The Vinnicombe Metric for Nonlinear Operators

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***Abstract*—This paper describes an extension of the Vinnicombe metric on linear operators to a pseudometric on nonlinear operators. A metric for finite-dimensional time-varying operators is shown to be capable of guaranteeing stability and performance robustness and reduces to the standard Vinnicombe metric for the time-invariant operator case, which is known to be less conservative than the gap metric. The analysis exploits the time-varying operator equivalents of unstable poles and normalized coprime fractional descriptions. In addition, a time-varying operator equivalent of the winding number is defined.**

***Index Terms*—Gap metric, nonlinear operator metric, Nu-gap metric, time-varying winding number, Vinnicombe metric.**

I. Introduction

The study of metrics on operator space is both theoretically interesting and of practical importance. Such metrics define notions of distance between input-output maps and allows the evaluation of quantities such as modeling approximation error and behavioral robustness. Operator metrics that are relevant to control problems are of particular interest. It is well known that the gap metric on linear operators [11] induces the weakest topology such that feedback stability is a robust property. Since the gap metric was introduced into the control literature as early as [7] it has been studied extensively. The paper [11] develops the properties of the gap metric for linear systems in detail and a procedure for calculating this metric appears in [8]. In [10] the metric is extended to time-varying linear plants and in [12] an extension of the gap metric to a pseudometric on nonlinear operators is given, based on earlier work on the parallel projection operator for nonlinear systems [6] and its relationship to the differential stabilizability of nonlinear feedback systems [9].

Building on work on the gap metric, a related metric on linear operators, the Vinnicombe (or ν-gap) metric [24], which actually induces the same topology as the gap metric, was developed. The Vinnicombe metric is less conservative than the gap metric in the sense that if a proximity in the Vinnicombe metric is *unable to guarantee* that a perturbed plant will be stabilized by a controller which results in a certain minimum performance level for a given original plant, then there exists some controller which achieves that minimum performance level on the original plant and some plant which achieves the Vinnicombe proximity condition, which is also *destabilized* by that controller (see [25, Ch. 4]). Such lack of conservatism is not a property which is enjoyed by the gap metric.

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Motivated by the superior control relevant characteristics of the Vinnicombe metric to the gap metric in the case of linear operators and springboarding from techniques presented in the extension of the gap metric to a nonlinear operator framework in [6], [9], [10], [12] we are motivated to seek a nonlinear extension of the Vinnicombe metric.

The majority of this paper is concerned with the development of a time-domain extension of the Vinnicombe metric to linear finite-dimensional time-varying systems. In Section IV, this metric is shown to have similar robustness properties to the linear time-invariant Vinnicombe metric. In the spirit of [9], [12], a metric on nonlinear operator space may then be defined by considering all linearizations of the nonlinear operators in question about their input-output trajectories. It remains to be ascertained whether the time-varying and the nonlinear metrics are also similarly nonconservative for the purposes of stability robustness analysis.

The rest of the paper is organized in the following manner. We first revisit the concept of differential stability for nonlinear systems [9] and see that linearizations of the nonlinear operator play an important role in stability analysis. We then extend the concepts of stable and unstable poles and bounded gain for linear finite-dimensional time-varying plants. This allows the so-called "bounded real lemma" [3] to be extended into the time-varying context. We exploit the existence of normalized coprime descriptions of time-varying plants [22] to establish a time-varying analogue of "winding number", which is crucial to the definition of the Vinnicombe metric and closely related to the Fredholm index for linear time-invariant plants [24]. This allows the development of a time-varying analogue of the Vinnicombe metric, with time-domain interpretation, which is then shown to have desirable properties in terms of analysis of the robustness of both stability and performance. Finally, a function on nonlinear operators is defined in terms of the time-varying Vinnicombe metric of linearizations of the nonlinear operators. This function is shown to be a pseudometric as it inherits the metric properties of the time-varying Vinnicombe metric function from which it is constructed.

A. Notation

The symbol $\otimes$ allows us to represent a product space of two sets A and B as $A \otimes B$. In the case where A and B are both vector spaces and the product space can be interpreted as a direct sum, we write this direct sum as $A \oplus B$. The notation X^* denotes the adjoint of an operator X. If M is a matrix, then M^* denotes the matrix adjoint, that is, its conjugate transpose.

The symbol δ_ν is used to denote the standard Vinnicombe metric [24] on linear time-invariant operators. In this paper, we also define a metric on linear *time-varying* operators, which we also denote δ_ν. Sometimes, in order to emphasise the fact that

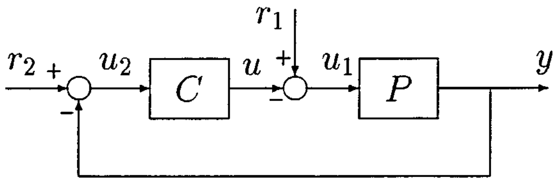

Fig. 1. Closed-loop system (P, C).

this metric is defined on linear time-varying operators, we use the notation $\delta_\nu^{\rm ltv}$. Finally, in Section V, we define a metric on nonlinear operators, which is denoted $\delta_\nu^{\rm nl}$.

II. Differential Stability of Nonlinear Systems

A. System and Stability Definition

In this section, we review key results of [9] and [6] which relate properties of input-output operators to the operator graphs. Some fundamental results on the graphs of nonlinear operators and coprime fractional representation, also appear in early papers such as [15] and [16]. We consider the standard feedback configuration of Fig. 1 and draw attention to the case when the input of the plant P is $\mathcal{U} \stackrel{\rm def}{:=} \mathcal{L}_2^m[t_0, \infty)$ and the output space is $\mathcal{Y} \stackrel{\rm def}{:=} \mathcal{L}_2^p[t_0, \infty)$. The direct sum of the input and output spaces is $\mathcal{W} \stackrel{\rm def}{:=} \mathcal{U} \oplus \mathcal{Y} = \mathcal{L}_2^{m+p}[t_0, \infty)$. Note that the choice $t_0 = -\infty$ is permitted.

We think of P as a possibly unbounded operator

$$P : \mathcal{D}_p \to \mathcal{Y} \tag{1}$$

with domain $\mathcal{D}_p \subset \mathcal{U}$. We assume that $0 \in \mathcal{D}_p$ with

$$P0 = 0 \tag{2}$$

and that P is causal. Formally, if T_τ denotes the truncation operator, that is for $f(t), t_0 \le t < \infty$ there holds $T_\tau f(t) = f(t)$ for $t \in [t_0, \tau]$ and $T_\tau f(t) = 0$ for $t > \tau$, then

$$T_\tau P T_\tau = T_\tau P.$$

The graph of P is defined in the usual way as

$$\mathcal{G}_p \stackrel{\rm def}{:=} \begin{bmatrix} I_m \\ P \end{bmatrix} \mathcal{D}_p \subset \mathcal{W} \tag{3}$$

and we capture a smoothness property for P by the assumption that $\mathcal{G}_p$ is a closed differentiable manifold with no boundary. (Since the graph has no boundary, it is both open and closed in the product space $\mathcal{W}$, so that both every u, y pair is surrounded by an open ball within the graph, *and* the graph contains all its limit points.) The class of all systems satisfying (1) to (3) is denoted by $\mathcal{P}_{\mathcal{U},\mathcal{Y}}$ and the class satisfying the differentiability property is denoted by $\mathcal{P}_{{\rm diff}\ \mathcal{U},\mathcal{Y}}$.

Definition II.1: A system $P = \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ is defined as respectively stable, finite-gain (fg) stable and incrementally finite gain (ifg) stable accordingly as

$$\mathcal{D}_p = \mathcal{U}$$
$$\|P\| \stackrel{\rm def}{:=} \sup_{\substack{x \in \mathcal{U} \\ x \neq 0}} \frac{\|Px\|}{\|x\|} < \infty$$

and $$\|P\|_\Delta \stackrel{\rm def}{:=} \sup_{\substack{x_1, x_2 \in \mathcal{U} \\ x_1 \neq x_2}} \frac{\|Px_1 - Px_2\|}{\|x_1 - x_2\|} < \infty.$$

Moreover $P$is termed respectively causally finite gain stable or causally incrementally finite gain stable if $P$is stable and there exists some M, independent of τ, such that, for all τ,

$$\|P\|_\tau \stackrel{\rm def}{:=} \sup_{\substack{x \in \mathcal{U} \\ x \neq 0}} \frac{\|T_\tau P x\|}{\|T_\tau x\|} \le M < \infty$$
$$\|P\|_{\Delta\tau} \stackrel{\rm def}{:=} \sup_{\substack{x_1, x_2 \in \mathcal{U} \\ T_\tau x_1 \neq T_\tau x_2}} \frac{\|T_\tau P x_1 - T_\tau P x_2\|}{\|T_\tau x_1 - T_\tau x_2\|} \le M < \infty.$$

When $P \in \mathcal{P}_{{\rm diff}\ \mathcal{U},\mathcal{Y}}$, the derivative of P is denoted dP. The derivative operation on $\mathcal{P}$ maps $\mathcal{U}$ into the space of continuous *linear* operators dP from $\mathcal{U}$ to $\mathcal{Y}$. Evaluating the derivative at a vector $w = [\,u^T \quad y^T\,]^T \in \mathcal{W}$ yields the linear operator dP, which is often referred to as the linearization of P around the trajectory w. If P is described by a nonlinear (affine in the control) finite dimensional equation

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

with $f(0) = 0$, $h(0) = 0$ and f, g smooth functions, then $d_u P$ is given by

$$\dot{\tilde{x}} = F(t)\tilde{x} + G(t)\tilde{u}$$
$$\tilde{y} = H(t)\tilde{x}$$
$$\text{where } F(t) = \frac{\partial}{\partial x} f\bigg|_{x(t)} + \frac{\partial}{\partial x} g\bigg|_{x(t)} u(t)$$
$$G(t) = g[x(t)]$$
$$\text{and } H(t) = \frac{\partial}{\partial x} h\bigg|_{x(t)}.$$

We shall say that P is differentially stable (causally differentially stable) if it is both ifg stable (causally ifg stable) and differentiable [9].

Consider now the standard feedback arrangement of Fig. 1 and define the nonlinear mapping

$$F_{P,C} : \mathcal{D}_P \oplus \mathcal{D}_C \to \mathcal{W}$$

by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} I_m & C \\ P & I_p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{4}$$

We do not treat the question of loop well-posedness in this paper. Generally, to secure this, it is assumed that at least one of P and C is strictly causal (has zero instantaneous gain). The topic is well-treated in [20] and [23].

Of more interest than $F_{P,C}$ is the mapping $F_{P,C}^{-1}$, which constructs a response from external inputs

$$F_{P,C}^{-1} : \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \to \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5}$$

We say that the interconnection $[P, C]$ in Fig. 1 is stable if $F_{P,C}^{-1}$ exists and is stable. Similarly, we can talk about causal incremental finite gain stability and so on.

An important result of [9] is the following.

Proposition II.1: Consider the standard feedback system of Fig. 1, with $P \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ and $C \in \mathcal{P}_{\mathcal{Y},\mathcal{U}}$. If $[P, C]$ is differentially stable, then it follows that $P \in \mathcal{P}_{\text{diff}\,\mathcal{U},\mathcal{Y}}$ and that $C \in \mathcal{P}_{\text{diff}\,\mathcal{Y},\mathcal{U}}$.

Thus, to have any chance of a smooth closed loop, the individual components, P and C, must themselves be smooth.

B. Nonlinear Stability

The paper, [6], sets out an important way of looking at closed-loop stability, based on linking stability to two other ideas. These are

- the existence of a decomposition of $\mathcal{W}$ into a sum of two manifolds $\mathcal{M}$ and $\mathcal{N}$ and
- the existence of a related parallel projection operator.

For our purposes, we shall need only the first idea.

Let $\mathcal{M}$ and $\mathcal{N}$ be two nonlinear submanifolds [18] of $\mathcal{W} = \mathcal{U} \oplus \mathcal{Y}$, such that each includes the origin. The pair $\mathcal{M}$, $\mathcal{N}$ is said to induce a coordinatization of $\mathcal{W}$ if the following two conditions hold: first

$$\mathcal{W} = \mathcal{M} \otimes \mathcal{N}$$

and second, given any $m + n = \bar{m} + \bar{n}$ with $m, \bar{m} \in \mathcal{M}$ and $n, \bar{n} \in \mathcal{N}$, it follows that $m = \bar{m}$ and $n = \bar{n}$. Recall that the graph of the "plant" $\mathcal{P}$ is defined as

$$\mathcal{G}_p = \begin{bmatrix} I_m \\ P \end{bmatrix} \mathcal{D}_p \subset \mathcal{U} \oplus \mathcal{Y} = \mathcal{W}$$

and define the graph of a "controller" operator $\mathcal{C}$ as

$$\bar{\mathcal{G}}_C = \begin{bmatrix} C \\ I_p \end{bmatrix} \mathcal{D}_C \subset \mathcal{U} \oplus \mathcal{Y} = \mathcal{W} \tag{6}$$

A key result of [6] is the following.

Proposition II.2: Consider the standard feedback system of Fig. 1 with $P \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ and C satisfying $C : \mathcal{D}_C \to \mathcal{U}$, $C0 = 0$ and (6). Then the interconnection is stable if and only if $\mathcal{G}_P$, $\bar{\mathcal{G}}_C$ induce a coordinatization on $\mathcal{W} = \mathcal{U} \oplus \mathcal{Y}$.

The idea behind the proposition can be gleaned from (4) rewritten as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} I_m \\ P \end{bmatrix} u_1 + \begin{bmatrix} C \\ I_p \end{bmatrix} u_2$$

For a stable system, any r_1, r_2 pair gives rise to a unique u_1, u_2 pair and thus a unique pair $[\,(I_m u_1)^T \quad (Pu_1)^T\,]^T \in \mathcal{G}_p$ and $[\,(Cu_2)^T \quad (I_p u_2)^T\,]^T \in \bar{\mathcal{G}}_C$, summing to $[\,r_1^T \quad r_2^T\,]^T$.

C. Differential Stability Characterization

The coordinatization $\mathcal{M}, \mathcal{N}$ is called differentiable if the associated coordinate map $\mathcal{F} : \mathcal{W} \to \mathcal{M} \otimes \mathcal{N}$ is differentiable. In terms of the remarks at the end of the last section, if the map from $[\,r_1^T \quad r_2^T\,]^T$ to u_1 is differentiable and P is differentiable, then $\mathcal{M}$ (and thus $\mathcal{N}$) is differentiable. Given $m \in \mathcal{M}$, $n \in \mathcal{N}$, it is possible to form the tangent space [9] $T_m(\mathcal{M})$ at the point m and likewise $T_n(\mathcal{N})$ at the point n. Then we can state the following proposition ([9, Corollary 1]).

Proposition II.3: Suppose that $P \in \mathcal{P}_{\text{diff}\,\mathcal{U},\mathcal{Y}}$, $C \in \mathcal{C}_{\text{diff}\,\mathcal{Y},\mathcal{U}}$, $\mathcal{M} = \mathcal{G}_p$ and $\mathcal{N} = \bar{\mathcal{G}}_c$. Let $M_m \overset{\text{def}}{:=} T_m(\mathcal{M})$ and $N_n \overset{\text{def}}{:=} T_n(\mathcal{N})$. If $[P, C]$ is differentiably stable and if P_m is a system with $\mathcal{G}_{P_m} = M_m$ and C_n is a system with $\bar{\mathcal{G}}_{C_n} = N_n$, then $[P_m, C_n]$ is finite gain stable with a uniform bound on the input-to-error norm, for all m, n.

Conversely, if for all $m, n \in \mathcal{M} \otimes \mathcal{N}$ we have that $[P_m, C_n]$ is finite gain stable with a uniform bound on the input-to-error norm, then $[P, C]$ is differentiably stable.

Roughly speaking, the nonlinear system is differentially stable if and only if all linearizations give (linear) stable closed loops (with uniformly bounded gain). A further extension is the following.

Corollary II.1: Adopt the same hypotheses as in Proposition II.3. If $[P, C]$ is differentially stable, then $[P_m, C]$ is differentially stable and $[P, C_n]$ is differentially stable, for all m, n.

The ultimate goal of this paper is to study the question of robust stability, or the retention of stability when a plant, or controller, is varied. The key to doing this will be to find conditions for robust stability of linear-time-varying systems. We aim to build up the argument with the following chain:

- $[P_\alpha, C]$ is differentially stable, …
- $\iff$ $[P_{\alpha x}, C_y]$ is finite gain stable with uniformly bounded gain (using Proposition II.3 of this section), …
- $\iff$ $[P_{\beta x}, C_y]$ is finite gain stable with uniformly bounded gain (using a linear-time-varying robust stability theorem presented subsequently and knowledge of the closeness of P_{1x} and P_{2x} in some metric, …
- $\iff$ $[P_\beta, C]$ is differentially stable (using Proposition II.3 again).

III. The Vinnicombe Metric for Linear Time-Varying Finite-Dimensional Systems

In this section, we define a metric on operator space, where the operators in question are linear finite-dimensional systems, which are possibly time-varying. This will be achieved by firstly, giving the Vinnicombe metric [24] for linear time-invariant systems a time-domain interpretation and then extending that metric to time-varying systems. Before this, however, we quickly review the definition and properties of the Vinnicombe metric for time-*invariant* systems.

A. Vinnicombe Metric for Time-Invariant Systems

Let $P_1(j\omega)$, $P_2(j\omega)$ be two rational transfer functions matrices of the same dimension. The Vinnicombe metric is a measure of the distance between P_1 and P_2. It can be defined as follows [24], [25]:

$$\delta_\nu(P_1, P_2) = \left\| (I + P_2 P_2^*)^{-1/2} (P_2 - P_1) (I + P_1^* P_1)^{-1/2} \right\|_\infty$$

provided the following two conditions are satisfied:

$$\det (I + P_1 P_2^*)(j\omega) \neq 0 \qquad \forall \omega \tag{7}$$

$$\text{and wno}\,[\det (I + P_1 P_2^*)] + \eta(P_1) - \bar{\eta}(P_2) = 0. \tag{8}$$

If the conditions of (7) and (8) are not both satisfied then $\delta_\nu(P_1, P_2) = 1$. In the above, $\eta(P_i)$ denotes the number of poles of P_i in the *open* right half complex plane $\text{Re}[s] > 0$ and $\bar{\eta}(P_j)$ is the number of poles of P_j in the *closed* right half plane $\text{Re}[s] \geq 0$, counted according to multiplicity. The symbol wno denotes the winding number evaluated on the standard Nyquist contour indented into the right half plane around any imaginary axis poles of P_1 and P_2.

There are various equivalent expressions. We draw attention here to some. Let $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i$ denote normalized right and left coprime fractional descriptions of P_i [27]. Define

$$G_i(s) = \begin{bmatrix} N_i(s) \\ M_i(s) \end{bmatrix}$$
$$\tilde{G}_i(s) = [\tilde{M}_i(s) \quad -\tilde{N}_i(s)].$$

Then $\delta_\nu(P_1, P_2) = \|\tilde{G}_2 G_1\|_\infty = \|\tilde{G}_1 G_2\|_\infty$ as long as $\det(G_2^* G_1(j\omega)) \neq 0$ for all ω and $\text{wno}[\det(G_2^* G_1(j\omega))] = 0$. Otherwise $\delta_\nu(P_1, P_2) = 1$. Also, for $\hat{G}_2 = [\hat{N}_2^T \quad \hat{M}_2^T]^T$ where $P_2 = \hat{N}_2 \hat{M}_2^{-1}$ is a right coprime fractional description of P_2, which is not necesarily normalized, we have

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathcal{RL}_\infty \\ \text{wno}[\det(Q)] = 0}} \left\| G_1 - \hat{G}_2 Q \right\|_\infty.$$

Of course, it is a nontrivial fact that the metric properties hold for $\delta_\nu(P_1, P_2)$, although the triangle inequality is the only property which requires much attention to prove.

B. Lyapunov Exponents, Dichotomies, and Linear Operators With Bounded Gain

This section defines some concepts which are the time-varying analogue of unstable poles. The Lyapunov exponents or order numbers, generalize the concept of poles to linear time-varying systems.

Definition III.1 (See also [14, Ch. 5]): Let $x_i(t)$ be a solution of the homogeneous equation $\dot{x} = A(t)x$, where $A(t)$ is continuous and bounded and has dimension $n \times n$. The order number of $x_i(t)$ is given by

$$\pi(x_i) = \limsup_{t \to \infty} \frac{\ln \|x_i(t)\|}{t}.$$

Every solution of $\dot{x} = A(t)x$ has a well-defined order number which is finite (except for the trivial solution $x(t) = 0$). If $\{x_1, x_2, \ldots, x_n\}$ is a fundamental system, that is, a set of linear independent solutions, it is called normal when $\sum \pi(x_i)$ is minimized. The associated order numbers, which are unique, are called the order numbers, or the Lyapunov exponents, of the differential equation.

We also, nonrigourously, say that a time-varying matrix $A(t)$ has a exponential dichotomy if the associated differential equation $\dot{x} = A(t)x$ possesses this property. The concept of an exponential dichotomy is a time-varying generalization of the condition that a system posesses no poles on the imaginary axis or at infinity.

Definition III.2: (See [5]). The homgeneous differential equation $\dot{x} = A(t)x$, with fundamental matrix $X(t)$, (that is, any solution of the matrix equation $\dot{X} = AX$ that is square and nonsingular) is said to possess an exponential dichotomy if there exists a projection matrix P (that is $P^2 = P$, P with rank, say, n_-) and positive constants K, L, α and β such that

$$\|X(t)PX^{-1}(s)\| \leq K \exp[-\alpha(t-s)] \text{ for } t \geq s$$
$$\|X(t)[I-P]X^{-1}(s)\| \leq L \exp[-\beta(s-t)] \text{ for } s \geq t.$$

Equivalently, for all ξ and for some positive constants $\bar{K}$, $\bar{L}$, $\bar{M}$,

$$\|X(t)P\xi\| \leq \bar{K} e^{[-\alpha(t-s)]} \|X(s)P\xi\| \text{ for } t \geq s$$
$$\|X(t)[I-P]\xi\| \leq \bar{L} e^{[-\beta(s-t)]} \|X(s)[I-P]\xi\| \text{ for } s \geq t$$
$$\|X(t)PX^{-1}(t)\| \leq \bar{M}.$$

These inequalities display the property that there are n_- independent solutions which are exponentially decaying (n_- negative order numbers) and $n_+ = n - n_-$ independent solutions which have exponential growth (n_+ positive order numbers); and also no solution which neither decays nor grows exponentially. Furthermore, the angle between the subspaces [5] of decaying and growing solutions remains bounded away from zero. The following proposition is the time-varying analogue of the fact that a time-invariant system with no poles on the imaginary axis or at infinity, has a bounded gain.

Proposition III.1: (See [5] and [21]). Suppose that $A(t)$ is an $n \times n$ bounded, continuous function of time. The nonhomogeneous equation $\dot{x} = A(t)x + f(t)$ has at least one bounded solution for every piecewise continuous $f(t)$, if and only if $\dot{x} = A(t)x$ has an exponential dichotomy. Furthermore, if $f(t) \in \mathcal{L}_2$ there exists a solution $x(t) \in \mathcal{L}_2$.

We remark that with $t_0 = -\infty$, the equation $\dot{x} = A(t)x$ has an exponential dichotomy if and only if $\dot{x} = -A^*(t)x$ has an exponential dichotomy. Indeed the latter equation will have n_- postive Lyapunov exponents and n_+ negative Lyapunov exponents. Note that the question of what are the initial conditions (or even what is the initial time) is unaddressed. Also, the proposition is silent on the question of uniqueness of the solution $x(\cdot)$, however, uniqueness of $x(\cdot) \in \mathcal{L}_2$ is fairly easy to prove. The same is true in the next proposition, which is known for the case in which $\dot{x} = A(t)x$ is exponentially stable [1]. The next proposition is a strengthening of Proposition III.1 to connect a bounded-input, bounded-output system property to an exponential dichotomy property.

Proposition III.2: Consider the system

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x \tag{9}$$

with bounded continuous A, B, C and with $[A, B]$ and $[A, C]$ respectively uniformly completely controllable and uniformly completely observable. Then for a continuous, bounded u, there exists at least one bounded piecewise continuous y satisfying the equations if and only if $\dot{x} = A(t)x$ has an exponential dichotomy.

Proof: If $\dot{x} = A(t)x$ has an exponential dichotomy, then application of Proposition III.1 immediately yields the property that any piecewise continuous bounded u, yields a piecewise continuous y. It remains to show the converse.

Let $\Phi(\cdot,\cdot)$ be the transition matrix [19] associated with the differential equation $\dot{x} = A(t)x$, that is $d/dt\Phi(t,s) = A(t)\Phi(t,s)$, $\Phi(t,t) = I$. In the interval $t \in [s, s+\delta]$ with δ

chosen so that $\int_s^{s+\delta}\Phi(t,s)^*C^*(t)C(t)\Phi(t,s)dt$ is uniformly nonsingular, there holds

$$y(t)=C(t)\Phi(t,s)x(s)+\int_s^t C(t)\Phi(t,s)B(s)u(s)ds$$

and

$$x(s)=\left[\int_s^{s+\delta}\Phi(t,s)^*C^*(t)C(t)\Phi(t,s)dt\right]^{-1} \times\left\{\int_s^{s+\delta}\Phi(t,s)^*C^*(t) \cdot\left[y(t)-\int_s^t C(t)\Phi(t,s)B(s)u(s)ds\right]dt\right\}.$$

It follows that when $[u(\cdot),y(\cdot)]$ is a bounded pair, then so is $[u(\cdot),x(\cdot)]$. That is, for every piecewise continuous bounded $u(\cdot)$, the equation $\dot{x}=A(t)x+B(t)u$ has a piecewise continuous (which is, in fact, continuous) solution $x(\cdot)$. Now let δ be such that $\int_s^{s+\delta}\Phi(s+\delta,t)B(t)B(t)^*\Phi(s+\delta,t)^*dt$ is uniformly nonsingular. Consider $u(t)$ defined by

$$u(t)=B^*(t)\Phi^*([n+1]\delta,t) \times\left[\int_{n\delta}^{\delta}\Phi([n+1]\delta,t)B(t)B(t)^*\Phi([n+1]\delta,t)^*dt\right]^{-1}u_n$$

for some n-vector u_n during the times t, when $n\delta\le t\le[n+1]\delta$. Set $A_n=\Phi([n+1]\delta,n\delta)$ and $x_n=x(n\delta)$. Then the differential (9) implies that

$$x_{n+1}=A_nx_n+u_n.$$

If the sequence u_n is an arbitrary bounded sequence, then $u(t)$ is piecewise continuous and bounded. We have shown that there exists a bounded $x(t)$ and therefore a bounded sequence x_n. It follows that the discrete time difference equation, $x_{n+1}=A_nx_n$ has an exponential dichotomy. From this and the boundedness and continuity of $A(t)$ it is easy to establish that the differential equation $\dot{x}=A(t)x$ also has an exponential dichotomy. ■

Remark III.1: A mild relaxation of the requirement that A,B,C are all bounded and continuous is almost certainly possible. Such a relaxation is achieved in [1] for the case where $\dot{x}=A(t)x$ is exponentially stable.

In the light of the previous proposition, we will sometimes say (nonrigourously), that an operator mapping the signal u to y has an exponential dichotomy if it can be written as a time-varying linear differential equation of the form in (9), where the equation $\dot{x}=A(t)x$ has an exponential dichotomy.

We shall also need the following two results of [5, Chs. 5 and 7]. The first result demonstrates that if a linear differential equation has an exponential dichotomy, then it can be split into a strictly stable and a strictly unstable part. The second proposition is a time-varying analogue of the Lyapunov Lemma [13] and shows the existence of a bounded solution to a Lyapunov-like differential inequality corresponding to differential equations with an exponential dichotomy.

Proposition III.3: Let $\dot{x}=A(t)x$ with $A(t)$ bounded and continuous, have an exponential dichotomy. Then there exists a Lyapunov transformation $\hat{x}=T(t)x$, that is, one where T,T^{-1} and $\dot{T}$ are bounded and continuous, such that the equation

$$\dot{\hat{x}}=(TAT^{-1}+\dot{T}T^{-1})\hat{x}$$

takes the form

$$\dot{\hat{x}}=\begin{bmatrix}A_1&0\\0&A_2\end{bmatrix}\hat{x}$$

with, for some positive K, L, α, and β,

$$\begin{aligned}\hat{x}_1(t)\le&K\exp[-\alpha(t-s)]\hat{x}_1(s),\forall t\ge s\\ \hat{x}_2(t)\le&L\exp[-\beta(s-t)]\hat{x}_2(s),\forall s\ge t.\end{aligned}$$

We will require a special case of Proposition III.3 showing that triangular decomposition is almost as informative as diagonal decomposition.

Corollary III.1: Consider the equation

$$\dot{x}=\begin{bmatrix}A_\alpha&B\\0&A_\beta\end{bmatrix}x \tag{10}$$

where bounded and continuous A_α and A_β separately have exponential dichotomies. Then this equation has an exponential dichotomy and the number of positive Lyapunov exponents is the sum of the same for A_α and A_β.

Proof: Without loss of generality, by Proposition III.3, we may assume that each of the A_α and A_β is block diagonal with

$$\begin{bmatrix}A_\alpha&B\\0&A_\beta\end{bmatrix}=\begin{bmatrix}A_{\alpha-}&0&B_{--}&B_{-+}\\0&A_{\alpha+}&B_{+-}&B_{++}\\0&0&A_{\beta-}&0\\0&0&0&A_{\beta+}\end{bmatrix} \tag{11}$$

with $A_{\alpha-}$, $A_{\beta-}$ having negative exponents and $A_{\alpha+}$, $A_{\beta+}$ having positive ones. It is clear that the Lyapunov exponents remain the same if we simply reorder the entries. Thus, the following matrix has the same Lyapunov exponents:

$$\hat{A}=\begin{bmatrix}A_{\alpha-}&B_{--}&0&B_{-+}\\0&A_{\beta-}&B_{+-}&0\\0&0&A_{\alpha+}&B_{++}\\0&0&0&A_{\beta+}\end{bmatrix} =\begin{bmatrix}\hat{A}_-&\hat{B}\\0&\hat{A}_+\end{bmatrix}. \tag{12}$$

The Lyapunov exponents of $\hat{A}_-$ are negative and those of $\hat{A}_+$ are positive. Define the time-varying matrix $X(t)$ by the Lyapunov differential equation

$$\dot{X}=\hat{A}_-X-X\hat{A}_+-\hat{B},X(-\infty)=0$$
$$\text{or } X=-\int_{-\infty}^t\hat{\Phi}_-(t,s)\hat{B}(s)\hat{\Phi}_+(s,t)ds$$
$$\text{and define } T=\begin{bmatrix}I&X\\0&I\end{bmatrix}. \tag{13}$$

Observe that the Lyapunov transformation based on T produces

$$T\hat{A}T^{-1} + \dot{T}T^{-1} = \begin{bmatrix} \hat{A}_- & 0 \\ 0 & \hat{A}_+ \end{bmatrix}. \tag{14}$$

The result is now obvious. ■

The second result from [5], is the following.

Proposition III.4: The equation $\dot{x} = A(t)x$ with $A(t)$ continuous and bounded, has an exponential dichotomy if and only if there exists a bounded, continuously differentiable symmetric $H(t)$ satisfying

$$\dot{H} + HA + A^*H \leq -\epsilon I$$

for some positive ϵ.

There is an important corollary to Proposition III.4 which links the signs of the Lyapunov exponents of A to the inertia of H.

Corollary III.2: Suppose that $\dot{x} = A(t)x$ with $A(t)$ continuous and bounded, has an exponential dichotomy. Let H be as defined in the statement of the Proposition III.4. Then if $A(t)$ has n_+ positive and n_- negative Lyapunov exponents, H has n_- positive and n_+ negative eigenvalues. The converse is also true.

Proof: First observe that if T is a Lyapunov transformation and $\hat{A} = TAT^{-1} + \dot{T}T^{-1}$, $\hat{H} = T^{-T}HT^{-1}$, then the inequality of Proposition III.4 implies and is implied by

$$\dot{\hat{H}} + \hat{H}\hat{A} + \hat{A}^*\hat{H} \leq -\hat{\epsilon}I$$

for some positive $\hat{\epsilon}$. The Lyapunov exponents of $\hat{A}$ are the same as those of A and the inertia of $\hat{H}$ is the same as that of H. Now suppose that

$$\hat{A}(t) = \begin{bmatrix} \hat{A}_1(t) & 0 \\ 0 & \hat{A}_2(t) \end{bmatrix}$$

with $\hat{A}_1(t)$, $\hat{A}_2(t)$ as in Proposition III.3. Then

$$\dot{\hat{H}}_{11} + \hat{H}_{11}\hat{A}_1 + \hat{A}_1^*\hat{H}_{11} \leq -\hat{\epsilon}I$$

and it is not hard to check that

$$\hat{H}_{11}(t) \geq \hat{\epsilon}\int_t^{\infty} \Phi_1^*(\sigma,t)\Phi_1(\sigma,t)d\sigma$$

where $\Phi_1(\sigma,t)$ is the transition matrix of $\dot{x}_1 = A_1(t)x_1$. Also

$$\dot{\hat{H}}_{22} + \hat{H}_{22}\hat{A}_2 + \hat{A}_2^*\hat{H}_{22}, \leq -\bar{\epsilon}I$$

and one can check that

$$\hat{H}_{22}(t) \leq -\bar{\epsilon}\int_{-\infty}^{t} \Phi_2^*(\sigma,t)\Phi_2(\sigma,t)dt.$$

Hence, $\hat{H}_{11} > 0$ and $\hat{H}_{22} < 0$. The intertia of $\hat{H}$ is the same as that of

$$\begin{bmatrix} \hat{H}_{11} & 0 \\ 0 & \hat{H}_{22} - \hat{H}_{12}^*\hat{H}_{11}^{-1}\hat{H}_{12} \end{bmatrix}$$

that is, there are $\dim \hat{H}_{11}$ and $\dim \hat{H}_{22}$ positive and negative eigenvalues of $\hat{H}$ respectively. This completes the proof. ■

The following proposition, a key result of this paper, is a time-varying analog of the bounded real lemma [3], [13]. Actually, in the sense that the bounded real lemma is confined to treating exponentially stable systems, the result is also an extension, through replacing exponential stability by exponential dichotomy.

Proposition III.5: Consider the system $\dot{x} = A(t)x + B(t)u, y = C(t)x$ where $A(t), B(t), C(t)$ are continuous and bounded.Suppose that $\dot{x} = A(t)x$ has an exponential dichotomy. The bounded operator $\mathcal{H} : u \to y$, existing by Proposition III.4, has induced $\mathcal{L}_2$ gain less than unity if and only if there exists a bounded symmetric $P(t)$ satisfying, for suitably small positive ϵ_1 and ϵ_2, the equation

$$-\dot{P} = PA + A^*P - P(BB^* + \epsilon_1 I)P - C^*C - \epsilon_2 I. \tag{15}$$

Moreover, if $A(t)$ has n_+ positive and n_- negative Lyapunov exponents, it follows that $P(t)$ has n_+ positive and n_- negative eigenvalues.

Proof: Suppose first that the equation for P holds. Suppose also that $u \in \mathcal{L}_2(-\infty,\infty)$ and that the solution of $\dot{x} = A(t)x + B(t)u$ with which we work has $\lim_{t\to-\infty} x(t) = \lim_{t\to\infty} x(t) = 0$. This is equivalent to taking the solution to be the unique solution for which $x \in \mathcal{L}_2(-\infty,\infty)$. It also ensures that $y \in \mathcal{L}_2(-\infty,\infty)$. Now we obtain, for any finite t_0 and t_1,

$$\begin{aligned} &\int_{t_0}^{t_1} [u^*(s)u(s) - y^*(s)y(s) - \epsilon_2 x^*(s)x(s)]\, ds \\ &= \int_{t_0}^{t_1} \{u^*(s)u(s) - x^*(s)[C^*(s)C(s) + \epsilon_2]x(s)\}\, ds \\ &= \int_{t_0}^{t_1} \{[u^*(s) + x^*(s)P(s)B(s)][u(s) + B^*(s)P(s)x(s)] \\ &\quad + \epsilon_1 x^*(s)P^2(s)x(s)\}\, ds \\ &\quad - x^*(t_1)P(t_1)x(t_1) + x^*(t_0)P(t_0)x(t_0) \end{aligned}$$

Letting $t_0 \to -\infty$ and $t_1 \to \infty$ yields the main result.

The converse is a little harder. Observe that with $A(t)$ taking the form shown in Proposition III.3 the equation $\dot{x} = A(t)x + u$ has solution

$$x(t) = \begin{bmatrix} \int_{-\infty}^{t} \Phi(t,s)u_1(s)ds \\ -\int_t^{\infty} \Phi(t,s)u_2(s)ds \end{bmatrix} \tag{16}$$

and it is easily seen that the mapping from $u \in \mathcal{L}_2(-\infty,\infty)$ to $x \in \mathcal{L}_2(-\infty,\infty)$ has bounded induced $\mathcal{L}_2$ norm. It follows that the system given by

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u + \sqrt{\epsilon_1}\bar{u} \\ y &= C(t)x \\ \bar{y} &= \sqrt{\epsilon_2}x \end{aligned} \tag{17}$$

has a gain which approaches the gain of $\mathcal{H}$ as positive valued ϵ_1 and ϵ_2 approach zero. Consequently, there exist positive ϵ_1 and ϵ_2 so that the gain of (17) itself is less than unity. Define

$$\tilde{B} = [B \quad \sqrt{\epsilon_1}I]$$
$$\tilde{u} = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C \\ \sqrt{\epsilon_2} I \end{bmatrix}$$
$$\tilde{y} = \begin{bmatrix} y \\ \bar{y} \end{bmatrix}.$$

We shall temporarily study the system

$$\dot{x} = A(t)x + \tilde{B}(t)\tilde{u}$$
$$\tilde{y} = \tilde{C}(t)x$$

which has induced $\mathcal{L}_2$ gain less than unity and is obviously uniformly completely controllable and observable (through $\bar{u}$ and $\bar{y}$ alone, in fact.) Now for finite t_0 and t_1, consider the problem of minimizing, subject to $\dot{x} = A(t)x + \tilde{B}(t)\tilde{u}$, with arbitrary given $x(t_0)$ and with $x(t_1) = 0$, the integral

$$\int_{t_0}^{t_1} [\tilde{u}^*(s)\tilde{u}(s) - \tilde{y}^*(s)\tilde{y}(s)]\, ds.$$

Notice that for any $\delta > 0$ appearing in the uniform complete controllability definition, there exists $\tilde{u}(\cdot)$ defined over $(t_0 - \delta, t_0)$ and with $\|\tilde{u}(\cdot)\| \le K\|x(t_0)\|$ such that the differential equation

$$\dot{x} = Ax + \tilde{B}\tilde{u}, \text{ with } x(t_0 - \delta) = 0$$

solved forward in time from $t = t_0 - \delta$, leads to the given $x(t_0)$. We apply $\tilde{u}$ in the time interval $[t_0 - \delta, t_0]$ followed by a $\tilde{u}$ in $[t_0, t_1]$ which is arbitrary, save that $x(t_1) = 0$. Applying the gain condition to the interval $[t_0 - \delta, t_0]$ shows that

$$\int_{t_0}^{t_1} [\tilde{u}^*(s)\tilde{u}(s) - \tilde{y}^*(s)\tilde{y}(s)]\, ds \ge \alpha \|x(t_0)\|^2 \tag{18}$$

for some (possibly negative) constant α (independent of t_0 and $x(t_0)$). Also, provided that $t_1 > t_0 + \delta$, it is clear that we can always find a specific $\tilde{u}$, call it $\hat{u}(t)$, with $\|\hat{u}\| \le K\|x(t_0)\|$ taking $x(t_0)$ to $x(t_0 + \delta) = 0$, $\hat{u}(t) = 0$ for $t \notin [t_0, t_0 + \delta]$. Also

$$\begin{aligned} \min_{\substack{u \\ x(t_1)=0}} \int_{t_0}^{t_1} [\tilde{u}^*(s)\tilde{u}(s) - \tilde{y}^*(s)\tilde{y}(s)]\, ds &\le \min_{\substack{\tilde{u} \\ x(t_1)=0}} \int_{t_0}^{t_1} \tilde{u}^*(s)\tilde{u}(s) ds \\ &\le \int_{t_0}^{t_1} \hat{u}^*(s)\hat{u}(s) ds \\ &\le \beta \|x(t_0)\|^2 \end{aligned} \tag{19}$$

for some constant β. Standard quadratic variational arguments, (see, for example, [4] and [19]) coupled with existence of upper and lower bounds in (18) and (19), then guarantee that for $t_0 \le t_1 - \delta$, the minimum is $x^*(t_0)P_{t_1}(t_0)x(t_0)$ where

$$-\dot{P}_{t_1} = P_{t_1}A + A^*P_{t_1} - P_{t_1}\tilde{B}\tilde{B}^*P_{t_1} - \tilde{C}^*\tilde{C}$$

with boundary condition $P_{t_1}(t_1) = \infty$. This boundary condition is to be interpreted as follows. In the vicinity of t_1, one solves backward in time

$$\dot{X} = AX + XA^* - \tilde{B}\tilde{B}^* - X\tilde{C}^*\tilde{C}X$$

with $X(t_1) = 0$. For any $t_2 < t_1$, which is sufficiently close to t_1, there holds that $X_2(t)$ is nonsingular, because $\tilde{B}\tilde{B}^*$ is semi-definite and one sets $P_{t_1}(t_2) = X^{-1}(t_2)$. Then one solves for $P_{t_1}(t)$ in $t < t_2$.

The $P_{t_1}(t)$ just defined depends on t_1. We now let $t_1 \to \infty$. It is not hard to argue that $P_{t_1}(t)$ is monotone decreasing with t_1. Since by (18) it is bounded below, (by αI), it follows that $\lim_{t_1 \to \infty} P_{t_1}(t)$ exists and, following standard arguments (see, for example [13], [17]), the limiting $P(t)$ satisfies

$$-\dot{P} = PA + A^*P - P\tilde{B}\tilde{B}^*P - \tilde{C}^*\tilde{C}$$

which is the same as (15). The claim regarding the sign of the eigenvalues of $P(s)$ follows by using Corollary III.2 with the identification $P = -H$. ■

A further tool we shall need is an easy generalization of a time-invariant result, again achieved by using Corollary III.2.

Proposition III.6: Let Z be a finite-dimensional time-varying operator with state-space representation given by (9), where $A(t)$is continuous, A, B, and C are bounded, $[A, B]$ is uniformly completely controllable and $[A, C]$ is uniformly completely observable. Suppose that A has n_+, n_- and 0 positive, negative and zero Lyapunov exponents and $\|Z\| < 1$. Then the realization of $I + Z$ and $(I + Z)^{-1}$ formed in the obvious ways, have the same Lyapunov exponents.

Proof: The generalized time-varying bounded real Lemma (see Proposition III.5) guarantees the existence of a suitably small positive ϵ_1 and ϵ_2 and a bounded symmetric $P(t)$ with

$$\dot{P} + PA + A^*P - P(BB^* + \epsilon_1 I)P - C^*C - \epsilon_2 I = 0. \tag{20}$$

Because A has n_+, n_-, and 0 positive, negative and zero Lyapunov exponents, P has n_+, n_- and 0 positive, negative, and zero eigenvalues. Now, the state transition matrix, $A_{(I+Z)^{-1}}$ of the realization of $(I + Z)^{-1}$ formed in the obvious way is $A - BC$ and evidently, from (20) we have that

$$\dot{P} + P(A - BC) + (A^* - C^*B^*)P - (PB - C^*)(B^*P - C) - \epsilon_1 P^2 - \epsilon_2 I = 0.$$

The eigenvalue properties of P imply that $(A - BC)$ has n_+, n_- and 0, respectively, positive, negative, and zero Lyapunov exponents, as required. ■

The following corollary can be proved with effectively the same argument.

Corollary III.3: Let Z be a finite-dimensional, time-varying operator, with the state representation given by (9), with A continuous, and A, B, C bounded. Suppose that there exists some symmetric P with both P and P^{-1} bounded such that (20) holds. Then $\|Z\| < 1$ and the realization of $(I + Z)^{-1}$ found in the obvious way from that of $(I + Z)$ has the same Lyapunov exponents as that of $I + Z$.

Remark III.2: The result of Proposition III.6 is also valid in the case that Z is defined by a quadruple $\{A, B, C, D\}$ with $D(t) \neq 0$ instead of as in (9). A variant on Proposition III.5 may be derived as per [13] who prove a bounded real lemma for the case a strictly causal operators and extend to the nonstrictly causal case in Problem 4.16. The same is of course true for

Corollary III.3. Note that the Riccati equation (20) is changed when $D(t) \neq 0$.

C. Normalized Coprime Factorization Representation

To translate the Vinnicombe metric concept to time-varying, finite-dimensional systems, is relatively straightforward. Quantities like G_2^* are replaced by the adjoints of causal and stable operators and quantities such as $\|Q\|_\infty$ are replaced by the induced norm of operators on $\mathcal{L}_2$ (with appropriate dimension). It is also possible to define normalized coprime fraction descriptions [22]. We begin with the latter.

A finite dimensional time-varying plant P has equations of the form

$$\begin{aligned}\dot{x} &= A(t)x + B(t)u\\ y &= C(t)x + D(t)u\end{aligned}$$

where

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right].$$

We assume that A is continuous and that A, B, C, D are all bounded.

A right coprime fractional description of P is associated with a system

$$\begin{bmatrix} N \\ M\end{bmatrix} = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C}_N & \bar{D}_N \\ \bar{C}_M & \bar{D}_M\end{array}\right]$$

with the property that M^{-1} exists (in fact, $\bar{D}_M$ is square and nonsingular), $P = NM^{-1}$, $\dot{x} = \bar{A}x$ is exponentially stable, the matrix $\bar{A}$ is continuous and the matrices $\bar{A}$, $\bar{B}$ $\bar{C}$ and $\bar{D}$ are all bounded. A right coprime fraction description is called *normalized* if it also has the additional property that the operator

$$\begin{bmatrix} N \\ M\end{bmatrix} : z \to y$$

is an $\mathcal{L}_2$ isometry. Equivalently $N^*N + M^*M = I$, where N^* is the adjoint of N which has a state-space realization given by

$$\begin{aligned}\dot{\hat{x}} &= -\bar{A}^*\hat{x} + \bar{C}_N^*\hat{u}\\ \hat{y} &= \bar{B}^*x + \bar{D}_N^*\hat{u}.\end{aligned}$$

One should think of the differential equation for N^* being solved backward in time from a zero initial state in the infinitely distant future.

A left coprime factorization description of P may be associated with the system

$$[\tilde{M} \; -\tilde{N}] = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_M & -\tilde{B}_N \\ \hline \tilde{C} & \tilde{D}_M & -\tilde{D}_N\end{array}\right]$$

such that $\dot{x} = \tilde{A}x$ is exponentially stable, $\tilde{D}_M^{-1}$ exists and the matrices $\tilde{A}$, $\tilde{C}$ and so on, are bounded, with $\tilde{A}$ continuous, and $P = \tilde{M}^{-1}\tilde{N}$. It is normalized if $[\tilde{M} \quad -\tilde{N}]$ is a co-isometry, [22] that is, $\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$.

Theorem III.1: ([22, Th. 4.1]) Consider a plant P defined by (21) with A continuous and A, B, C, D bounded. Define

$$R(t) = I + D(t)^*D(t)$$

and

$$\tilde{R}(t) = I + D(t)D(t)^*.$$

Suppose that $(A,\, B)$ is uniformly completely stabilizable and $(C,\, A)$ is uniformly completely detectable [2]; then there is a normalized right coprime factorization description $P = NM^{-1}$ where

$$\begin{bmatrix} N \\ M\end{bmatrix} = \left[\begin{array}{c|c} A+BF & BR^{-1/2} \\ \hline F & R^{-1/2} \\ C+DF & DR^{-1/2}\end{array}\right]$$

with $\dot{x} = (A+BF)x$ exponentially stable and F bounded, with $F = -R^{-1}(B^*X + D^*C)$ and $X(t)$ the solution at time t of

$$\begin{aligned}\dot{X} = X\left(A - BR^{-1}D^*C\right) &+ \left(A^* - C^*DR^{-1}B^*\right)X\\ &- XBR^{-1}B^*X + C^*\tilde{R}^{-1}C \quad (21)\end{aligned}$$

with boundary condition $X(t_f) = 0$, when $t_f \to \infty$. Likewise, there is a normalized left coprime factorization description $P = \tilde{M}^{-1}\tilde{N}$ where

$$[\tilde{M} \; -\tilde{N}] = \left[\begin{array}{c|cc} A+LC & L & -(B+LD) \\ \hline \tilde{R}^{-1/2}C & \tilde{R}^{-1/2} & -\tilde{R}^{-1/2}D\end{array}\right]$$

$L = -(BD^* + YC^*)\tilde{R}^{-1}$ *and* Y *is the limiting solution of*

$$\begin{aligned}\dot{Y} = (A - BD^*\tilde{R}^{-1}C)Y &+ Y(A^* - C^*\tilde{R}^{-1}DB^*)\\ &- YC^*\tilde{R}^{-1}CY + BR^{-1}B^*\end{aligned}$$

with boundary condition $Y(t_s) = 0$, *as* $t_s \to -\infty$.

Proof: In [22], the proof is given for the normalized left coprime-fractional representation. The proof for the normalized right coprime-fractional representation follows similarly. For completeness, Appendix II gives an alternative proof for the normalized right coprime-fractional representation. ■

D. The "Time-Varying" Winding Number

In preparation for defining the Vinnicombe metric for time-varying systems, we need to establish the equivalent of the winding number condition.

Let P_1, P_2 be two linear finite-dimensional, time-varying plants as in Theorem III.1, each with the same input and output dimensions. Let normalized coprime fractional descriptions be found as

$$G_i = \begin{bmatrix} N_i \\ M_i\end{bmatrix} = \left[\begin{array}{c|c} A_i+B_iF_i & B_iR_i^{-1/2} \\ \hline F_i & R_i^{-1/2} \\ C_i+D_iF_i & D_iR_i^{-1/2}\end{array}\right].$$

Given a state-space representations of linear time-invariant systems, it is straightforward to give corresponding state-space representations of multiplication, addition and inversion of the original systems: see Section 3.6 of [26]. The formulae given there are also valid for state-space representations of linear time-varying systems. Using such manipulations, the state evolution matrix $A_{G_1^* G_2}$ of one particular representation of $G_1^* G_2$ may be shown to be

$$A_{G_1^* G_2} = \begin{bmatrix} -(A_1 + B_1 F_1)^* & \diamond \\ 0 & A_2 + B_2 F_2 \end{bmatrix} \tag{22}$$

and that of $(G_1^* G_2)^{-1}$ is found to be (23), shown at the bottom of the page. Note that the matrix in (23) is also the state-evolution matrix of $(I + P_1^* P_2)^{-1}$.

In the time-invariant case, requiring $\det[G_1^* G_2]$ to have zero winding number is equivalent to requiring that $G_1^* G_2$ and $(G_1^* G_2)^{-1}$ have the same number of poles in the closed right hand plane, Re$[s] \geq 0$ (see (8)). Note also that $G_1^* G_2$ is bounded on the $j\omega$-axis and that a requirement associated with the winding number condition, given by (7), is the property that $\det[G_1^* G_2]$ is nonzero on the $j\omega$-axis. For both (7) and (8) to hold is equivalent to each of $G_1^* G_2$ and $(G_1^* G_2)^{-1}$ having the same number of poles in the *open* right hand plane, Re$[s] > 0$ and none on the $j\omega$-axis. In terms of a state-space realization $\{A, B, C, D\}$ of $G_1^* G_2$, irrespective of whether it is minimal, the zero winding number condition (8) is equivalent to requiring that A and $A - BD^{-1}C$ have the same number of eigenvalues in Re$[s] > 0$.

In the time-varying case, then, a requirement equivalent to (8) is that the matrices in (22) and (23) have the same number of positive order numbers or Lyapunov exponents ([14], see also Section III-B). The time-varying equivalent of the winding number is the number $(= \dim(A_2))$ of positive Lyapunov exponents of (22) minus the number of positive Lyapunov exponents of (23). Equation (7) is equivalent to the condition that $(G_2^* G_1)^{-1}$ is bounded on the $j\omega$-axis, so that the associated (two-sided) time domain impulse response maps $\mathcal{L}_2(-\infty, \infty)$ into $\mathcal{L}_2(-\infty, \infty)$. In the time-varying case, this will be guaranteed if the Lyapunov exponents of $A_{(G_1^* G_2)^{-1}}$ are all nonzero, that is, provided that this matrix has an exponential dichotomy.

By the properties of normalized coprime fractional descriptions, the number of positive Lyapunov exponents of $G_2^* G_1$ is apparent from (22) and is precisely $\dim(A_1)$, using Corollary III.1 and the fact that $A_1 + B_1 F_1$ and $A_2 + B_2 F_2$ are exponentially stable. There is additional insight possible due to (23). If $P_1 = P_2$, then (23) becomes the second equation shown at the bottom of the page. This is the Hamiltonian matrix linked with the Riccati equation (21) and as such, it has an equal number of positive and negative Lyapunov exponents. So if $P_1 \neq P_2$, but P_2 has the same state-space dimension as P_1, then with state-space realizations of P_1 and P_2 that are not very different, the winding number property will hold. We remark that it is known that if a time-varying matrix has no zero Lyapunov exponents, then the numbers of positive and negative Lyapunov exponents are invariant under small perturbations of the matrix (see [5]).

Formally we state the following definition.

Definition III.3: Let X be a time-varying operator with state-space (21), where A is continuous and A, B, C, D and D^{-1} are bounded. Then the winding number of the operator X is defined if A and $A - BD^{-1}C$ have an exponential dichotomy, as the number of positive Lyapunov exponents associated with A minus the number of positive Lyapunov exponents associated with $A - BD^{-1}C$. The operator X has zero winding number if both A and $A - BD^{-1}C$ have the same number of positive (and therefore negative) Lyapunov exponents. If either A or $A - BD^{-1}C$ does not have an exponential dichotomy, the winding number is not defined.

Notice that this definition is consistent whether minimal or nonminimal realizations are used, so long as any nonminimal realization has no zero Lyapunov exponents for the A matrix.

It is also not hard to establish that the products of operators with zero winding number also have zero winding number.

Proposition III.7: Let X_i, $i = 1, 2$ be two invertible, finite dimensional operators, such that the dimension of the input of X_1 is equal to the dimension of the output of X_2 and with state-space representation

$$\begin{aligned} \dot{x}_i &= A_i(t) x_i + B_i(t) u_I \\ y_i &= C_i(t) x_i + D_i(t) u_i \end{aligned}$$

with A_i continuous and A_i, B_i, C_i, D_i, and D_i^{-1} bounded for all time t. Suppose that for $i = 1, 2$, each pair A_i and $A_i - B_i D_i^{-1} C_i$ have exponential dichotomies with the same number of postive and negative Lyapunov exponents as each other. Then a state-space realization $\{A, B, C, D\}$ of the operator $X = X_1 X_2$ is given by

$$\begin{aligned} A &= \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} \\ B &= \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} \\ C &= \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \\ D &= D_1 D_2 \end{aligned}$$

$$A_{(G_1^* G_2)^{-1}} = \begin{bmatrix} -A_1^* + C_1^* D_2 (I + D_1^* D_2)^{-1} B_1 & C_1^* (I + D_2 D_1^*)^{-1} C_2 \\ B_2 (I + D_2^* D_1)^{-1} B_1^* & A_2 - B_2 (I + D_1^* D_2)^{-1} D_1^* C_2 \end{bmatrix}. \tag{23}$$

$$A_{(G_1^* G_1)^{-1}} = \begin{bmatrix} -A_1^* + C_1^* D_1 (I + D_1^* D_1)^{-1} B_1 & C_1^* (I + D_1 D_1^*)^{-1} C_1 \\ B_1 (I + D_1^* D_1)^{-1} B_1^* & A_1 - B_1 (I + D_1^* D_1)^{-1} D_1^* C_2 \end{bmatrix}.$$

with the property that A and $A-BD^{-1}C$ have exponential dichotomies with the same number of positive and negative Lyapunov exponents as each other.

Proof: Notice that

$$A-BD^{-1}C=\begin{bmatrix} A_1-B_1D_1^{-1}C_1 & 0 \\ -B_2C_1(D_1D_2)^{-1}C_1 & A_2-B_2D_2^{-1}C_2 \end{bmatrix}.$$

The hypothesis that for each i, A_i and $A_i - B_iD_i^{-1}C_i$ have the same number of positive and negative Lyapunov exponents as each other, together with Corollary III.1 establishes the result. ■

E. Time-Varying Metric: Definition

We can now give a definition for a metric on time-varying finite dimensional operators, which reduces to the standard Vinnicombe metric for the time-invariant case.

Definition III.4: Suppose that P_α and P_β are two plants with the same number of inputs and outputs and with finite-dimensional bounded and continuous state-space realizations,

$$P_i=\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right].$$

Suppose further that each $\{A_i, B_i, C_i, D_i\}$ is uniformly completely stabilizable and detectable and that normalized coprime factorization descriptions $P_i = N_iM_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i$ are fixed with

$$G_i=\begin{bmatrix} N_i \\ M_i \end{bmatrix}$$
$$\tilde{G}_i=[\,\tilde{M}_i \quad -\tilde{N}_i\,].$$

Define $\delta_\nu(P_\alpha,P_\beta)=\|\tilde{G}_\alpha G_\beta\|$ provided that the matrix in (23), with A_1, A_2 replaced by A_α, A_β respectively, has $\dim A_\beta$ positive Lyapunov exponents and therefore $\dim A_\alpha$ negative Lyapunov exponents. Otherwise, define $\delta_\nu(P_\alpha,P_\beta)=1$.

Theorem III.2: The function $\delta_\nu(\cdot,\cdot)$ in Definition III.4 defines a metric on the space of linear finite-dimensional time-varying operators.

Here, for a linear time varying system operator X, the notation $\|X\|$ is taken to be the induced $\mathcal{L}_2$ norm.

Proof: Obviously $\delta_\nu(P_\alpha,P_\beta)=0$ if and only if $P_\alpha = P_\beta$. Next, for an operator $X:\mathcal{L}_2(-\infty,\infty)\to\mathcal{L}_2(-\infty,\infty)$ for which X^{-1} exists define $\underline{\sigma}(X)=\|X^{-1}\|^{-1}$. It is easy to check that $\underline{\sigma}(X)=\inf_{\|u\|=1}\|Xu\|$. Now observe that because $G_i^*G_i = I = \tilde{G}_i\tilde{G}_i{}^*$ while also $\tilde{G}_iG_i=0$, there holds

$$\begin{bmatrix} \tilde{G}_i \\ G_i^* \end{bmatrix}[\,\tilde{G}_i^* \quad G_i\,]=[\,\tilde{G}_i^* \quad G_i\,]\begin{bmatrix} \tilde{G}_i \\ G_i^* \end{bmatrix}=I$$

so that $G_iG_i^*+\tilde{G}_i^*\tilde{G}_i = I$. Since $G_i^*G_i=I$, there follows $(G_\alpha^*G_\beta)^*(G_\alpha^*G_\beta)+(\tilde{G}_\alpha G_\beta)^*(\tilde{G}_\alpha G_\beta) = I$ and so it is easy to see, using the minimum gain characterization of $\underline{\sigma}(X)$ that $\underline{\sigma}^2(G_\alpha^*G_\beta)+\|\tilde{G}_\alpha G_\beta\|^2 = 1$. Now $\underline{\sigma}(G_\alpha^*G_\beta) = \|(G_\alpha^*G_\beta)^{-1}\|^{-1} = \|(G_\beta^*G_\alpha)^{-1}\|^{-1} = \underline{\sigma}(G_\beta^*G_\alpha)$. Hence $\|\tilde{G}_\alpha G_\beta\|^2=\|\tilde{G}_\beta G_\alpha\|^2$. This establishes that $\delta_\nu(P_\alpha,P_\beta)=\delta_\nu(P_\beta,P_\alpha)$.

Finally, that $\delta_\nu(P_\alpha,P_\Omega)\le\delta_\nu(P_\alpha,P_\theta)+\delta_\nu(P_\theta,P_\Omega)$ is proved in Appendix I. ■

F. Alternative Characterization of the Time-Varying Vinnicombe Metric

It will be useful to generalize a formula available in the time-invariant case characterising the Vinnicombe metric as the solution of a certain optimization problem.

Proposition III.8: Suppose that P_1 and P_2 are two plants with the same number of inputs and outputs as each other, with linear, finite-dimensional state space realizations and with normalized coprime factorization descriptions $P_i = N_iM_i^{-1}$ and where $G_i = [\,N_i^T \quad M_i^T\,]^T$. Let $\mathcal{Q}$ be the set of linear operators Q with finite-dimensional realization with exponential dichotomy and zero winding number. Then

$$\delta_\nu(P_1,P_2)=\inf_{Q\in\mathcal{Q}}\|G_1-G_2Q\|. \tag{24}$$

Proof: This follows the proof of Lemma 6.1 in [24] almost exactly. Since $G_2G_2^*+\tilde{G}_2^*\tilde{G}_2=I$, there holds

$$\|G_1-G_2Q\|=\left\|\begin{bmatrix} G_2^* \\ \tilde{G}_2 \end{bmatrix}(G_1-G_2Q)\right\| = \left\|\begin{bmatrix} G_2^*G_1-Q \\ \tilde{G}_2G_1 \end{bmatrix}\right\|.$$

As for the time-invariant case, if $\delta_\nu(P_1,P_2)<1$ then $G_2^*G_1$ is invertible and has a realization with exponential dichotomy and the operator has zero winding number. Obviously, $Q=G_2^*G_1$ achieves the infimum.

It remains to show that the formula is true for the case where $G_1^*G_2$ does not satisfy the zero winding number condition. We show by contradiction that $\delta_\nu(P_1,P_2)=1$ implies that (24) has value unity. First assume that we have a $Q\in\mathcal{Q}$, satisfying

$$\|G_1-G_2Q\|=\|I-G_1^*G_2Q\|<1.$$

The first equality in the above is obtained by left-multiplication by the unitary operator G_1^*. Now, by appealing to arguments in Section III-D we see that the operator $G_1^*G_2$ has a state-space realization in which the state evolution matrix A has exponential dichotomy. By proposition hypothesis, Q also has a state-space realization with an exponential dichotomy and hence so does the operator $G_1^*G_2Q$. Now, apply the main result on gain bounded operators (Corollary III.3) extended to the case where Z has a direct feedthrough term, identifying $Z=-I+G_1^*G_2Q$. It follows that $I+Z=G_1^*G_2Q$ has zero winding number. Accordingly $G_1^*G_2=(I+Z)Q^{-1}$ has zero winding number (see Proposition III.7) and hence $\delta_\nu(P_1,P_2)=\|\tilde{G}_2G_1\|$.

Therefore if $G_1^*G_2$ does not satisfy the zero winding number condition we have that

$$\|G_1-G_2Q\|=\|I-G_1^*G_2Q\|\ge 1$$

for all zero winding number Q. Setting $Q=0$ makes the above equation an equality and the theorem is proved. ■

The following corollary places a bound on the Vinnicombe metric between two time-varying plants which are related by a multiplicative error.

Corollary III.4: Let $P=NM^{-1}$ be a normalized coprime fraction description of a time-varying finite-dimensional linear plant and let Δ be a stable bounded linear operator with bound $\|\Delta\|<1$, so that an output multiplicative perturbation of P

is given by $(I+\Delta)P=(I+\Delta)NM^{-1}$. Then, $\delta_\nu^{\rm ltv}(P,(I+\Delta)P)\le\|\Delta\|$

Proof: First, observe that $(I+\Delta)N$ and M are coprime. To see this, note that the coprimeness of N, M ensures the existence of X, Y such that $XN+MY=I$. The norm condition on Δ ensures that $I+\Delta$ has zero winding number and being stable, it also has a stable inverse. Hence, $X(I+\Delta)^{-1}(I+\Delta)N+MY=I$ where $X(I+\Delta)^{-1}$ is also stable. This shows that $(I+\Delta)N$ and M are coprime. Now let N_Δ, M_Δ define a normalized coprime fractional description of $(I+\Delta)P$. As a consequence, there exists a stable and stably invertible L such that

$$\begin{bmatrix}(I+\Delta)N\\ M\end{bmatrix}=\begin{bmatrix}N_\Delta\\ M_\Delta\end{bmatrix}L$$

and so we have that

$$\begin{aligned}\delta_\nu(P,(I+\Delta)P)&=\inf_{Q\in\mathcal{Q}}\left\|\begin{bmatrix}N\\ M\end{bmatrix}-\begin{bmatrix}(I+\Delta)N\\ M\end{bmatrix}L^{-1}Q\right\|\\ &=\inf_{Q\in\mathcal{Q}}\left\|\begin{bmatrix}N\\ M\end{bmatrix}-\begin{bmatrix}(I+\Delta)N\\ M\end{bmatrix}\hat{Q}\right\|\\ &\le\|\Delta N\|\\ &\le\|\Delta\|.\end{aligned}$$

■

IV. Robustness for Linear Time-Varying Systems

In this section, we are going to prove the central result on robustness for linear time-varying systems. Suppose that an interconnection $[P_1, C]$ is stable and that P_1 is changed to P_2. We shall show that $[P_2, C]$ is also stable provide that $\delta_\nu(P_1,P_2)$ is exceeded by a quantity, defined for $[P_1, C]$, termed the generalized stability margin.

This condition is very much in the spirit of the corresponding result for time-invariant systems [24] and a result for time-varying systems where a different metric (the gap metric) is used for the distance between two controllers [11], [12]. The gap metric is known to lead to more conservative results in the time-invariant case [24].

Definition IV.1: Consider the standard feedback configuration of Fig. 1 in which P and C are finite dimensional, linear, time-varying systems. Suppose that at least one of P, C is strictly causal. Denote the input–output operator

$$T_{P,C}:\begin{bmatrix}r_2\\ r_1\end{bmatrix}\to\begin{bmatrix}y\\ u\end{bmatrix}\tag{25}$$

and suppose that $T_{P,C}$ has bounded induced $\mathcal{L}_2$ norm, that is, the closed-loop is stable. The generalized stability margin of the interconnection is defined to be $b_{P,C}=\|T_{P,C}\|^{-1}$. If the closed-loop is unstable, or if the induced $\mathcal{L}_2$ norm of $T_{P,C}$ is unbounded, then define $b_{P,C}=0$.

An alternative expression for $b_{P,C}$ is given in terms of normalized coprime fractional descriptions of P and C and well-known in the time-invariant case [24].

Proposition IV.1: Adopt the hypotheses of Definition IV.1. Suppose that $P=NM^{-1}$ and $C=\tilde{Y}^{-1}\tilde{X}$ are normalized right and left coprime fractional descriptions of P and C and set

$$\begin{aligned}G&=\begin{bmatrix}N\\ M\end{bmatrix}\\ \tilde{K}&=[-\tilde{X}\quad\tilde{Y}].\end{aligned}$$

It follows that $b_{P,C}=\underline{\sigma}(\tilde{K}G)$.

Proof:

$$\begin{aligned}T_{P,C}&=\begin{bmatrix}P\\ I\end{bmatrix}(I-CP)^{-1}[-C\quad I]\\ &=\begin{bmatrix}NM^{-1}\\ I\end{bmatrix}\left(I-\tilde{Y}^{-1}\tilde{X}NM^{-1}\right)^{-1}[-\tilde{Y}^{-1}\tilde{X}\quad I]\\ &=\begin{bmatrix}N\\ M\end{bmatrix}\left(\tilde{Y}M-\tilde{X}N\right)^{-1}\begin{bmatrix}-\tilde{X}^T\\ \tilde{Y}^T\end{bmatrix}^T\\ &=G\left(\tilde{K}G\right)^{-1}\tilde{K}.\end{aligned}$$

Since $G^*G=I$ and $\tilde{K}\tilde{K}^*=I$ then $\|T_{P,C}\|=\|(\tilde{K}G)^{-1}\|$ and the result is immediate. ■

Now we can state the main result of this paper.

Theorem IV.1: Consider a feedback system of the form of Fig. 1 with plant P_1 and controller C being finite dimensional linear time-varying systems. Suppose that the closed loop is stable with generalized stability margin $b_{P,C}$. Let P_2 be a finite-dimensional linear time-varying plant, with $\delta_\nu(P_1,P_2)<b_{P_1,C}$. Then $[P_2, C]$ is a stable loop.

Proof: Define operators G_1, G_2 and $\tilde{K}$ using normalized fractional representations in an obvious manner. We have just observed that

$$T_{P_2,C}=G_2\left(\tilde{K}G_2\right)^{-1}\tilde{K}$$

and if $(\tilde{K}G_2)^{-1}$ has zero winding number, it is evident that $[P_2, C]$ will be a stable loop. Note that it is of course true that $\tilde{K}G_2$ is stable. If it has zero winding number then its inverse is stable. Now observe that

$$\begin{aligned}\tilde{K}G_2&=\tilde{K}[G_1\quad\tilde{G}_1^*]\begin{bmatrix}G_1^*\\ \tilde{G}_1\end{bmatrix}G_2\\ &=\tilde{K}G_1G_1^*G_2+\tilde{K}\tilde{G}_1^*\tilde{G}_1G_2.\end{aligned}\tag{26}$$

Now the stability of $[P_1, C]$ guarantees that $(\tilde{K}G_1)^{-1}$ is well-defined and is a bounded operator. Also, because $\delta_\nu(P_1,P_2)<b_{P_1,C}<1$ and since, by Theorem IV.1 we know that $\underline{\sigma}(G_1^*G_2)^2+\delta_\nu(P_1,P_2)^2=1$. It follows that $\underline{\sigma}(G_1^*G_2)\ge\sqrt{1-b_{P_1,C}^2}>0$ and hence $(G_1^*G_2)^{-1}$ is a bounded operator. Hence

$$\begin{aligned}\tilde{K}G_2=&\tilde{K}G_1\\ &\cdot\left[I+\left(\tilde{K}G_1\right)^{-1}\left(\tilde{K}\tilde{G}_1^*\right)\left(\tilde{G}_1G_2\right)(G_1^*G_2)^{-1}\right](G_1^*G_2).\end{aligned}$$

The right-hand side consists of the product of three operators, the first and third of which are zero winding number operators, by virtue of the assumptions, namely that $[P_1, C]$ is stable and $\delta_\nu(P_1,P_2)<1$. If we can prove that the middle operator is also a zero winding number operator, it follows by Theorem III.2

that $\tilde{K}G_2$ has this property and then the stability of $[P_2, C]$ is an immediate consequence.

Now the fact that, by assumption, $\delta_\nu(P_1,P_2) < b_{P_1,C}$ implies that

$$\left\|\tilde{G}_1 G_2\right\| < \underline{\sigma}\left[\tilde{K}G_1\right] \tag{27}$$

from which an immediate consequence is

$$\underline{\sigma}\left(G_1^* G_2\right) > \left\|\tilde{K}\tilde{G}_1^*\right\|. \tag{28}$$

Then

$$\begin{aligned}&\left\|\left(\tilde{K}G_1\right)^{-1}\left(\tilde{K}\tilde{G}_1^*\right)\left(\tilde{G}_1G_2\right)\left(G_1^*G_2\right)^{-1}\right\|\\ &\le\left\|\left(\tilde{K}G_1\right)^{-1}\right\|\cdot\left\|\tilde{K}\tilde{G}_1^*\right\|\cdot\left\|\tilde{G}_1G_2\right\|\cdot\left\|\left(G_1^*G_2\right)^{-1}\right\|\\ &=\left[\underline{\sigma}\left(\tilde{K}G_1\right)\right]^{-1}\left\|\tilde{K}\tilde{G}_1^*\right\|\cdot\left\|\tilde{G}_1G_2\right\|\cdot\left\|\left[\underline{\sigma}\left(G_1^*G_2\right)\right]^{-1}\right\|\\ &<1.\end{aligned}$$

Accordingly, by Proposition III.6, $I+(\tilde{K}G_1)^{-1}(\tilde{K}G_1^*)(\tilde{G}_1G_2)(G_1^*G_2)^{-1}$ is a zero winding number operator. ∎

Corollary IV.1: For any P_1, P_2 and C such that the loops $[P_1, C]$, and $[P_2, C]$ can be defined, there holds

$$\sin^{-1} b_{P_2,C} \ge \sin^{-1} b_{P_1,C} - \sin^{-1}\delta_\nu\left(P_1,P_2\right). \tag{29}$$

Proof: The result is trivial in the case that $[P_1, C]$ is unstable, or $\delta_\nu(P_1,P_2)\ge b_{P_1,C}$. Therefore, suppose that $[P_1, C]$ is stable. From (26), we see that

$$\begin{aligned}&\underline{\sigma}\left(\tilde{K}G_2\right)\\ &\quad\ge\underline{\sigma}\left(\tilde{K}G_1\right)\cdot\underline{\sigma}\left(G_1^*G_2\right)-\left\|\tilde{K}\tilde{G}_1^*\right\|\cdot\left\|\tilde{G}_1G_2\right\|\\ &\quad=b_{P_1,C}\sqrt{1-\delta_\nu\left(P_1,P_2\right)^2}-\sqrt{1-b_{P_1,C}^2}\cdot\delta_\nu\left(P_1,P_2\right)\\ &\quad=\sin\left[\sin^{-1}b_{P_1,C}\right]\cos\left[\sin^{-1}\delta_\nu\left(P_1,P_2\right)\right]\\ &\qquad-\sin\left[\sin^{-1}\delta_\nu\left(P_1,P_2\right)\right]\cos\left[\sin^{-1}b_{P_1,C}\right]\\ &\quad=\sin\left[\sin^{-1}b_{P_1,C}-\sin^{-1}\delta_\nu\left(P_1,P_2\right)\right].\end{aligned}$$

Taking the inverse sine of both sides of the above equation yields (29) by noting that $\sin^{-1}$ is monotonic increasing. ∎

In the time-invariant case, we can also bound the closed loop error. This carries over to the time-varying case.

Corollary IV.2: Adopt the same hypotheses as Theorem IV.1. Then

$$\begin{aligned}\delta_\nu\left(P_1,P_2\right)&\le\left\|T_{P_1,C}-T_{P_2,C}\right\|\\ &\le\frac{\delta_\nu\left(P_1,P_2\right)}{b_{P_1,C}b_{P_2,C}}.\end{aligned}$$

Proof: Note first that

$$T_{P_1,C}=\begin{bmatrix}P_1\\ I\end{bmatrix}\left(I-CP_1\right)^{-1}\begin{bmatrix}-C & I\end{bmatrix}$$

and, with minor calculations

$$I-T_{P_1,C}=\begin{bmatrix}I\\ C\end{bmatrix}\left(I-P_1C\right)^{-1}\begin{bmatrix}I & -P_1\end{bmatrix}.$$

Now, we argue as in [24] that

$$\begin{aligned}T_{P_1,C}-T_{P_2,C}&=\left\{\begin{bmatrix}P_1\\ I\end{bmatrix}\left(I-CP_1\right)^{-1}\begin{bmatrix}P_2\\ I\end{bmatrix}\left(I-CP_2\right)^{-1}\right\}\\ &\quad\cdot\begin{bmatrix}-C^T\\ I\end{bmatrix}^T\\ &=\left[G_1\left(\tilde{K}G_1\right)^{-1}-G_2\left(\tilde{K}G_2\right)^{-1}\right]\tilde{K} \qquad (30)\\ &=\left[G_1\left(\tilde{K}G_1\right)^{-1}\tilde{K}-I\right]G_2\left(\tilde{K}G_2\right)^{-1}\tilde{K}\\ &=\left[T_{P_1,C}-I\right]G_2\left(\tilde{K}G_2\right)^{-1}\tilde{K}\\ &=-K\left(\tilde{G}_1K\right)^{-1}\tilde{G}_1G_2\left(\tilde{K}G_2\right)^{-1}\tilde{K}. \qquad (31)\end{aligned}$$

It follows from (30) that

$$T_{P_1,C}-T_{P_2,C}=-\left[G_2-G_1Q\right]\left(\tilde{K}G_2\right)^{-1}\tilde{K} \tag{32}$$

where $Q=(\tilde{K}G_1)^{-1}(\tilde{K}G_2)$. Furthermore, it follows from (31) that

$$\begin{aligned}&\left\|T_{P_1,C}-T_{P_2,C}\right\|\\ &\le\|K\|\cdot\left\|\left(\tilde{G}_1K\right)^{-1}\right\|\cdot\left\|\tilde{G}_1G_2\right\|\cdot\left\|\left(\tilde{K}G_2\right)^{-1}\right\|\cdot\left\|\tilde{K}\right\|\\ &\le\frac{\delta_\nu}{b_{P_1,C}b_{P_2,C}}.\end{aligned} \tag{33}$$

Evidently, from (32)

$$-\left(G_2-G_1Q\right)=\left(T_{P_1,C}-T_{P_2,C}\right)\tilde{K}^*\left(\tilde{K}G_2\right)$$

and so $\|G_2-G_1Q\|\le\|T_{P_1,C}-T_{P_2,C}\|$. Now recall that $\delta_\nu(P_1,P_2)=\inf_{Q\in\mathcal{Q}}\|G_2-G_1Q\|$, where $\mathcal{Q}$ is the set of finite dimensional operators with exponential dichotomy and zero winding number of the A matrix. Since $\tilde{K}G_1$, $\tilde{K}G_2$ and $(\tilde{K}G_1)^{-1}\tilde{K}G_2$ are each in $\mathcal{Q}$, by stability and as a composition of zero winding number operators, it follows that

$$\begin{aligned}\delta_\nu\left(P_1,P_2\right)&\le\left\|G_2-G_1Q\right\|\\ &\le\left\|T_{P_1,C}-T_{P_2,C}\right\|.\end{aligned}$$

∎

V. Pseudometric on Nonlinear Operators

Similarly to the construction of an extension of a linear time-varying gap metric to a nonlinear gap pseudometric in [9], we define the following function for nonlinear operators $\mathcal{P}_\alpha$, $\mathcal{P}_\beta$ by considering all linearizations (see Section II-C) of the nonlinear operators about their trajectories.

$$\begin{aligned}\delta_\nu^{\mathrm{nl}\uparrow}\left(\mathcal{P}_\alpha,\mathcal{P}_\beta\right)&=\sup_{u_\alpha\in\mathcal{U}}\inf_{u_\beta\in\mathcal{U}}\delta_\nu^{\mathrm{ltv}}\left(T_{u_\alpha}\mathcal{P}_\alpha,T_{u_\beta}\mathcal{P}_\beta\right)\\ \delta_\nu^{\mathrm{nl}}&=\max\left[\delta_\nu^{\mathrm{nl}\uparrow}\left(\mathcal{P}_\alpha,\mathcal{P}_\beta\right),\delta_\nu^{\mathrm{nl}\uparrow}\left(\mathcal{P}_\beta,\mathcal{P}_\alpha\right)\right]\end{aligned} \tag{34}$$

Theorem V.1: The function $\delta_\nu^{\mathrm{nl}}(\mathcal{P}_\alpha,\mathcal{P}_\beta)$ defines a pseudometric on the space of differentially stabilizable operators.

Proof: This follows the proof of [9, Th. 1]. We need to show

- $\delta_\nu(\mathcal{P}_\alpha, \mathcal{P}_\alpha) = 0$
- $\delta_\nu(\mathcal{P}_\alpha, \mathcal{P}_\beta) = \delta_\nu(\mathcal{P}_\beta, \mathcal{P}_\alpha)$
- $\delta_\nu(\mathcal{P}_\alpha, \mathcal{P}_\Omega) \leq \delta_\nu(\mathcal{P}_\alpha, \mathcal{P}_\theta) + \delta_\nu(\mathcal{P}_\theta, \mathcal{P}_\Omega)$

The first conditions are inherited directly from the properties of the linear time-varying metric from which the nonlinear metric is composed and the second follows from the symmetry of the definition of the nonlinear pseudometric. The triangle inequality is not difficult to prove either. To see this consider the linearization of $\mathcal{P}_\alpha$, $\mathcal{P}_\Omega$ about arbitrary inputs $\hat{u}_\alpha$ and $\hat{u}_\Omega$, the linearization of $\mathcal{P}_\theta$ about $\hat{u}_\theta$. From the properties of the linear time-varying metric

$$\delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\Omega}\mathcal{P}_\Omega\right) \leq \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\theta}\mathcal{P}_\theta\right) + \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\theta}\mathcal{P}_\theta, T_{\hat{u}_\Omega}\mathcal{P}_\Omega\right).$$

To avoid complicated arguments involving ϵ and δ, we shall assume that the infima and suprema in (34) are attainable minima and maxima, however the conclusions are still valid even if this is not the case. Choose $\hat{u}_\Omega^\star(\hat{u}_\theta)$ to be that which results in the minimum $\inf_{u_\Omega} \delta_\nu^{\text{ltv}}(T_{\hat{u}_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega)$, then the aforementioned implies

$$\begin{aligned}\delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\Omega^\star}\mathcal{P}_\Omega\right) \leq& \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\theta}\mathcal{P}_\theta\right) \\ &+ \inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right) \\ \inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{u_\Omega}\mathcal{P}_\Omega\right) \leq& \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\theta}\mathcal{P}_\theta\right) \\ &+ \inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right).\end{aligned}$$

But the above is true for all $\hat{u}_\theta$, so taking $\hat{u}_\theta^\star(\hat{u}_\alpha)$ as that which results in the minimum $\inf_{u_\theta} \delta_\nu^{\text{ltv}}(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta)$ we have that

$$\begin{aligned}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{\hat{u}_\Omega}\mathcal{P}_\Omega\right) \leq& \inf_{u_\theta} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta\right) \\ &+ \inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\theta^\star}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right) \\ \leq& \inf_{u_\theta} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta\right) \\ &+ \sup_{u_\theta}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{u_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right).\end{aligned}$$

Again, the above is true for all $\hat{u}_\alpha$, so taking $\hat{u}_\alpha^\star$ as that which results in the maximum $\sup_{u_\alpha}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}(T_{u_\alpha}\mathcal{P}_\alpha, T_{u_\Omega}\mathcal{P}_\Omega)$ we have

$$\begin{aligned}\sup_{u_\alpha}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{u_\alpha}\mathcal{P}_\alpha, T_{u_\Omega}\mathcal{P}_\Omega\right) \leq& \inf_{u_\theta} \delta_\nu^{\text{ltv}}\left(T_{\hat{u}_\alpha^\star}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta\right) \\ &+ \sup_{u_\theta}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{u_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right) \\ \leq& \sup_{u_\alpha}\inf_{u_\theta} \delta_\nu^{\text{ltv}}\left(T_{u_\alpha}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta\right) \\ &+ \sup_{u_\theta}\inf_{u_\Omega} \delta_\nu^{\text{ltv}}\left(T_{u_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right)\end{aligned}$$

that is

$$\delta_\nu^{\text{nl}\uparrow}\left(T_{u_\alpha}\mathcal{P}_\alpha, T_{u_\Omega}\mathcal{P}_\Omega\right) \leq \delta_\nu^{\text{nl}\uparrow}\left(T_{u_\alpha}\mathcal{P}_\alpha, T_{u_\theta}\mathcal{P}_\theta\right) + \delta_\nu^{\text{nl}\uparrow}\left(T_{u_\theta}\mathcal{P}_\theta, T_{u_\Omega}\mathcal{P}_\Omega\right).$$

The definition of the nonlinear pseudometric as the maximum of two directed nonlinear metrics and elementary algebraic manipulation proves the triangle inequality. ■

VI. Robustness Properties of the Nonlinear Operator Pseudometric

For the nonlinear system of Fig. 1, define $T_{P,C}^{\text{nl}}$ as the operator from $[\,r_2^T \quad r_1^T\,]^T \rightarrow [\,y^T \quad u^T\,]^T$. Take the norm $\|\cdot\|_\Delta$ as defined for incrementally finite gain stability in Definition II.1 and define the nonlinear stability margin for an interconnection $[P, C]$ as $b_{P,C}^{\text{nl}} = \|T_{P,C}\|_\Delta^{-1}$ if $T_{P,C}$ is incrementally finite gain stable and zero otherwise. It is easy to see that

$$b_{P,C}^{\text{nl}} = \inf_{\substack{[y,u],[\bar{y},\bar{u}]\in\mathcal{R}T_{P,C} \\ [y,u]\neq[\bar{y},\bar{u}]}} \frac{\|[r_2, r_1] - [\bar{r}_2, \bar{r}_1]\|}{\|[y, u] - [\bar{y}, \bar{u}]\|}. \tag{35}$$

where $[r_1, r_2]$ give rise to signals $[y, u]$ and $[\bar{r}_1, \bar{r}_2]$ give rise to signals $[\bar{y}, \bar{u}]$.

Theorem VI.1: Let P_α and P_β be two nonlinear plants that satisfy the differentiability property and let C be a controller which stabilizes P_α, so that the operator $T_{P_\alpha,C}^{\text{nl}}$ is incrementally finite gain stable. Suppose also that $\delta_\nu^{\text{nl}}(P_\alpha, P_\beta) < b_{P_\alpha,C}^{\text{nl}}$. Then the interconnection $[P_\beta, C]$ is incrementally finite gain stable.

Proof: Let $P_\alpha^m = T_m P_\alpha$, $P_\beta^{\hat{m}} = T_{\hat{m}} P_\alpha$ and $C^n = T_n C$, be linearizations of P_α, P_β and C about trajectories $u_1 = m$, $u_1 = \hat{m}$ and $u_2 = n$ respectively. Let $\eta = b_{P_\alpha,C}^{\text{nl}} - \delta_\nu^{\text{nl}}(P_\alpha, P_\beta) > 0$.

Now observe that $\inf_m \delta_\nu^{\text{ltv}}(P_\alpha^m, P_\beta^{\hat{m}}) \leq \delta_\nu^{\text{nl}}(P_\alpha, P_\beta) < b_{P_\alpha,C}^{\text{nl}} \leq b_{P_\alpha^m,C^n}^{\text{ltv}}$ and in fact $b_{P_\alpha^m,C^n}^{\text{ltv}} - \inf_m \delta_\nu^{\text{ltv}}(P_\alpha^m, P_\beta^{\hat{m}}) \geq \eta$. Hence $[P_\beta^{\hat{m}}, C^n]$ is stable and this is true for arbitrary linearizations $\hat{m}$ and n. In order to apply the results of Proposition II.3, we need to show that the stability margin resulting from the interconnection $[P_\beta^{\hat{m}}, C^n]$ is bounded away from zero.

As in the proof of Theorem V.1, in order to avoid complicated ϵ-δ arguments, we assume that the following infimum is an attainable minimum. By letting $m^\star(\hat{m})$ be the minimizer of $\inf_m \delta_\nu^{\text{ltv}}(P_\alpha^m, P_\beta^{\hat{m}})$, then by Theorem IV.1 and Corollary IV.1

$$\begin{aligned}&\sin^{-1} b_{P_\beta^{\hat{m}},C^n}^{\text{ltv}} \\ &\quad\geq \sin^{-1} b_{P_\alpha^{m^\star},C^n}^{\text{ltv}} - \sin^{-1}\inf_m \delta_\nu^{\text{ltv}}\left(P_\alpha^m, P_\beta^{\hat{m}}\right) \\ &\quad\geq \sin^{-1}\inf_m b_{P_\alpha^m,C^n}^{\text{ltv}} \\ &\qquad - \sin^{-1}\inf_m \delta_\nu^{\text{ltv}}\left(P_\alpha^m, P_\beta^{\hat{m}}\right).\end{aligned} \tag{36}$$

Since $b_{P_\alpha,C}^{\text{nl}} \leq b_{P_\alpha^m,C^n}^{\text{ltv}}$ and $\inf_m \delta_\nu^{\text{ltv}}(P_\alpha^m, P_\beta^{\hat{m}}) \leq \delta_\nu^{\text{nl}}(P_\alpha, P_\beta)$ it can easily be seen from (36) that

$$\sin^{-1} b_{P_\beta^{\hat{m}},C^n}^{\text{ltv}} \geq \sin^{-1} b_{P_\alpha,C}^{\text{nl}} - \sin^{-1}\delta_\nu^{\text{nl}}\left(P_\alpha, P_\beta\right).$$

However, recall that this is true for arbitrary $\hat{m}$, n so then

$$\begin{aligned}\inf_{\hat{m},n} \sin^{-1} b_{P_\beta^{\hat{m}},C^n}^{\text{ltv}} =& \sin^{-1} b_{P_\beta,C}^{\text{nl}} \\ &\geq \sin^{-1} b_{P_\alpha,C}^{\text{nl}} - \sin^{-1}\delta_\nu^{\text{nl}}\left(P_\alpha, P_\beta\right) > 0.\end{aligned} \tag{37}$$

The second inequality follows from the theorem hypothesis and the monotonicity of $\sin^{-1}$. From (37), it follows that the gain of every linearised combination of $[P_\beta^{\hat{m}}, C^n]$ gives rise to a $T_{P_\beta^{\hat{m}},C^n}$ that is uniformly bounded by $(b_{P_\beta,C}^{\text{nl}})^{-1}$ and so, by Proposition II.3, the interconnection $[P_\beta, C]$ is incrementally finite gain stable. ■

VII. Conclusion

In this paper, we have developed a metric on linear finite-dimensional time-varying operators which has a time-domain interpretation and which reduces to the Vinnicombe metric [24] in the time-invariant case. The metric on time-varying systems enjoys the same robustness property as the time-invariant Vinnicombe metric. The time-varying Vinnicombe metric on linear operators may be used as the basis for constructing a pseudometric on nonlinear operators in the spirit of [12]. In the process of developing a time-varying Vinnicombe metric we have exploited existing results on time-varying systems, as well as extended existing results on time-invariant systems. We have required time-varying analogues of normalized coprime factorizations, unstable and stable poles and zeros, winding number and the bounded real lemma. Further work in this area will involve investigation of the relationship between the winding number and the Fredholm index of linear time-varying operators. It remains to be investigated whether the metric which was defined on time-varying operators and the pseudometric (34) which was defined on nonlinear operators, guarantee robustness nonconservatively.

Appendix I
Proofs

Proof of Triangle Inequality

Lemma 1: Let P_α, P_θ and P_Ω be three plants with the same input-output dimensions and with linear finite-dimensional realization. With Definition III.4 we have $\underline{\sigma}(G_\alpha^* G_\Omega) \geq \underline{\sigma}(G_\alpha^* G_\theta)\underline{\sigma}(G_\theta^* G_\Omega) - \|\tilde{G}_\theta G_\alpha\| \|\tilde{G}_\theta G_\Omega\|$.

Proof: Since $G_\theta G_\theta^* + \tilde{G}_\theta^* \tilde{G}_\theta = I$, there holds $G_\alpha^* G_\Omega = G_\alpha^* G_\theta G_\theta^* G_\Omega + G_\alpha^* \tilde{G}_\theta^* \tilde{G}_\theta G_\Omega$. Thus, for any u we have

$$\begin{aligned}
&\|G_\alpha^* G_\Omega u\| \\
&\geq \|G_\alpha^* G_\theta G_\theta^* G_\Omega u\| - \left\|G_\alpha^* \tilde{G}_\theta^* \tilde{G}_\theta G_\Omega u\right\| \\
&\geq \underline{\sigma}\left(G_\alpha^* G_\theta\right) \|G_\theta^* G_\Omega u\| - \left\|G_\alpha^* \tilde{G}_\theta^*\right\| \cdot \left\|\tilde{G}_\theta G_\Omega u\right\| \\
&\geq \underline{\sigma}\left(G_\alpha^* G_\theta\right) \underline{\sigma}\left(G_\theta^* G_\Omega\right) \|u\| - \left\|G_\alpha^* \tilde{G}_\theta^*\right\| \cdot \left\|\tilde{G}_\theta G_\Omega\right\| \cdot \|u\|.
\end{aligned}$$

Now let u_k be a sequence with $\|u_k\| = 1$ such that $G_\alpha^* G_\Omega u_k \to \underline{\sigma}(G_\alpha^* G_\Omega)$. The claim of the lemma is an immediate consequence. ■

Lemma 2: Under the hypotheses of Lemma .1 we have

$$\sin^{-1}\left\|\tilde{G}_\alpha G_\Omega \leq \sin^{-1}\left\|\tilde{G}_\theta G_\alpha\right\| + \sin^{-1}\left\|\tilde{G}_\theta G_\Omega\right\|.\right. \quad (38)$$

Proof: Recognize that $\underline{\sigma}(G_\alpha^* G_\theta) = \sqrt{1 - \|\tilde{G}_\alpha G_\theta\|^2}$ and $\underline{\sigma}(G_\theta^* G_\Omega) = \sqrt{1 - \|\tilde{G}_\theta G_\Omega\|^2}$. Hence, from Lemma .1 we have

$$\begin{aligned}
&\underline{\sigma}\left(G_\alpha^* G_\Omega\right) \\
&\geq \sqrt{1 - \left\|\tilde{G}_\alpha G_\theta\right\|^2}\sqrt{1 - \left\|\tilde{G}_\theta G_\Omega\right\|^2} - \left\|\tilde{G}_\theta G_\alpha\right\| \cdot \left\|\tilde{G}_\theta G_\Omega\right\| \\
&= \cos\left[\sin^{-1}\left\|\tilde{G}_\theta G_\alpha\right\|\right] \cos\left[\sin^{-1}\left\|\tilde{G}_\theta G_\Omega\right\|\right] \\
&\quad - \sin\left[\sin^{-1}\left\|\tilde{G}_\theta G_\alpha\right\|\right] \sin\left[\sin^{-1}\left\|\tilde{G}_\theta G_\Omega\right\|\right] \\
&= \cos\left[\sin^{-1}\left\|\tilde{G}_\theta G_\alpha\right\| + \sin^{-1}\left\|\tilde{G}_\theta G_\Omega\right\|\right]
\end{aligned}$$

with $\sin^{-1}$ taken to lie within [0, $1/2\pi$]. In the case $\sin^{-1}\|\tilde{G}_\theta G_\alpha\| + \sin^{-1}\|\tilde{G}_\theta G_\Omega\| > 1/2\pi$ the cosine is negative and the inequality of the lemma is then trivial. In the case that $\sin^{-1}\|\tilde{G}_\theta G_\alpha\| + \sin^{-1}\|\tilde{G}_\theta G_\Omega\| \leq 1/2\pi$, then we obtain again the lemma inequality since $\sin^{-1}\|\tilde{G}_\alpha G_\Omega\| = \cos^{-1}\underline{\sigma}(\tilde{G}_\alpha G_\Omega)$. ■

Proof [Proof of the triangle Inequality of Theorem III.2]: Let $P_\alpha, P_\theta, P_\Omega$ be three different plants with

$$P = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right].$$

and with $\{A_i, B_i, C_i\}$ uniformly completely stabilizable and detectable. Let the associated normalized coprime factorization descriptions be $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i$ with

$$\begin{aligned}
G_i(s) &= \begin{bmatrix} N_i(s) \\ M_i(s) \end{bmatrix} \\
\tilde{G}_i(s) &= \left[\tilde{M}_i(s) \quad -\tilde{N}_i(s)\right].
\end{aligned}$$

The triangle inequality is almost immediate from (38) by taking the sine of each side

$$\begin{aligned}
\left\|\tilde{G}_\alpha G_\Omega\right\| &\leq \left\|\tilde{G}_\theta G_\alpha\right\| \underline{\sigma}\left(\tilde{G}_\theta^* G_\Omega\right) + \left\|\tilde{G}_\theta G_b\right\| \underline{\sigma}\left(\tilde{G}_\theta^* G_\alpha\right) \\
&\leq \left\|\tilde{G}_\theta G_\alpha\right\| + \left\|\tilde{G}_\theta G_\Omega\right\|.
\end{aligned} \quad (39)$$

We only need to consider what happens if $\delta_\nu(P_i, P_j) \neq \|\tilde{G}_i G_j\|$ for the different i, j combinations, but is unity instead. In the case $\delta_\nu(P_\alpha, P_\theta)$ or $\delta_\nu(P_\theta, P_\Omega)$ is unity the triangle inequality is trivial. Therefore, suppose that neither is unity. If $\sin^{-1}\|\tilde{G}_\theta G_\alpha\| + \sin^{-1}\|\tilde{G}_\theta G_\Omega\| > 1/2\pi$, there is nothing to prove, for then $\|\tilde{G}_\theta G_\alpha\| + \|\tilde{G}_\theta G_\Omega\| \geq 1$ and obviously $\delta_\nu(G_\alpha, G_\Omega) \leq 1$. So suppose that $\sin^{-1}\|\tilde{G}_\theta G_\alpha\| + \sin^{-1}\|\tilde{G}_\theta G_\Omega\| < 1/2\pi$. We want to show that the time-varying winding number condition is satisfied for $G_\alpha^* G_\Omega$, for then (39) is equivalent to the triangle inequality $\delta_\nu(P_\alpha, P_\Omega) \leq \delta_\nu(P_\alpha, P_\theta) + \delta_\nu(P_\Omega, P_\theta)$ and the theorem proof will be completed. Now note from the proof of Lemma .1 that the condition $\sin^{-1}\|\tilde{G}_\theta G_\alpha\| + \sin^{-1}\|\tilde{G}_\theta G_\Omega\| < 1/2\pi$ was shown to be equivalent to

$$\sqrt{1 - \left\|\tilde{G}_\alpha G_\theta\right\|^2}\sqrt{1 - \left\|\tilde{G}_\theta G_\Omega\right\|^2} - \left\|\tilde{G}_\alpha G_\theta\right\| \cdot \left\|\tilde{G}_\theta G_\Omega\right\| > 0$$

that is

$$\underline{\sigma}\left(G_\alpha^* G_\theta f\right) \underline{\sigma}\left(G_\theta^* G_\Omega\right) > \left\|G_\alpha^* \tilde{G}_\theta^*\right\| \cdot \left\|\tilde{G}_\theta G_\Omega\right\|.$$

Denote by X and Y, respectively, the operators $G_\alpha^* G_\theta G_\theta^* G_\Omega$ and $G_\alpha^* \tilde{G}_\theta^* \tilde{G}_\theta G_\Omega$. As observed in the course of the proof of Lemma 1,

$$\begin{aligned}
G_\alpha^* G_\Omega &= X + Y \\
&= X\left(I + X^{-1}Y\right).
\end{aligned}$$

Since $X = G_\alpha^* G_\theta G_\theta^* G_\Omega$ and $Y = G_\alpha^* \tilde{G}_\theta^* \tilde{G}_\theta G_\Omega$, there holds

$$\underline{\sigma}(X) \geq \underline{\sigma}\left(G_\alpha^* G_\theta\right) \underline{\sigma}\left(G_\theta^* G_\Omega\right) > \left\|G_\alpha^* \tilde{G}_\theta^*\right\| \cdot \left\|\tilde{G}_\theta \tilde{G}_\Omega\right\| \geq \|Y\|.$$

Equivalently $\|X^{-1}\|^{-1} > \|Y\|$ and so $\|X^{-1}Y\| \leq \|X^{-1}\| \cdot \|Y\| < 1$.

Since $\delta_\nu(P_\alpha, P_\theta)$ and $\delta_\nu(P_\Omega, P_\theta)$ is each less than unity, the time-varying winding number condition is satisfied for $G_\alpha^* G_\theta$ and $G_\theta^* G_\Omega$. Therefore uniformly completely controllable and observable realizations of X and X^{-1} have the same number of positive as negative Lyapunov exponents, by Proposition III.7. By Proposition III.6, realizations of $I + X^{-1}Y$ and $[I + X^{-1}Y]^{-1}$ have the same Lyapunov exponents for the state evolution matrix A, that is the time-varying winding number condition is satisfied for $I + X^{-1}Y$. Hence it is satisfied for the product $X[I + X^{-1}Y] = G_\alpha^* G_\Omega$ by Theorem III.2. ■

Appendix II
Normalized Coprime Factorization for Time-Varying Systems

Proof [Proof of Theorem III.1; see also [22]]: We shall only prove the result relating to the right coprime fractional description. The proof for the left coprime fractional description is similar. The assumptions on $A(t)$, $B(t)$, $C(t)$, $D(t)$ guarantee that $\{A - BR^{-1}D^*C, B, C\}$ is a uniformly completely stabilizable and detectable triple (see [2] and note that despite the words discrete time in the title, as the reference makes clear, the ideas are valid for continuous time systems). By considering the problem of minimizing, for the system $\dot{x} = \left(A - BR^{-1}D^*C\right)x + Bu$, the quadratic index $\int_{t_0}^{\infty} u^*u + x^*C^*Cx dt$ with arbitrary $x(t_0)$, one concludes that the limiting solution of the Riccati equation (21) exists, is bounded and is stabilizing in the sense that

$$\begin{aligned}\dot{x} &= \left(A - BR^{-1}D^*C - BR^{-1}B^*X\right)x \\ &= (A + BF)x\end{aligned}$$

is exponentially stable. Next consider

$$\begin{aligned}\dot{x} &= (A + BF)x + BR^{-1/2}z \\ y_1 &= Fx + R^{-1/2}z \\ y_2 &= (C + DF)x + DR^{-1/2}z\end{aligned}$$

and note that all matrices are bounded. Realize that, in the light of the theorem statement, the operator linking z to $[y_1^* \quad y_2^*]^*$ is just G, the constituents of which are M and N. We must show that $P = NM^{-1}$, that is, if y_1 and y_2 are identified with u and y, the equations are consistent with the equations for P. Because $R^{-1/2}$ is nonsingular, the operator M has a well-defined inverse. Denote $M^{-1}u$ by z. Since $M : z \to u$ it is easily verified that $M^{-1} : u \to z$ is given by

$$\begin{aligned}\dot{x}_1 &= Ax_1 + Bu,\ x_1(t_0) = 0 \\ z &= -R^{-1/2}Fx_1 + R^{-1/2}u.\end{aligned}$$

Here, we are assuming that u, x_1 and z are defined on $[t_0, \infty)$. We can let t_0 go to $-\infty$ if we wish. The mapping $N : z \to y$ is given by

$$\begin{aligned}\dot{x}_2 &= (A + BF)x_2 + BR^{-1/2}z,\ x_2(t_0) = 0 \\ y &= (C + DF)x_2 + DR^{-1/2}z.\end{aligned}$$

If $z = -R^{-1/2}Fx_1 + R^{-1/2}u$ is the output of M^{-1} with input u then

$$\begin{aligned}\dot{x}_2 &= (A + BF)x_2 - BFx_1 + Bu. \\ \text{Evidently } \frac{d}{dt}(x_1 - x_2) &= (A + BF)(x_1 - x_2)\end{aligned}$$

and since $x_1(t_0) = x_2(t_0)$ it follows that $x_1 = x_2$ for all time t. Hence

$$\begin{aligned}y &= (C + DF)x_1 + DR^{-1/2}\left(-R^{-1/2}Fx_1 + R^{-1/2}u\right) \\ &= Cx_1 + Du\end{aligned}$$

that is, y agrees with the output of P when the input is u. This shows that $P = NM^{-1}$. We shall now establish the normalization property. Evidently

$$\begin{aligned}y_1^*y_1 + y_2^*y_2 &= x^*F^*Fx + 2x^*F^*R^{-1/2}x \\ &\quad + z^*R^{-1}z + x^*\left(C^* + F^*D^*\right)(C + DF)x \\ &\quad + 2x^*\left(C^* + F^*D^*\right)DR^{-1/2}z \\ &\quad + z^*R^{-1/2}D^*DR^{-1/2}z.\end{aligned}$$

Several lines of algebra show that

$$\begin{aligned}F^*F + \left(C^* + F^*D^*\right)(C + DF) &= C^*\tilde{R}^{-1}C \\ &\quad + XBR^{-1}B^*X \\ R^{-1} + R^{-1/2}D^*DR^{-1/2} &= I \\ \text{and } F^*R^{-1} + \left(C^*F^*D^*\right)DR^{-1/2} &= -XBR^{-1/2}.\end{aligned}$$

Thus

$$\begin{aligned}y_1^*y_1 + y_2^*y_2 &= x^*C^*\tilde{R}^{-1}Cx \\ &\quad + x^*XBR^{-1}B^*Xx - 2x^*XBR^{-1/2}z + z^*z.\end{aligned}$$

The Riccati (21) can also be written as

$$\begin{aligned}-\dot{X} &= X(A + BF) + \left(A^* + F^*B^*\right)X \\ &\quad + XBR^{-1}B^*X + C^*\tilde{R}^{-1}C\end{aligned}$$

and so

$$\begin{aligned}y_1^*y_1 + y_2^*y_2 &= -x^*\left[\dot{X} + X(A + BF) + \left(A^* + F^*B^*\right)X\right]x \\ &\quad - 2x^*XBR^{-1/2}z + z^*z \\ &= \frac{d}{dt}\left[x^*Xx\right] + z^*z.\end{aligned}$$

Now suppose that $z \in \mathcal{L}_2[t_0, \infty)$. The exponential stability of $A + BF$ and the boundedness of the various matrices ensures that $\lim_{t_f \to \infty} x(t_f) = 0$. With $x(t_0) = 0$ it follows that $\int_{t_0}^{\infty} y_1^*y_1 + y_2^*y_2 dt = \int_{t_0}^{\infty} z^*z dt$. This is the normalization property. ■

References

[1] B. D. O. Anderson and J. B. Moore, "New results in linear systems stability," *SIAM J Control*, pp. 398–414, 1969.

[2] ——, "Detectability and stabilizability of time-varying discrete-time linear systems," *SIAM J. Control Optim.*, vol. 19, pp. 20–32, 1981.

[3] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Upper Saddle River, NJ: Prentice-Hall, 1973.

[4] D. J. Clements and B. D. O. Anderson, *Singular Optimal Control: The Linear Quadratic Problem*. Berlin, Germany: Springer-Verlag, 1978.
[5] W. Coppel, *Dichotomies in Stability Theory*. Berlin, Germany: Springer-Verlag, 1978, vol. 629, Lecture Notes in Mathematics.
[6] J. C. Doyle, T. T. Georgiou, and M. C. Smith, "The parallel projection operator of a nonlinear feedback system," *Syst. Control Lett.*, vol. 20, pp. 79–85, 1993.
[7] A. K. El-Sakkary, "The gap metric: Robustness of stabilization of feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 240–247, Mar. 1985.
[8] T. T. Georgiou, "On the computation of the gap metric," *Syst. Control Lett.*, vol. 11, pp. 253–257, 1988.
[9] ——, "Differential stability and robust control of nonlinear systems," *Math. Control Signals Syst.*, pp. 289–306, 1993.
[10] T. T. Georgiou, A. Pascoal, and P. P. Khargonekar, "On the robust stabilizability of uncertain linear time-invariant plants using nonlinear time-varying controllers," *Automatica*, vol. 35, no. 5, pp. 617–624, Sept. 1987.
[11] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 673–687, June 1990.
[12] ——, "Robustness analysis of nonlinear feedback systems: An input-output approach," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1200–1219, Sept. 1997.
[13] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Upper Saddle River, NJ: Prentice-Hall, 1995.
[14] W. Hahn, *Theory and Application of Lyapunov's Direct Method*. Upper Saddle River, NJ: Prentice-Hall, 1963.
[15] J. Hammer, "Non-linear systems, stabilization and coprimeness," *Int. J. Control*, vol. 42, no. 1, pp. 1–20, 1985.
[16] ——, "Fraction representations of nonlinear systems: A simplified approach," *Int. J. Control*, vol. 46, no. 2, pp. 455–472, 1987.
[17] R. E. Kalman, "Contributions to the theory of optimal control," *Boletin de la Sociedad Matematica Mexicana*, pp. 102–119, 1960.
[18] H. W. Knobloch, A. Isidori, and D. Flockerzi, *Topics in Control Theory*. Boston, MA: Birkhäuser, 1993.
[19] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972.
[20] M. Araki and M. Saeki, "A quantitative condition for the well-posedness of interconnected dynamical system," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 569–577, May 1983.
[21] C. Perron, "Die stabilitätsfrage bei differentialgleichungen," *Math Zeitschrift*, vol. 30, pp. 702–738, 1930.
[22] R. Ravi, A. M. Pascoal, and P. P. Khargonekar, "Normalized coprime factorizations for linear time-varying systems," *Syst. Control Lett.*, vol. 18, no. 6, pp. 455–465, June 1992.
[23] M. Vidyasagar, "On the well-posedness of large-scale interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 413–421, June 1980.
[24] G. Vinnicombe, "Frequency domain uncertainty and the graph topology," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1371–1383, Sept. 1993.
[25] ——, *Uncertainty and Feedback $\mathcal{H}_\infty$ Uncertainty and the ν-Gap Metric*. London, U.K.: Imperial College Press, 2001.
[26] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.
[27] M. Vidyasagar, *Control Systems Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

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