton and James (1999) exploits information state ideas, leading to an infinite dimensional filter equation, which nevertheless, leads to necessary and sufficient conditions for solving the H_∞ output feedback problem. Each of Isidori (1994), Van der Schaft (1996) and Helton and James (1999), however, deals with the standard H_∞ problem. In this paper, we investigate the constant disturbance rejection problem. Not surprisingly, the H_∞ constant disturbance rejection problem that we consider for the nonlinear case inherits the difficulty of the linear case: the existence of un-stabilisable states makes the problem nonstandard. In this paper, we will present a method which can simplify (by order reduction) the Hamiton Jacobi (HJ) Partial differential equation (PDE) for the nonlinear disturbance rejection problem by using the concept of *comprehensive stability*, which is extended from the linear case (see Mita et al., 1998). Furthermore, we can show that the controller for output feedback control contains an integrator in a sense defined later, in Section 5.

In the next section, we examine, for linear systems, the mixed sensitivity H_∞ method, and in particular, the so-called extended H_∞ method, which can deal with the robust constant disturbance suppression problem. In Section 3, we set up the disturbance suppression problem for the nonlinear case. Section 4, the main part, gives an order reduction theorem for the state feedback HJ PDE arising from the nonlinear constant disturbance suppression problem. Finally in Section 5, we probe the structure of the output feedback H_∞ controller of the system under consideration, and show that it normally contains an integrator.

2. H_∞ treatment of the classical disturbance suppression problem for linear systems

Let us consider a classical disturbance rejection problem as shown in Fig. 1. This depicts a linear time-invariant single-input single-output (SISO) system. It consists of the interconnection of a plant $P_0(s)$ and controller $C(s)$ forced by a command signal r, as well as an input disturbance d_u and an output disturbance d_y. The disturbance rejection problem is interpreted as making $e_y(t)$ tend asymptotically to zero, i.e., $\lim_{t\to\infty} e_y(t) = 0$, in the presence of external disturbances. The classical disturbance suppression technique that involves introduction of an integrator demands separate theoretical consideration of stability, and certainly does not deal directly with the robust stability issue. To guarantee robust stability, we

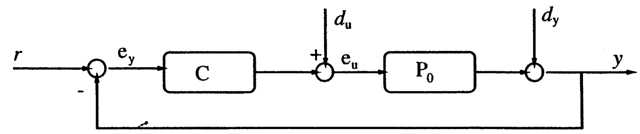

Fig. 1. A classical disturbance suppression problem.

need to rely on a theory of robust control, such as H_∞ theory. There are at least two ways (Zhou et al., 1996; Mita et al., 1998) to design an H_∞ controller for the linear disturbance suppression problem. The main methodological device is to introduce an integrator into a selected weight function. In Zhou et al. (1996) an integrator is introduced into one of the output weights W_z (see Fig. 2 left), while in Mita et al. (1998) there is an integrator in one of the input weights W_d (Fig. 2 right). It can be checked, for the linear case, that the two H_∞ problems are duals of each other. So, without loss of generality, we can choose the mixed sensitivity problem described in Fig. 2 (right) as the basis for our discussions. In this diagram, P_0 represents the given plant, $1/s$ and W_{d2} are input weights, W_z is an output weight, and C is the controller which needs to be constructed in such a way that it can stabilise the plant P_0, and make the infinity norm of the transfer function from $[\hat{w}_1\ \hat{w}_2]^{\mathrm{T}}$ to z less than some given bound γ. Note that at zero frequency the integrator ensures that the gain from the integrator output to z will be zero, and this is the mechanism for achieving constant disturbance suppression. Given the plant and the weights, the standard approach is to seek to formulate the problem as an H_∞ problem. However, this problem does not satisfy all the pre-requisite conditions of the standard H_∞ control problem (which includes a stability condition (Green & Limebeer, 1995; Zhou et al., 1996), because of an un-stabilisable mode at the origin. Therefore, this problem is termed nonstandard. More precisely, consider the state-variable realisation of the "generalised plant" with input $\hat{w}_1, \hat{w}_2$ and u and output z and y in Fig. 2 (right). The entire state is not stabilisable from u, because the integrator driven by $\hat{w}_1$ is unaffected by u. The so-called extended H_∞ controller (Mita et al., 1998) will solve the mixed sensitivity problem described in Fig. 2 (right). The synthesis of the extended H_∞ controller requires a "quasi-stabilising" solution (Mita et al., 1998) of the "X"-Riccati equation. The original $(n+1)$th order Riccati equation can be constructed from the solution of a reduced order nth order equation, n being the degree of P_0. Similarly we can use extended H_∞ controller design to solve the mixed sensitivity problem of Fig. 2 (left). Not surprisingly, for this dual formulation, it is possible to simplify the controller synthesis by constructing the solution to the original $(n+1)$th order "Y" Riccati equation from the solution of a reduced nth order equation.

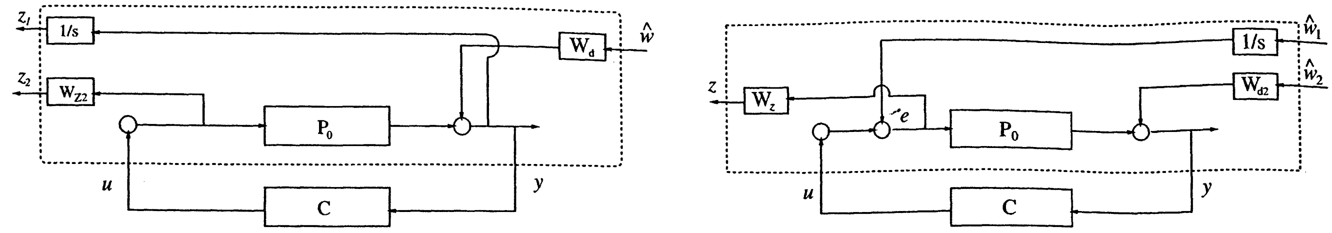

Fig. 2. The linear H_∞ framework for the disturbance suppression problem. Left: an integral in the output weight. Right: an integral in the input weight.

3. Setting up the disturbance suppression problem formulation in the nonlinear case

We consider the classical disturbance problem shown as in Fig. 1, except that the plant may be nonlinear. In order to give a more explicit description, we suppose that the SISO nonlinear plant, P_0, is modelled as follows:

$$P_0: \begin{cases} \dot{x}_0 = A(x_0) + B_1(x_0)w_1 + B_2(x_0)u, \\ y = C_2(x_0) + w_2. \end{cases} \quad (1)$$

We assume that the functions appearing in systems of this paper are smooth with bounded first and second order partial derivatives. Here, $w = [w_1\ w_2]^{\mathrm{T}}$, and $w_1 \in R$ corresponds to a plant input disturbance, while $w_2 \in R$ corresponds to a plant output disturbance. The introduction of the disturbance w_2 can be interpreted as a way of capturing additive norm bounded modelling uncertainty for output feedback H_∞ control. In order that the constant disturbance rejection problem is solvable, it is necessary that the range space of $B_1(x_0)$ is within the range space of $B_2(x_0)$. It is then in principle possible to find an input transformation $Gv = u$ such that $B_2(x_0)G = B_1(x_0)$, so that we suppose the matching condition $B_1(x_0) = B_2(x_0)$ without loss of generality (For the input disturbance rejection problem, this condition is always satisfied). As mentioned in the previous section, for the linear case Zhou et al. (1996), Mita et al. (1998), there are two ways to perform this step. The first one is depicted in Fig. 2 (left), and the second one in Fig. 2 (right). The first way, as stated in the last section, leads to a solution allowing order reduction of the "Y"-Riccati equation (or observer Riccati equation) in the linear case. However, for the nonlinear case, there is no simple and explicit "filter" HJ PDE which is equivalent to the "Y" Riccati equation of linear H_∞ system theory. If we choose the second formulation, it turns out that we can reduce the order of the control HJ PDE for the state feedback problem (which is equivalent to the "X" Riccati equation in the linear case). Therefore, we elect to extend this nonlinear disturbance suppression problem to an H_∞ problem along the lines of Mita et al. (1998). The framework of this nonlinear H_∞ problem is shown in Fig. 3. In order to slightly extend the application scope of our method, we choose $(1/s)G_{w1}(s)$ instead of only $1/s$ as the weight of $\hat{w}_1$. Here $G_{w1}(s)$ is a stable and proper transfer function. Based on linear classical control theory, $(1/s)G_{w1}(s)$ can be written as $\alpha/s + G_{w12}(s)$, where α is real and $G_{w12}(s)$ is a stable and strictly proper transfer function. The state equation of the weight transfer function becomes

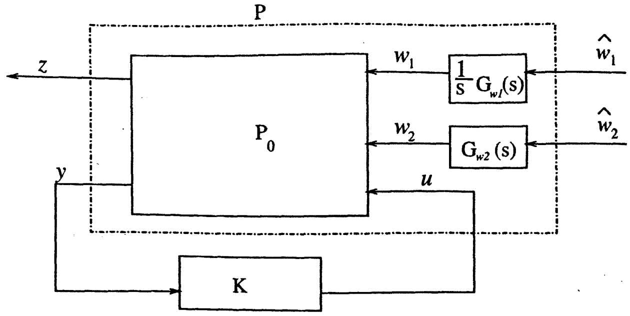

Fig. 3. A nonlinear disturbance suppression problem.

$$\dot{x}_{w_{11}} = \alpha \hat{w}_1,$$
$$\dot{x}_{w_{12}} = A_{w_{12}} x_{w_{12}} + B_{w_{12}} \hat{w}_1,$$
$$w_1 = w_{11} + w_{12} = x_{w_{11}} + C_{w_{12}} x_{w_{12}}. \quad (2)$$

We assume that $G_{w_2}(s)$ in Fig. 3 is a rational stable proper transfer function with no finite and infinite $j\omega$-axis zeros, which can be expressed as

$$\dot{x}_{w_2} = A_{w_2} x_{w_2} + B_{w_2} \hat{w}_2,$$
$$w_2 = C_{w_2} x_{w_2} + D_{w_2} \hat{w}_2. \quad (3)$$

The equations of the generalised system are now:

$$\dot{x}_{w_{11}} = \alpha \hat{w}_1,$$
$$\dot{x}_{w_{12}} = A_{w_{12}} x_{w_{12}} + B_{w_{12}} \hat{w}_1,$$
$$\dot{x}_{w_2} = A_{w_2} x_{w_2} + B_{w_2} \hat{w}_2,$$
$$\dot{x}_0 = A(x_0) + B_1(x_0)(x_{w_{11}} + C_{w_{12}} x_{w_{12}} + u),$$
$$y = C_2(x_0) + C_{w_2} x_{w_2} + D_{w_2} \hat{w}_2. \quad (4)$$

The choice of z is important. We elect to set $z = e'$, a small difference from the linear case. [1] That is, we split the disturbance w_1 into two components, $w_{11} = x_{w11}$ and $w_{12} = C_{w2} x_{w2}$

$$z = e' = x_{w_{11}} + u. \quad (4^*)$$

[1] For the linear case, $z = e = y_{w_1} + u$. We also could choose $z = e$ here, but it is more convenient to provide another choice of z.

4. Simplification of the state feedback HJ PDE for disturbance suppression problem

Here, we extend the concept of the so-called comprehensive stability (Mita et al., 1998) to the nonlinear H_∞ problem. This includes the nonlinear disturbance rejection problem, which contains un-stabilisable states. The constant disturbance rejection problem (as we have formulated it) is a nonstandard H_∞ problem, because $x_{w_{11}}$ is not stabilisable from u.

First we introduce the standard nonlinear state feedback H_∞ control problem: See Fig. 4, let the state space model for plant P be

$$\dot{x} = A(x) + B_1(x)\hat{w} + B_2(x)u,$$

$$z = C_1(x) + D_{12}(x)u,$$

$$y = x. \tag{5}$$

The standard state feedback H_∞ control problem is to find a controller $u = K(x)$ which makes the closed loop (P,K) γ-dissipative and internally stable, see Helton and James (1999). Internal stability is the condition that $x(t) \to 0$ as $t \to 0$ for all $x(0)$ and $\hat{w} \in L_2[0,\infty)$.

Theorem 1. *Consider the system defined by Eq.* (5), *and suppose that* $\exists \alpha, \beta : \alpha I \geqslant E_1 = D_{12}^{\mathrm{T}} D_{12} \geqslant \beta I > 0$ *for all* x. *Suppose one can find a strictly positive proper smooth function* V *of* x, *such that* $V(x) > 0$ *for* $x \neq 0$, $V(0) = 0$ *and which* (*a*) *satisfies the state feedback Hamilton–Jacobi PDE* (*HJ PDE*)

$$\begin{aligned} &\nabla_x V(A - B_2 E_1^{-1} D_{12}^{\mathrm{T}} C_1) \\ &\quad + \frac{1}{2} \nabla_x V(\gamma^{-2} B_1 B_1^{\mathrm{T}} - B_2 E_1^{-1} B_2^{\mathrm{T}}) \nabla_x V^{\mathrm{T}} \\ &\quad + \frac{1}{2} C_1^{\mathrm{T}} (I - D_{12} E_1^{-1} D_{12}^{\mathrm{T}}) C_1 = 0. \end{aligned} \tag{6}$$

and (*b*) *makes the vector field*

$$A - B_2 E_1^{-1} D_{12}^{\mathrm{T}} C_1 + (\gamma^{-2} B_1 B_1^{\mathrm{T}} - B_2 E_1^{-1} B_2^{\mathrm{T}}) \nabla_x V^{\mathrm{T}}$$

asymptotically stable. Then the central controller for the state feedback problem, which guarantees γ-*dissipativity and internal stability, is defined as*: $K^\star(x) = -E_1(x)^{-1}[D_{12}(x)^{\mathrm{T}} C_1(x) + B_2(x)^{\mathrm{T}} \nabla_x V(x)^{\mathrm{T}}]$. *Furthermore, even if* (*b*) *is not fulfilled, the closed-loop satisfies the dissipation inequality for all* T, $x(0)$, *and* $\hat{w}(.)$:

$$V(x(t)) + \frac{1}{2} \int_0^T |z(t)|^2 \,\mathrm{d}t \leqslant \gamma^2 \frac{1}{2} \int_0^T |\hat{w}(t)|^2 \,\mathrm{d}t + V(x(0)).$$

Proof. See Helton and James (1999).

For some mixed sensitivity problems, such as (4), there exist some un-stabilisable states, so it is obvious that no stabilising solution for the HJ PDE exists. In order to get around this obstacle, we extend the concepts of comprehensive stability and essential stability (Mita et al., 1997) to nonlinear systems.

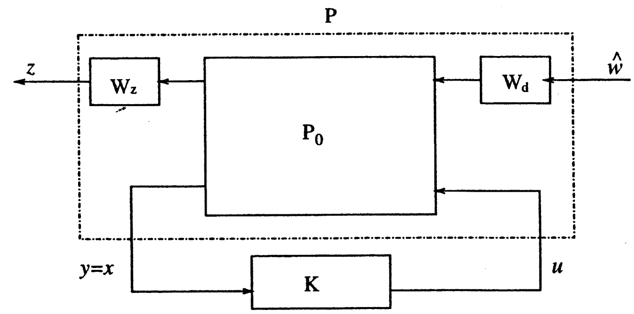

Fig. 4. The structure of mixed sensitivity problem.

Definition 2. The closed-loop system (P, K) in Fig. 4 is essentially stable if the interconnection of the physical plant P_0 and controller K is internally stable, or equivalently, if the only noninternally stable modes of (P, K) are those associated with the weighting.

The motivation is that the weighting is not present in any physical sense, while P_0 and K are physically present.

Definition 3. The closed-loop system (P, K) in Fig. 4 is said to be comprehensively stable if it is essentially stable, and the closed-loop from $\hat{w}$ to z is γ-dissipative. When this is the case, K is called a comprehensively stabilising controller.

Definition 4. The system (of Fig. 4) with input $\hat{w}$ and output z is said to be zero-detectable if the conditions that $\hat{w}(t) = 0$ and $z(t) = 0$ for all $t \geqslant 0$, are sufficient to imply that $\lim_{t \to \infty} x(t) = 0$.

We present a lemma as follows, which will be needed for the main stability theorem. It comes from a simple extension of La Salle's invariance principle (La Salle & Lefschetz, 1961).

Definition 5. Let $x = [x_w^{\mathrm{T}}\ x_s^{\mathrm{T}}]^{\mathrm{T}}$. Define Π as the projection from $\Re^{\dim(x)}$ to $\Re^{\dim(x_s)}$ in the obvious way by $\Pi([x_w^{\mathrm{T}}\ x_s^{\mathrm{T}}]^{\mathrm{T}}) = x_s$.

Lemma 6. *Let* $V(x)$ *be a scalar function with continuous partial derivatives. Let* B_r *be the set defined as* $\{x \colon V(x) < r\}$. *Assume that for a fixed but arbitrary* $r \in \Re$, ΠB_r *is a bounded set and that also within* B_r *the following conditions hold*

- $\dot{V}(x) \leqslant 0$,
- $V(x) > 0$ *for* $x_s \neq 0$ *and* $V(x) = 0$ *for* $x_s = 0$,
- *for every trajectory of* x *starting from* $x(0)$ *within* B_r, *there is a bound for* $x(t)$ (*which may possibly depend on* $x(0)$).

Let $\mathcal{N}$ be the set of all points within B_r where $\dot{V}(x) \equiv 0$ and let M be the largest invariant set within $\mathcal{N}$. Then for every possible $x(0)$ in B_r, as $t \to \infty$, $x(t) \to M$ and consequently every associated projection $x_s = \Pi(x)$ tends to $M_s = \Pi(M)$.

Proof. Since $\dot{V}(x) \leqslant 0$ then $V(x(t)) \leqslant V(x(0)) = v$ as $t \to \infty$ so that $x(t) \in B_v$ for $t \geqslant 0$. Since $V \geqslant 0$, it follows that $V(x(t))$ has a limit l as $t \to \infty$, where $l \leqslant v$. Let Γ be the (positive) limiting set of $x(t)$. Note that Γ is not empty due to the boundedness of the trajectories of $x(t)$ on B_r. By the continuity of $V(x)$ we conclude that $V(x_\Gamma) = l$ for all $x_\Gamma \in \Gamma$ and that therefore $\Gamma \subset B_l$ and $\dot{V}(x) \equiv 0$ on Γ. Since Γ is an invariant set it follows that $\Gamma \subset M$. Since $x(t)$ remains bounded within B_v, it follows that $x \to M$ as $t \to \infty$ and the theorem conclusion follows.

Theorem 7. *Consider the system defined by Eq.* (5), *and suppose that $\exists \alpha, \beta : \alpha I \geqslant E_1 = D_{12}^{\mathrm{T}} D_{12} \geqslant \beta I > 0$ for all x. Suppose also that $u = K(x)$ for some K such that $K(0) = 0$. Suppose that the state vector of P is of the form $[x_w^{\mathrm{T}}\ x_s^{\mathrm{T}}]^{\mathrm{T}}$, in which the components x_s are stabilisable from u and the components x_w are associated only with weights and are not necessarily stabilisable. Then the closed-loop system (P, K) will be comprehensively stable, given the following conditions are satisfied*:

- *There exists a storage function V, such that $V(x) > 0$ if $x_s \neq 0$ and $V(x) = 0$ if $x_s = 0$, which satisfies the dissipative inequality*:

$$\dot{V}(x) \leqslant \tfrac{1}{2}[\gamma^2 \|\hat{w}\|^2 - \|z\|^2]. \tag{7}$$

- *The states x_s are zero-detectable.*

Proof. From Eq. (7), we can calculate that the closed loop from $\hat{w}$ to z is γ-dissipative. Now we only need to prove that the state x_s is asymptotically stabilised. Because inequality (7) is satisfied for all $\hat{w}$, for the case when $\hat{w} = 0$; we have $\dot{V}(x) \leqslant -\|z\|^2$. Now we appeal to Lemma 6. The set of trajectories for which $\dot{V} \equiv 0$ is the set for which $z(t) \equiv 0$. By the theorem hypothesis, $\hat{w} \equiv 0$ and $z = 0$ imply $\lim_{t \to \infty} x_s(t) = 0$.

Theorem 8. *Consider the system described by Eq.* (4). *Suppose that the state vector of the plant P is of the form $[x_{w_{11}}\ x_s^{\mathrm{T}}]^{\mathrm{T}} = [x_{w_{11}}\ x_{w12}^{\mathrm{T}}\ x_{w_2}^{\mathrm{T}}\ x_0^{\mathrm{T}}]^{\mathrm{T}}$, where the sub-state x_s is zero-detectable. If there exists a function $\bar{V}(x)$, such that $\bar{V}(x) > 0$ if $x_s \neq 0$ and $\bar{V}(x) = 0$ if $x_s = 0$, which satisfies the following HJ PDE*:

$$\nabla_{x_s} \bar{V} \bar{A} + \tfrac{1}{2} \nabla_{x_s} \bar{V} (\gamma^{-2} \bar{B}_1 \bar{B}_1^{\mathrm{T}} - \bar{B}_2 \bar{E}_1^{-1} \bar{B}_2^{\mathrm{T}}) \nabla_{x_s} \bar{V}^{\mathrm{T}} = 0, \tag{8}$$

then system (4) *can be comprehensively stabilised by the central controller $K^\star(x_s)$, defined as*

$$K^\star(x_s) = -\bar{E}_1(x_s)^{-1}[\bar{D}_{12}(x_s)^{\mathrm{T}} C_1(x) + \bar{B}_2(x_s)^{\mathrm{T}} \nabla_{x_s} \bar{V}(x)^{\mathrm{T}}].$$

In the above equations the terms are given by

$$\bar{A} = \bar{A}(x_s) = \begin{bmatrix} A_{w_{12}} x_{w_{12}} \\ A_{w_2} x_{w_2} \\ A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} B_{w_{12}} & 0 \\ 0 & B_{w_2} \\ 0 & 0 \end{bmatrix},$$

$$\bar{B}_2 = \begin{bmatrix} 0 \\ 0 \\ B_1(x_0) \end{bmatrix}, \quad C_1(x) = x_{w_{11}}, \bar{D}_{12} = D_{12} \quad \textit{and}$$

$$\bar{E}_1 = \bar{D}_{12}^{\mathrm{T}} \bar{D}_{12}.$$

Proof. The proof is in Appendix A.

5. The structure of output feedback controller

In the linear case, the disturbance rejection output feedback controller necessarily includes an integrator. We now investigate the output feedback controller structure for the nonlinear case. We first define a notion of internal stability.

Definition 9. A closed-loop system is internally stable around all constant operating points if when subjected to inputs composed of the sum of a constant signal plus signal in L_2, all internal states $x(t)$ of the closed-loop system become the sums of constant signals plus a signals in L_2. Output stability is defined similarly.

Definition 10. A nonlinear system contains an integrator iff there exist some initial conditions for the state, and some input signal in L_2 which results in the output being the sum of a non-zero constant signal plus a signal in L_2.

Theorem 11. *Consider the constant disturbance suppression problem described by Eqs.* (4) *and* (4*). *Suppose that an output feedback $\mathcal{H}_\infty$ controller exists, such that the resultant closed loop is both internally stable and output stable around all constant operating points in the sense of Definition* 9. *Then assuming that the plant P_0 does not contain an integrator in the sense of Definition* 10, *then the controller must contain an integrator.*

Proof. Consider the case where $\hat{w}_1, \hat{w}_2 \in L_2$ and the input w_{11} is composed of the sum of a nonzero constant signal plus a signal in L_2, and $w_2 \in L_2$. Since there is an $\mathcal{H}_\infty$ controller, the signal z must obey the dissipation inequality with respect to $\hat{w}_1$ and $\hat{w}_2$, and hence $z \in L_2$. By the properties of w_{11} and z, it follows that the demanded controller output u can be described as a *nonzero* constant signal plus a signal in L_2 Since the closed-loop system is stable, it follows that both the plant states x_p and the observed outputs must be able to be described as constant signals plus signals in L_2. Because of our assumption that the nonlinear plant does not contain an integrator, the observed output y must be a *zero* constant signal plus a signal in L_2. We now observe that the controller K has an input $y \in L_2$ and an output u which is a *nonzero* constant signal plus a signal in L_2. By Definition 10, the controller contains an integrator.

6. Conclusion

This paper presents a modest extension of nonlinear H_∞ theory in order to solve the constant disturbance rejection problem. We have suggested a nonlinear extension of a concept introduced for the corresponding linear problem, that of the "*comprehensively stabilising*" controller, and have achieved an order reduced HJ PDE for the state feedback problem. Furthermore, we draw the conclusion that the output feedback controller normally must contain an integrator for constant disturbance suppression. This method improves our intuitive understanding of the linear problem.

Appendix A. Proof of Theorem 8

Let $u = \tilde{u} + \hat{u}$, and

$$\hat{u} = -E_1(x)^{-1} D_{12}(x)^{\mathrm{T}} C_1(x) = x_{w_{11}}. \tag{A.1}$$

Then, Eqs. (4) and (4*) together become

$$\begin{aligned}
\dot{x}_{w_{11}} &= \alpha \hat{w}_1,\\
\dot{x}_{w_{12}} &= A_{w_{12}} x_{w_{12}} + B_{w_{12}} \hat{w}_1,\\
\dot{x}_{w_2} &= A_{w_2} x_{w_2} + B_{w_2} \hat{w}_2,\\
\dot{x}_0 &= A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} + B_1(x_0)\tilde{u},\\
z &= \tilde{u},\\
y &= C_2(x_0) + C_{w_2} x_{w_2} + D_{w_2} \hat{w}_2.
\end{aligned} \tag{A.2}$$

For the system defined by Eq. (A.2), set

$$\hat{A}(x) = [0 \quad A_{w_{12}} x_{w_{12}} \quad A_{w_2} x_{w_2} \quad A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}}]^{\mathrm{T}},$$

$$\hat{B}_1(x) = \begin{bmatrix} \alpha & B_{w_{12}} & 0 & 0 \\ 0 & 0 & B_{w_2} & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\hat{B}_2(x) = [0 \quad 0 \quad 0 \quad B_1(x_0)]^{\mathrm{T}}, \quad \hat{C}_1(x) = 0,$$

$$\hat{D}_{12}(x) = 1, \quad \text{and} \quad \hat{E}_1 \triangleq \hat{D}_{12}^{\mathrm{T}} \hat{D}_{12}.$$

Then according to Theorem 1, the HJ PDE for the above system is

$$\begin{aligned}
&V_{x_{w_{12}}} A_{w_{12}} x_{w_{12}} + V_{x_{w_2}} A_{w_2} x_{w_2} + V_{x_0}(A(x_0) + B_1(x_0) C_{w_{12}} x_{w_2})\\
&+ \frac{1}{2}\gamma^{-2} \begin{bmatrix} V_{x_{w_{11}}} \\ V^{\mathrm{T}}_{x_{w_{12}}} \\ V^{\mathrm{T}}_{x_{w_2}} \\ V^{\mathrm{T}}_{x_0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \alpha^2 & \alpha B_{w_{12}} & 0 & 0 \\ \alpha B_{w_{12}} & B_{w_{12}} B^{\mathrm{T}}_{w_{12}} & 0 & 0 \\ 0 & 0 & B_{w_2} B^{\mathrm{T}}_{w_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\\
&\times \begin{bmatrix} V_{x_{w_{11}}} \\ V^{\mathrm{T}}_{x_{w_{12}}} \\ V^{\mathrm{T}}_{x_{w_2}} \\ V^{\mathrm{T}}_{x_0} \end{bmatrix}\\
&- \frac{1}{2} \begin{bmatrix} V_{x_{w_{11}}} \\ V^{\mathrm{T}}_{x_{w_{12}}} \\ V^{\mathrm{T}}_{x_{w_2}} \\ V^{\mathrm{T}}_{x_0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1(x_0) B_1^{\mathrm{T}}(x_0) \end{bmatrix} \begin{bmatrix} V_{x_{w_{11}}} \\ V^{\mathrm{T}}_{x_{w_{12}}} \\ V^{\mathrm{T}}_{x_{w_2}} \\ V^{\mathrm{T}}_{x_0} \end{bmatrix}\\
&= 0.
\end{aligned} \tag{A.3}$$

Now with V the solution of Eq. (8), we can verify that $V(x_{w_{11}}, x_{w_{12}}, x_{w_2}, x_0) = \bar{V}(x_{w_{12}}, x_{w_2}, x_0)$ satisfies (A.3). For with this identification, $\bar{V}_{x_{w_{11}}} = 0$, and Eq. (A.3) becomes

$$\begin{aligned}
&\begin{bmatrix} \bar{V}^{\mathrm{T}}_{x_{w_{12}}} \\ \bar{V}^{\mathrm{T}}_{x_{w_2}} \\ \bar{V}^{\mathrm{T}}_{x_0} \end{bmatrix} \begin{bmatrix} A_{w_{12}} x_{w_{12}} \\ A_{w_2} x_{w_2} \\ A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} \end{bmatrix}\\
&+ \frac{\gamma^{-2}}{2} [\bar{V}_{x_{w_{12}}} \ \bar{V}_{x_{w_2}} \ \bar{V}_{x_0}] \begin{bmatrix} B_{w_{12}} B^{\mathrm{T}}_{w_{12}} & 0 & 0 \\ 0 & B_{w_2} B^{\mathrm{T}}_{w_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{V}^{\mathrm{T}}_{x_{w_{12}}} \\ \bar{V}^{\mathrm{T}}_{x_{w_2}} \\ \bar{V}^{\mathrm{T}}_{x_0} \end{bmatrix}\\
&- \frac{1}{2} [\bar{V}_{x_{w_{12}}} \ \bar{V}_{x_{w_2}} \ \bar{V}_{x_0}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1(x_0) B_1^{\mathrm{T}}(x_0) \end{bmatrix} \begin{bmatrix} \bar{V}^{\mathrm{T}}_{x_{w_{12}}} \\ \bar{V}^{\mathrm{T}}_{x_{w_2}} \\ \bar{V}^{\mathrm{T}}_{x_0} \end{bmatrix}\\
&= 0.
\end{aligned} \tag{A.4}$$

This is identical to Eq. (8), which is true by hypothesis. Since the solution $\bar{V}(x)$ has the property that $\bar{V}(x) > 0$ if $x_s \neq 0$, $\bar{V}(x) = 0$ if $x_s = 0$, then from Theorem 1 we conclude that equation (A.3) has a solution that makes the closed-loop $(P, K^\star)$ γ-dissipative without necessarily satisfying the closed-loop asymptotic stability condition of Theorem 1. Because x_s is zero-detectable by hypothesis, then from the γ-dissipativity property that we have demonstrated immediately previously, and from application of Theorem 7, we conclude that the closed-loop (P, K) is comprehensively stabilised.

References

Byrnes, C. I., Priscoli, F. D., & Isidori, A. (1997). *Output regulation of uncertain nonlinear system.* USA: Birkhauser.

Green, M., & Limebeer, D. J. N. (1995). *Linear robust control.* Englewood Cliffs, NJ: Prentice-Hall.

Helton, J. W., & James, M. R. (1999). *Extending $\mathscr{H}_\infty$ control to nonlinear systems: control of nonlinear systems to achieve performance objectives.* Philadelphia, PA: SIAM.

Isidori, A. (1994). $\mathscr{H}_\infty$ control via measurement feedback for affine nonlinear systems. *International Journal of Robust and Nonlinear control, 4*, 553–574.

La Salle, J., & Lefschetz, S. (1961). *Stability by Liapunov's direct method with applications.* New York: Academic Press.

Mita, T., Hirata, M., Murata, K., & Zhang, H. (1998). $\mathscr{H}_\infty$ control versus disturbance-observer-based control. *IEEE Transactions on Industrial Electronics, 45*(3) 488–495.

Mita, T., Xin, X., & Anderson, B. D. O. (1997). Extended $\mathscr{H}_\infty$ control—solving $\mathscr{H}_\infty$ servo and estimation problems. *Proceedings of the 36th conference on decision and control* (pp. 4653–4658).

Van der Schaft, A. J. (1996). *L_2 Gain and passivity techniques in nonlinear control.* New York: Springer.

Zhou, K., Doyle, J. C., & Glover, K. (1996). *Robust and optimal control.* Englewood Cliffs, NJ: Prentice-Hall.