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HOMOTOPY FOR THE ν-GAP*

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Abstract. The main result of this paper is to show that a "winding number condition" which relates to two strictly multivariable linear operators, and which is used in the definition of the Vinnicombe (ν-gap) metric, is equivalent to the existence of a multivariable homotopy in the same metric between the two operators. This allows a characterisation of the Vinnicombe metric which is independent of the linearity of the underlying operators, and suggests possible extension of the metric to nonlinear operators.

1. Introduction. This paper presents a result that provides a building block for the development of a control-relevant metric for nonlinear operators. Metrics on operator space define notions of distance between input-output maps and allows the quantification of concepts such as modeling approximation error and behavioural robustness.

Here we investigate a particular property of a specific metric on linear operators, known as the Vinnicombe (Nu-gap or ν-gap) metric [24], which is closely related to the gap metric [12, 13] that was introduced into the control literature as early as [8].

The Vinnicombe metric has an advantage over the gap metric in that it is less conservative in the following sense. Let $[P_0, C_0]$ be a stable plant-controller interconnection. Then there exists upper bounds $\bar{g}$ and $\bar{\nu}$ in, respectively, the gap metric $\delta_g(P_0, P)$ and the Vinnicombe metric $\delta_\nu(P_0, P)$, between plants with transfer function P_0 and P such that if the bounds are not met or exceeded, then $[P, C_0]$ is guaranteed to be stable. In the case of the Vinnicombe metric, for any prescribed $\epsilon > 0$, one can find a P such that $\delta_\nu(P_0, P) \leq \nu + \epsilon \leq 1$ and $[P, C_0]$ is not stabilising. Such a result is not available for the gap metric (see page 104 Chapter 4, [26]).

Motivated by the superior control relevant characteristics of the Vinnicombe metric to the gap metric in the case of linear operators, an ultimate goal, not achieved in this paper, is a nonlinear extension of the Vinnicombe metric. However, the definition of the Vinnicombe metric involves checking a property known as the "winding number condition" [24, 26] related to counting encirclements of the origin of a particular frequency domain function. Because it is defined in terms of an operator in the frequency domain, such a condition is an inherently linear systems concept. If the Vinnicombe metric is to be extended to nonlinear systems, then an obvious stepping stone is to characterise the winding number condition in a way that does not depend upon the linearity of the underlying operators.

There have been several candidate functionals proposed to extend the Vinnicombe

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metric to apply to nonlinear operators. In [3] a time-domain definition of the Vinnicombe metric was proposed, which would be suitable to apply to nonlinear operators, although a complete characterisation was lacking due to the challenges of defining the equivalent of a winding number. There, it was hypothesised that the winding number condition is equivalent to the existence of a particular multivariable homotopy, an hypothesis which this paper shows to be true, although for the case of scalar plants, the particular homotopy in question is multivariable, and arises by embedding the scalar plants in a higher-dimensional operator space.

In [1] a nonlinear metric is defined in terms of the linearisations of the operators about all operating points. This is similar in character to the extension of the (linear) gap metric to a nonlinear version in [14] based on earlier work on the parallel projection operator for nonlinear systems [7] and its relationship to the differential stabilisability of nonlinear feedback systems [11].

Reference [18] studies various alternative definitions for nonlinear operator metrics and presents some results on the relationship between them. Feedback stability and performance are characterised in terms of the well-posedness and norms of the parallel projection operators of [7]. It presents a formula for the nonlinear gap metric involving a product of left and right graph representations of nonlinear operators, but since that definition is for a generalisation of the classical (traditional) gap [13] rather than the Vinnicombe gap, the nonlinear equivalent of winding numbers is not touched upon.

Another definition of a metric which reduces to the linear Vinnicombe metric in the case of Linear Time Invariant (LTI) systems was also given in [25]. The definition was given in terms of a directed distance between the graph spaces of the two nonlinear operators to be compared. It was also shown that the existence of a gap-metric homotopy is a sufficient additional condition for closeness in the metric to guarantee closeness of closed loop performance. It was also asserted that in the case of LTI systems, the existence of such a homotopy can be determined by a simple winding number test.

The main result of this paper is to demonstrate that indeed the "winding number condition" for linear operators is equivalent to the existence of a homotopy between the two operators of interest. The existence of the homotopy is proved by explicit construction. This alternative characterisation of the condition, involving the existence of a homotopy, hence does not require the operators to be linear. This allows for a nonlinear version of the winding number condition to be developed, and consequently completes the task begun in [3] of defining one possible nonlinear extension of the Vinnicombe metric.

The rest of the paper is organised in the following manner. We first give a definition of the Vinnicombe metric for linear operators, as well as some alternative characterisations of the quantity. We then state the main theorem that the winding number condition used in the definition of the Vinnicombe metric is equivalent to the existence of a particular homotopy and prove that falsity of the winding number condition implies that such a homotopy cannot exist. Before proving the converse by construction, we introduce a number of results related to the problem of all-pass embedding and Nevanlinna interpolation which are derived from well-known results in the $\mathcal{H}_\infty$ literature [6, 15, 17, 23, 27]. Using these results, we then proceed at length to construct a homotopy and show that it satisfies the conditions in the statement of the main theorem, before finishing with some concluding remarks.

2. Preliminaries.

2.1. Notation and Terminology. For a transfer matrix G, the paraconjugate transfer matrix will be denoted by G^*, and is to be interpreted as $G(-s)^T$. For real rational transfer matrix G, the paraconjugate is the Hermitian conjugate (or conjugate transpose) on the imaginary axis. The notation $\|\cdot\|_\infty$ will refer to the $\mathcal{L}_\infty$ norm. The $\mathcal{H}_\infty$ norm will be explicitly distinguished as $\|\cdot\|_{\mathcal{H}_\infty}$. Transfer function matrices will usually be denoted in standard font X, although in Theorem 5.4 and its proof, they are denoted in bold $\mathbf{X}$, in order to provide notational distinction from a (static) complex matrix. Dimensions of matrices will sometimes appear as superscripts: $X^{p\times m}$ has p rows and m columns. While the notation for the maximum and minimum singular value of a matrix X is standard (and is given by $\overline{\sigma}(X)$ and $\underline{\sigma}(X)$ respectively), in this paper we also use the same notation to apply to a transfer matrix, in which case

$$\begin{aligned}\overline{\sigma}(\mathbf{G}) &\stackrel{\text{def}}{:=} \sup_\omega \overline{\sigma}[\mathbf{G}(j\omega)]\\ &= \|\mathbf{G}\|_\infty\\ \underline{\sigma}(\mathbf{G}) &\stackrel{\text{def}}{:=} \inf_\omega \underline{\sigma}[\mathbf{G}(j\omega)]\end{aligned}$$

The notation $F_l(P,K)$ represents the lower feedback interconnection $P_{11} + P_{21}K(I - P_{22}K)^{-1}P_{21}$ of compatibly dimensioned transfer matrices P and K as in [27].

The spaces $\mathcal{H}_\infty$ and $\mathcal{L}_\infty$ denote the set of transfer matrices which are respectively analytic on the closed right half-plane and bounded on the imaginary axis. The spaces $\mathcal{RH}_\infty$ and $\mathcal{RL}_\infty$, denote subspaces of respectively $\mathcal{H}_\infty$ and $\mathcal{L}_\infty$ with elements that are matrices that can be expressed as finite dimensional rational transfer functions with real coefficients.

As in [27] (see page 365) a transfer matrix $U \in \mathcal{RL}_\infty$ is called *paraunitary* if $U^*U = I$. A transfer matrix $\Pi \in \mathcal{RH}_\infty$ is called *inner* if it is paraunitary in addition to being stable, that is $\Pi^*\Pi = I$. A transfer matrix $\Omega \in \mathcal{RL}_\infty$ is called *all-pass* if it is square and paraunitary so that $\Omega^*\Omega = \Omega\Omega^* = I$ and therefore Ω^{-1} exists and equals Ω^* on the imaginary axis. A $p\times m$, $(p \le m)$ transfer matrix in $\mathcal{RH}_\infty$ is called *outer* if it has full row rank in the open right hand plane, that is, if it possesses a right-inverse that is analytic and in the *open* right-hand plane (that is, the right-inverse is stable, but possibly marginally stable and not necessarily proper). It is *strictly outer* if it possesses a right-inverse that is analytic and bounded in *closed* right-hand plane (that is, the right-inverse is proper and strictly stable). A transfer matrix is called *co-inner* (respectively *co-outer*) if it is the transpose of an *inner* (respectively *outer*) transfer matrix.

2.2. Transfer Matrix Factorisations. Transfer matrix factorisations will be used extensively in the following development. In this section we recall the existence of particular factorisations, appearing mainly in [19], [23], [27] for later use in the paper. The factorisations are numbered in order that we will be able to refer to them by number later on.

1. Every "tall" ($p \times m$ with $p \geq m$) transfer matrix $T(s) \in \mathcal{RH}_\infty$, such that $T^*T > 0$ on the imaginary axis, has a factorisation [27]

$$T(s) = \Pi(s)\bar{T}(s) \tag{2.1}$$

where $\bar{T}$ is square and strictly outer (and stable) and $\Pi(s)$ is (tall, $p \times m$) inner, and contains all the (strictly) right-hand-plane zeros of T. Such an inner-outer factorisation is unique up to a constant orthogonal factor.

2. Similarly (by duality) every "fat" $p \times m$, ($p \leq m$) transfer matrix $F(s) \in \mathcal{RH}_\infty$, such that $FF^* > 0$ on the imaginary axis, has a factorisation

$$F(s) = \bar{F}(s)\Xi(s) \tag{2.2}$$

where $\bar{F}$ is square and strictly outer (stable) and $\Xi(s)$ is (fat, $p \times m$) and co-inner.

3. Every full-rank "fat" $p \times m$, ($p \leq m$) transfer matrix $F(s) \in \mathcal{RH}_\infty$ can be factorised as $\begin{bmatrix} \hat{F} & 0 \end{bmatrix} H$ where $\hat{F}$ is square and H is square, stable and minimum-phase.

This can be easily seen [19] from the Smith form of F over the principal ideal domain of proper stable transfer matrices. If $\hat{F}$ has full rank and no imaginary axis zeros, then we can use the above inner-outer factorisation (2.1) to express $\hat{F}(s) = \Theta(s)\tilde{F}(s)$ where Θ is square and inner (and therefore all-pass) and $\tilde{F}$ is outer. It then follows, that if $FF^* > 0$ on the imaginary axis, then it may also be factorised as

$$F(s) = \Theta(s)\bar{F}(s) \tag{2.3}$$

where $\Theta(s)$ is stable and all-pass (square) and $\bar{F} = \tilde{F}H$ is fat ($p \times m$) and outer.

4. Similarly every "tall" $p \times m (p \geq m)$ transfer matrix $T(s) \in \mathcal{RH}_\infty$, such that $T^*T > 0$ on the imaginary axis, has a factorisation

$$T(s) = \bar{T}(s)\Theta(s) \tag{2.4}$$

where $\bar{T}$ is "tall" ($p \times m$) and co-outer, and $\Theta(s)$ is stable all-pass (square).

5. Any "tall" transfer matrix $T(s) \in \mathcal{RL}_\infty$ may be expressed as a right coprime factorisation [23] over the ring of proper stable transfer matrices as $T(s) =$

$\mathsf{N}(s)\mathsf{M}(s)^{-1}$. Inner-outer factorisation of $\mathsf{M}(s) = \Theta_{\mathsf{M}}(s)\bar{\mathsf{M}}(s)$ is in equation (2.3) results in $\mathsf{T}(s) = \mathsf{N}(s)\bar{\mathsf{M}}(s)^{-1}\Theta_{\mathsf{M}}(s)^{-1}$, where Θ is stable all-pass. We can further define $\hat{\mathsf{N}}(s) = \mathsf{N}(s)\bar{\mathsf{M}}(s)^{-1} \in \mathcal{RH}_\infty$ so that

$$\mathsf{T}(s) = \hat{\mathsf{N}}(s)\Theta_{\mathsf{M}}(s)^* \tag{2.5}$$

is a right coprime factorisation where the denominator matrix is stable all-pass.

If we further assume that $\mathsf{T}^*\mathsf{T} > 0$ on the imaginary axis then $\hat{\mathsf{N}}(s)$ is also full rank on the imaginary axis. The tall matrix $\hat{\mathsf{N}}(s)$ may be further factored as a co-outer-inner factorisation $\hat{\mathsf{N}}(s) = \bar{\mathsf{N}}(s)\Theta_{\mathsf{N}}(s)$ (see equation (2.4)) with $\Theta_{\mathsf{N}}(s)$ stable and all-pass, and $\bar{\mathsf{N}}(s)$ co-outer, so that

$$\mathsf{T} = \bar{\mathsf{N}}(s)\Theta_{\mathsf{N}}(s)\Theta_{\mathsf{M}}(s)^*. \tag{2.6}$$

Here $\Theta_{\mathsf{N}}(s)$ contains the right hand plane zeros of $\mathsf{T}(s)$ and $\Theta_{\mathsf{M}}(s)^*$ contains its right hand plane poles.

6. Similarly, a fat transfer matrix $\mathsf{F}(s) \in \mathcal{RL}_\infty$ has a factorisation

$$\mathsf{F}(s) = \Theta_{\mathsf{Fp}}(s)^*\hat{\mathsf{F}}(s) \tag{2.7}$$

where $\Theta_{\mathsf{Fp}}(s)$ is stable and all-pass and $\hat{\mathsf{F}}(s)$ is outer. If $\mathsf{F}(s)$ is full-rank on the imaginary axis, then

$$\mathsf{F}(s) = \Theta_{\mathsf{Fp}}(s)^*\Theta_{\mathsf{Fz}}(s)\bar{\mathsf{F}}(s) \tag{2.8}$$

where $\Theta_{\mathsf{Fz}}(s)$ is stable and all-pass and $\bar{\mathsf{F}}(s)$ is outer.

7. If an *all-pass* matrix $\Omega \in \mathcal{RL}_\infty$ is factorised in the above manner then

$$\begin{aligned}\Omega(s) &= \tilde{\Theta}_p(s)^{-1}\tilde{\Theta}_z(s)\\ &= \tilde{\Theta}_p(s)^*\tilde{\Theta}_z(s)\end{aligned} \tag{2.9}$$

where both $\tilde{\Theta}_p(s)$ and $\tilde{\Theta}_z(s)$ are stable and all-pass and $\tilde{\Theta}_p(s)^*$ contains the right half-plane poles and $\tilde{\Theta}_z(s)$ contains the right half-plane zeros of Ω.

8. Alternatively

$$\begin{aligned}\Omega(s) &= \Theta_z(s)\Theta_p(s)^{-1}\\ &= \Theta_z(s)\Theta_p(s)^*\end{aligned} \tag{2.10}$$

where $\Theta_p(s)^*$ contain the right half-plane poles and $\Theta_z(s)$ contains the right half-plane zeros of $\Omega(s)$, and $\Theta_p(s)$ and $\Theta_z(s)$ are each *stable* and all-pass. These factorisations (which are coprime factorisations, with constituents which are stable and all-pass) are also unique up to a constant orthogonal factor.

9. Finally, the paraconjugate of a full-rank fat transfer matrix $F \in \mathcal{RL}_\infty$ may be factorised as an inner-outer factorisation $F^*(s) = \Pi_F(s)\hat{F}(s)$ where $\Pi_F(s)$ is inner and $\hat{F}(s)$ is square and outer. We can perform factorisation as in equation (2.5) on the square transfer matrix $\hat{F}^* = \tilde{F}(s)\Theta_F(s)^*$ where $\tilde{F}(s)$ is stable and $\Theta_F(s)^*$ is anti-stable and all-pass.
   Setting $\Pi_{Fp}(s) = \Pi_F(s)\Theta_F(s)$ gives us a factorisation of the original fat transfer matrix F as

$$F(s) = \tilde{F}(s)\Pi_{Fp}(s)^*, \tag{2.11}$$

   where $\tilde{F}(s) \in \mathcal{RH}_\infty$ and square, and Π_{Fp} is inner (so that $\Pi_{Fp}(s)^*$ is anti-stable and its transpose is paraunitary). If F is full rank on the imaginary axis then so is $\tilde{F}(s)$ which can be factorised as in equation (2.3) or (2.4) to give

$$F(s) = \Theta_{FzR}\bar{F}_R(s)\Pi_{Fp}(s)^*, \tag{2.12}$$

$$F(s) = \bar{F}_L(s)\Theta_{FzL}\Pi_{Fp}(s)^*, \tag{2.13}$$

   where $\bar{F}_R(s)$ and $\bar{F}_L(s)$ are square and outer and Θ_{FzR} and Θ_{FzL} are stable and all-pass.

3. The Linear Vinnicombe Metric.

3.1. Definition. Let $P_\alpha^{p\times m}, P_\beta^{p\times m}$ be two real rational transfer functions matrices of the same dimension. The Vinnicombe metric [26] (also known as the Nu-gap or ν-gap metric) is a measure of the distance between P_α and P_β. It can be defined as follows. The chordal distance between P_α and P_β at a frequency $\hat{\omega}$ is given by

$$\kappa(P_\alpha, P_\beta, \hat{\omega}) \stackrel{\text{def}}{:=} \lim_{\omega\to\hat{\omega}} \bar{\sigma}\Big\{[I + P_\beta(j\omega)P_\beta(j\omega)^*]^{-\frac{1}{2}}\,[P_\beta(j\omega) - P_\alpha(j\omega)]\,, \\ [I + P_\alpha(j\omega)^*P_\alpha(j\omega)]^{-\frac{1}{2}}\Big\},$$

and

$$\bar{\kappa}(P_\alpha, P_\beta) \stackrel{\text{def}}{:=} \sup_{\omega} \kappa(P_\alpha, P_\beta, \omega). \tag{3.1}$$

In the definition of the chordal distance, the inverse square root $X^{-\frac{1}{2}}$ is understood to be a matrix square root of the inverse square matrix X^{-1}, where X is positive definite Hermitian. The reason for the limit operation in the definition is to account for the possibility that either P_α or P_β has poles on the imaginary axis.

The Vinnicombe metric distance between P_α and P_β is then defined as

$$\delta_\nu(P_\alpha, P_\beta) = \bar{\kappa}(P_\alpha, P_\beta)$$

provided the following two conditions are satisfied:

$$\det[I + P_\alpha P_\beta^*](j\omega) \neq 0, \quad \forall\omega, \text{ and} \tag{3.2}$$

$$\text{wno}\left[\det(I + P_\alpha P_\beta^*)\right] + \breve{\eta}(P_\alpha) - \bar{\eta}(P_\beta) = 0. \tag{3.3}$$

If the conditions of equations (3.2) and (3.3) are not both satisfied then $\delta_\nu(P_\alpha, P_\beta) = 1$. In the above, $\breve{\eta}(P_i)$ denotes the number of poles of P_i in the *open* right half complex plane $\mathrm{Re}(s) > 0$ and $\bar{\eta}(P_j)$ is the number of poles of P_j in the *closed* right half-plane $\mathrm{Re}[s] \geq 0$, counted according to multiplicity. indented, if necessary, into the right half-plane around any imaginary axis poles of P_α and P_β.

The symbol $\mathrm{wno}[f(\cdot)]$ denotes the winding number of holomorphic complex function $f(\cdot)$, defined as the number of net encirclements of the origin in the *clockwise* direction of $f(s)$ as s as traverses the *clockwise* Nyquist D-contour from $-j\infty$ to $j\infty$ on the imaginary axis, indented, if necessary, into the right half-plane around any imaginary axis poles of P_α and P_β, and then around at infinity in the positive complex plane [26].

REMARK 3.1. *For the clockwise Nyquist traversal, the net change in the argument of* $f(s)$ *will be* 2π rad *in the clockwise direction for each non-minimum-phase zero and in a counter-clockwise direction for each unstable pole of* $f(s)$ *[9]. The winding number is therefore the number of right half-plane zeroes of* $f(s)$ *minus the number of right half-plane poles.*

There are various equivalent expressions for the Vinnicombe metric. We draw attention here to some. Let $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1} \tilde{N}_i$ denote normalised right and left coprime fractional descriptions [23] of P_i. Define

$$
\begin{aligned}
G_i^{(p+m)\times m} &= \begin{bmatrix} N_i \\ M_i \end{bmatrix}, \\
\text{and } \tilde{G}_i^{p\times(p+m)} &= \begin{bmatrix} -\tilde{M}_i & \tilde{N}_i \end{bmatrix}.
\end{aligned} \tag{3.4}
$$

It follows that $\begin{bmatrix} G_i & \tilde{G}_i^* \end{bmatrix}$ is all-pass. The chordal distance at a frequency ω is given by

$$
\begin{aligned}
\kappa(P_\alpha, P_\beta, \omega) &= \bar{\sigma}\left[\tilde{G}_\beta(j\omega) G_\alpha(j\omega)\right] \\
&= \bar{\sigma}\left\{\tilde{M}_\beta(j\omega)\left[P_\alpha(j\omega) - P_\beta(j\omega)\right] M_\alpha(j\omega)\right\}
\end{aligned} \tag{3.5}
$$

where the last equality holds only for $j\omega$ not a pole of either P_α or P_β. The Vinnicombe metric may then be alternatively expressed as

$$
\delta_\nu(P_\alpha, P_\beta) = \|\tilde{G}_\beta G_\alpha\|_\infty = \|\tilde{G}_\alpha G_\beta\|_\infty \tag{3.6}
$$

(where $\|\cdot\|_\infty$ represents the $\mathcal{L}_\infty$ norm), provided that

$$
\det\left[G_\beta^* G_\alpha(j\omega)\right] \neq 0 \text{ for all } \omega \text{ and} \tag{3.7}
$$

$$
\mathrm{wno}\left\{\det\left[G_\beta^* G_\alpha\right]\right\} = 0. \tag{3.8}
$$

Otherwise $\delta_\nu(P_\alpha, P_\beta) = 1$. Note that conditions (3.2) and (3.7) are provably equivalent, as are conditions (3.3) and (3.8).

A third equivalent characterisation of the ν-gap metric is as follows. For $\hat{G}_\beta = \begin{bmatrix} \hat{N}_\beta^* & \hat{M}_\beta^* \end{bmatrix}^*$ where $P_\beta = \hat{N}_\beta \hat{M}_\beta^{-1}$ is *any* right coprime fractional description of P_β, that is, one which is not necesarily normalised, we have [24]

$$\delta_\nu(P_\alpha, P_\beta) = \inf_{\substack{Q,\ Q^{-1} \in \mathcal{RL}_\infty \\ \text{wno}[\det(Q)]=0}} \|G_\alpha - \hat{G}_\beta Q\|_\infty .$$

Of course, it is a nontrivial fact that the metric properties hold for $\delta_\nu(P_\alpha, P_\beta)$ [24, 26], although the triangle inequality is the only property which requires much attention to prove.

3.2. Properties of the Vinnicombe Metric. We now give two simple but useful results which relate the maximum and minimum singular values of various transfer matrices derived from the normalised coprime fraction descriptions of the two plants.

LEMMA 3.2. *Let P_α and P_β be two transfer matrices with respective right normalised coprime factorisations, G_α and G_β defined by equation (3.4). The chordal distance between P_α and P_β of equation (3.1), is related to the minimum singular value of $G_\beta(j\omega)^* G_\alpha(j\omega)$ as follows.*

$$\kappa(P_\alpha, P_\beta, \omega)^2 + \{\underline{\sigma}\,[G_\beta(j\omega)^* G_\alpha(j\omega)]\}^2 = 1.$$

Proof. Since $\begin{bmatrix} \tilde{G}_\beta^* & G_\beta \end{bmatrix}$ is all-pass, $\tilde{G}_\beta^* \tilde{G}_\beta + G_\beta G_\beta^* = I$ and so

$$\left[\tilde{G}_\beta(j\omega) G_\alpha(j\omega)\right]^* \tilde{G}_\beta(j\omega) G_\alpha(j\omega) + [G_\beta(j\omega)^* G_\alpha(j\omega)]^* \, G_\beta(j\omega)^* G_\alpha(j\omega) = G_\alpha(j\omega)^* G_\alpha(j\omega) = I.$$

Then it follows that

$$\left\{\bar{\sigma}\left[\tilde{G}_\beta(j\omega) G_\alpha(j\omega)\right]\right\}^2 + \{\underline{\sigma}\,[G_\beta(j\omega)^* G_\alpha(j\omega)]\}^2 = 1.$$

The lemma statement follows from equation (3.5). □

COROLLARY 3.3. *The condition that the square matrix $G_\beta^* G_\alpha(j\omega)$ has the property that* $\det[G_\beta^* G_\alpha(j\omega)] = 0$ *for some ω, is equivalent to $\kappa(P_\alpha, P_\beta, \omega) = \bar{\sigma}\left[\tilde{G}_\beta(j\omega)\right.$ $\left.G_\alpha(j\omega)\right] = 1$ at that ω and hence $\bar{\kappa}(P_\alpha, P_\beta) = 1$.*

4. Main Result. The main result in this paper is to show that the winding number condition (equations (3.3) and (3.8)) is equivalent to the existence of a homotopy, parametrised by, say λ, from P_α to P_β, such that the Vinnicombe distance $\delta_\nu(P_\alpha, P_\lambda)$ is arbitrarily close to monotonically non-decreasing and hence always strictly less than unity.

For the case that $P_\alpha = p_\alpha$ and $P_\beta = p_\beta$ are scalar, the equivalence only holds if multivariable homotopies are allowed, by embedding the scalar operators p_α and p_β in a higher dimensional operator, in for example, the obvious way by $p_\alpha \to \begin{bmatrix} p_\alpha & 0 \end{bmatrix}^T$. This is formalised in the following theorem.

If only scalar homotopies p_λ from p_α to p_β are considered, then in addition to the satisfaction of the winding number condition, an extra condition is necessary for the existence of a homotopy, namely, that the p_α and p_β end-points must share the same Cauchy index (see below). If they have a different Cauchy index, then no scalar homotopy between them exists, irrespective of whether the winding number condition holds.

THEOREM 4.1. *Let real rational* $p \times m$ *transfer functions* P_α *and* P_β *be given, with at least one of* p *and* m *strictly greater than one and with* $\delta_\nu(P_\alpha, P_\beta) < 1$*. Then for any given* η *(which will provide an upper bound for departure from monotonicity), there exists a Vinnicombe metric homotopy, parametrised by* $\lambda \in [\lambda_\alpha, \lambda_\beta]$*, given by* P_λ*, varying from* P_α *to* P_β *such that the following properties hold.*

- *Endpoint properties:* $P_\lambda = P_\alpha$ *for* $\lambda = \lambda_\alpha$*, and* $P_\lambda = P_\beta$ *for* $\lambda = \lambda_\beta$*.*
- *Vinnicombe Continuity Property: For every* $\hat{\lambda} \in [\lambda_\alpha, \lambda_\beta]$ *and* $\epsilon > 0$ *there exists* δ *such that* $\delta_\nu(P_{\hat{\lambda}}, P_\lambda) < \epsilon$ *for all* $\lambda \in [\lambda_\alpha, \lambda_\beta]$ *with* $\left|\hat{\lambda} - \lambda\right| < \delta$*.*
- *Subunitary Property:* $\bar{\kappa}(P_\alpha, P_\lambda) = \sup_\omega \kappa(P_\alpha, P_\lambda, \omega) < 1$ *for all* $\lambda \in [\lambda_\alpha, \lambda_\beta]$*.*
- *Monotonicity Property (Arbitrary Closeness to):* $\bar{\kappa}(P_\alpha, P_\lambda) \geq \bar{\kappa}(P_\alpha, P_{\hat{\lambda}}) - \eta$ *for all* $\hat{\lambda}, \lambda \in [\lambda_\alpha, \lambda_\beta]$ *such that* $\hat{\lambda} \leq \lambda$*.*

Conversely, if there exists a homotopy with the Endpoint and Vinnicombe Continuity properties as well as the Subunitary property, then $\delta_\nu(P_\alpha, P_\beta) < 1$*, which is equivalent to saying that if* $\delta_\nu(P_\alpha, P_\beta) = 1$ *then no homotopy satisfying those three properties exists.*

REMARK 4.2. *If* $P_\alpha = p_\alpha$ *and* $P_\beta = p_\beta$ *are single-input single-output (that is, if* $p = m = 1$*) and strictly proper, then a homotopy satisfying the above four properties can exist only if* $\delta_\nu(p_\alpha, p_\beta) < 1$ *and* $\mathcal{I}_{-\infty}^{+\infty}[p_\alpha] = \mathcal{I}_{-\infty}^{+\infty}[p_\beta]$*, where* $\mathcal{I}_l^u[f(\cdot)]$ *is the Cauchy index [10] of a real rational function* $f(x) : \mathbb{R} \to \mathbb{R}$ *over an interval* $[l, u]$ *of the real line (where either or both* l *or* u *can be at infinity) and is defined as* $\mathcal{I}_l^u[f(\cdot)] = \mathcal{P}_l^u[f(\cdot)] - \mathcal{N}_l^u[f(\cdot)]$*where* $\mathcal{P}_l^u$ *is the number of (positive) jumps that* $f(x)$ *makes from* $-\infty$ *to* $+\infty$ *as* s *increases in the open interval* (l, u) *and* $\mathcal{N}_l^u$ *is the number of (negative) jumps that* $f(x)$ *makes from* $+\infty$ *to* $-\infty$ *over the interval. This is demonstrated in the proof of Theorem 4.1 in [2]. It remains an open question whether equality of the Cauchy index of the end-point scalar plants is sufficient to deduce the equivalence between the winding number condition and the existence of a scalar homotopy.*

Before giving a detailed proof of the theorem, quite some development will be required. However, it is reasonably straightforward to prove the converse part of the theorem, both for multivariable and scalar plants, which we do in the immediately following text. The sufficiency part of the theorem statement will be proven by lengthy and detailed construction in Section 6.4 and following.

Proof. [Converse part of Theorem 4.1] In this part of the proof we show that if there exists a homotopy with the Endpoint and Vinnicombe Continuity properties as well as the Subunitary property, then the Winding Number and Determinant

Conditions hold and hence $\delta_\nu(P_\alpha, P_\beta) < 1$.

We will first show that for an arbitrary $\bar{\lambda}_\theta \in [\lambda_\alpha, \lambda_\beta]$ with $\delta_\nu(P_\alpha, P_\theta) < 1$, there exists an open λ ball around $\bar{\lambda}_\theta$ such that $\det[G_\alpha^* G_\lambda] \neq 0$ and has the same winding number as $\det[G_\alpha^* G_\theta]$ for all λ in the intersection of the open ball and the interval $[\lambda_\alpha, \lambda_\beta]$ (note that $\det[G_\alpha^* G_\theta] \neq 0$ because $\delta_\nu(P_\alpha, P_\theta) < 1$).

To demonstrate this, let $\bar{\lambda}_\theta$ be any scalar in the interval $[\lambda_\alpha, \lambda_\beta]$ with $\delta_\nu(P_\alpha, P_\theta) < 1$ and define the positive scalar $\epsilon(\theta)$ by $1 - 2\epsilon = \delta_\nu(P_\alpha, P_\theta) < 1$. Assume that G_λ is a Vinnicombe metric homotopy in λ passing through G_θ so that there exists a $\delta > 0$ such that the conditions $\left|\lambda - \bar{\lambda}_\theta\right| < \delta$ and $\lambda \in [\lambda_\alpha, \lambda_\beta]$ implies that $\delta_\nu(P_\theta, P_\lambda) \leq \epsilon$. Now define

$$\Delta_{\theta\lambda} \stackrel{\text{def}}{:=} G_\lambda - G_\theta G_\theta^* G_\lambda.$$

We can bound the magnitude of $\Delta_{\theta\lambda}$ because

$$\begin{aligned} \Delta_{\theta\lambda}^* \Delta_{\theta\lambda} &= I - G_\lambda^* G_\theta G_\theta^* G_\lambda \\ &= (\tilde{G}_\theta G_\lambda)^* (\tilde{G}_\theta G_\lambda), \end{aligned}$$

so that

$$\|\Delta_{\theta\lambda}\|_\infty = \delta_\nu(P_\theta, P_\lambda) \leq \epsilon.$$

By Lemma 3.2 we can see that

$$\begin{aligned} \inf_\omega \underline{\sigma}\{[G_\theta(j\omega)]^* G_\lambda(j\omega)\} &= \sqrt{1 - \delta_\nu(P_\theta, P_\lambda)^2}, \\ \bar{\sigma}\left[(G_\theta^* G_\lambda)^{-1}\right] &\leq \frac{1}{\sqrt{1 - \epsilon^2}}. \end{aligned}$$

Now let us investigate the winding number of $\det(G_\alpha^* G_\lambda)$.

$$\begin{aligned} G_\alpha^* G_\lambda &= G_\alpha^* (G_\theta G_\theta^* G_\lambda + \Delta_{\theta\lambda}), \\ &= G_\alpha^* (G_\theta + \Delta_{\theta\lambda} (G_\theta^* G_\lambda)^{-1}) G_\theta^* G_\lambda, \\ \text{wno}\{\det[G_\alpha^* G_\lambda]\} &= \text{wno}\{\det[G_\alpha^* (G_\theta + \Delta_{\theta\lambda} (G_\theta^* G_\lambda)^{-1})]\} + \text{wno}\{\det[G_\theta^* G_\lambda]\}, \\ &= \text{wno}\{\det[G_\alpha^* (G_\theta + \Delta_{\theta\lambda} (G_\theta^* G_\lambda)^{-1})]\}, \end{aligned} \tag{4.1}$$

and note that $\|\Delta_{\theta\lambda}(G_\theta^* G_\lambda)^{-1}\|_\infty \leq \epsilon(1 - \epsilon^2)^{-\frac{1}{2}}$. By the pointwise-in-frequency continuity of the right hand side of equation (4.1) with respect to λ, there exists an open ball around $\bar{\lambda}_\theta$ such that $\det[G_\alpha^* G_\lambda] \neq 0$ and has the same winding number as $\det[G_\alpha^* G_\theta]$ (where $\det[G_\alpha^* G_\theta] \neq 0$ because $\delta_\nu(P_\alpha, P_\theta) < 1$). Note that these properties hold for arbitrary $\bar{\lambda}_\theta \in [\lambda_\alpha, \lambda_\beta]$.

We now specify a particular value of $\bar{\lambda}_\theta$. Specifically, we define λ_θ as

$$\lambda_\theta \stackrel{\text{def}}{:=} \sup\left\{\bar{\lambda} \leq \lambda_\beta : \det[G_\alpha^* G_\lambda] \neq 0 \text{ and } \text{wno}\{\det[G_\alpha^* G_\lambda]\} = 0,\ \forall \lambda \in [\lambda_\alpha, \bar{\lambda}]\right\}.$$

It is obvious that λ_θ exists because clearly $\det[G_\alpha^* G_\lambda] \neq 0$ and $\text{wno}\{\det[G_\alpha^* G_\lambda]\} = 0$ for all λ in the singleton set $[\lambda_\alpha]$, and indeed, by the Vinnicombe continuity property, for some finite interval close to λ_α so that the set in the definition of λ_θ is non-empty. Furthermore, because of the Subunitary property, it also follows that $\delta_\nu(P_\alpha, P_\lambda) < 1$ for $\lambda = \lambda_\theta$.

From the definition of λ_θ it follows that $\det[G_\alpha^* G_\lambda] \neq 0$ and $\mathrm{wno}\{\det[G_\alpha^* G_\lambda]\} = 0$, for all λ on the semi-open interval $[\lambda_\alpha, \lambda_\theta)$. We now show, by contradiction, that $\lambda_\theta = \lambda_\beta$. If it were the case that $\lambda_\theta < \lambda_\beta$, then at $\lambda = \lambda_\theta$ either $\det[G_\alpha^* G_\theta] = 0$, which contradicts the Subunitary Property, or the Winding Number Condition fails. The failure of the Winding Number Condition at $\lambda = \lambda_\theta$ is in contradiction with the argument in the previous paragraph, that establishes the existence of an open λ ball around λ_θ, with $\det[G_\alpha^* G_\lambda] \neq 0$ having the same winding number as $\det[G_\alpha^* G_\theta]$. Therefore, $\lambda_\theta \geq \lambda_\beta$ and the Determinant and Winding Number Conditions must hold for $G_\alpha^* G_\lambda$ for all λ on the semi-open interval $[\lambda_\alpha, \lambda_\beta)$. In addition, since the Vinnicombe Continuity and Subunitary Properties also hold *at* $\lambda = \lambda_\beta$, we have the Determinant and Winding Number Conditions holding on the *closed* interval $[\lambda_\alpha, \lambda_\beta]$.

The Subunitary and Endpoint properties imply that $\bar{\kappa}(P_\alpha, P_\beta) < 1$, which, with the Winding Number and Determinant Conditions, gives $\delta_\nu(P_\alpha, P_\beta) < 1$ as required. The contrapositive statment is that if $\delta_\nu(P_\alpha, P_\beta) = 1$ then no homotopy satisfying the Endpoint, Vinnicombe Continuity and Subunitary Properties exists. □

We will demonstrate the sufficiency part of the theorem statement by construction. However, we first need several results which are important for solving the Nevanlinna interpolation problem, which will turn up in the procedure to construct this homotopy.

5. Some Mathematical Machinery. We present several results reasonably well understood in the $\mathcal{H}_\infty$ literature, which will be required in the construction of our homotopy.

LEMMA 5.1. *Let* T_{11}, T_{12}, T_{21} *be given with respective dimensions* $p \times m$, $p \times p$ *and* $m \times m$ *such that* $T_{11}, T_{12}, T_{21}, T_{12}^{-1}, T_{21}^{-1} \in \mathcal{RL}_\infty$. *We consider only tall (or square)* T_{11} *with* $p \geq m$. *Consider the model matching problem*

$$\gamma^\star = \inf_{Q \in \mathcal{RH}_\infty} \|E\|_{\mathcal{H}_\infty}$$

where E *is defined as*

$$E^{p\times m} = T_{11}^{p\times m} - T_{12}^{p\times p} Q^{p\times m} T_{21}^{m\times m}.$$

Then an optimal $Q \in \mathcal{RH}_\infty$ *exists, call it* $Q^\star$ *and it results in an error function* $E = T_{11} - T_{12} Q^\star T_{21}$, *which is a scalar multiple* $\gamma^\star$ *of an paraunitary function [27], that is* $E^* E = \gamma^{\star 2} I$. *Furthermore, if the minimum achievable norm is* $\gamma^\star$ *then for any* $\gamma \geq \gamma^\star$ *it is possible to find* $Q \in \mathcal{RH}_\infty$ *such that the error* $E = T_{11} - T_{12} Q T_{21}$ *has the property that* $E^* E = \gamma^2 I$.

Proof. This is proved using a standard solution [23] to the model matching problem by converting it to an all-pass embedding problem [17]. See Appendix A.1. □

We will next show the equivalence of the above $\mathcal{H}_\infty$ model matching problem to a corresponding Nevanlinna interpolation problem. First however, we will present a lemma about the properties of certain all-pass transfer matrices with prescribed zeros and zero directions.

LEMMA 5.2. *Let a set $\mathcal{Z}$ of K distinct complex numbers z_k for $k = 1 \ldots K$, with $Re(z_k) > 0$ be given, as well as a set $\mathcal{L}$ of $p \times 1$ unit norm direction vectors l_k, where if z_k is real, then l_k is real and if z_k is not real, then there is some $z_j = \overline{z_k} \in \mathcal{Z}$ with corresponding unit norm direction vector $l_j = \overline{l_k} \in \mathcal{L}$.*

Then the following is true.

1. *There exists $\Theta_Z(s) \in \mathcal{RH}_\infty^{p\times p}$, a stable real-rational all-pass (square) transfer matrix of McMillan degree [19] K such that $l_k^* \Theta_Z(z_k) = 0$ for all z_k, that is, with zeros $z_k, k = 1 \ldots K$ with output zero directions l_k^*.*
2. (a) *Furthermore, Θ_Z can be factored, for each k, as*

$$\Theta_Z(s) = \Theta_{zk}(s)\Theta_{Z/k}(s),$$

where

$$\Theta_{zk}(s) = \begin{bmatrix} l_k & L_k \end{bmatrix} \begin{bmatrix} \frac{s-z_k}{s+\overline{z_k}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} l_k^* \\ L_k^* \end{bmatrix}$$

and L_k is a (constant) unitary completion of l_k so that $\begin{bmatrix} l_k & L_k \end{bmatrix}$ is a constant unitary matrix. Also $\Theta_{Z/k}(s)$ is a rational (though not necessarily real rational) stable matrix with zeros z_j for $j = 1, \ldots, k-1, k+1, \ldots, K$.

(b) *Furthermore, if $z_j = \overline{z_k}$ for complex z_k it holds that*

$$\Theta_{zj}(s) = \begin{bmatrix} \overline{l_k} & \overline{L_k} \end{bmatrix} \begin{bmatrix} \frac{s-\overline{z_k}}{s+z_k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} l_k^T \\ L_k^T \end{bmatrix},$$

that is, $l_j = \overline{l_k}$ and $L_j = \overline{L_k}$.

3. (a) *An alternative factorisation of the following form exists.*

$$\Theta_Z(s) = \tilde{\Theta}_{Z/k}(s)\tilde{\Theta}_{zk}(s),$$

where

$$\tilde{\Theta}_{zk}(s) = \begin{bmatrix} \tilde{l}_k & \tilde{L}_k \end{bmatrix} \begin{bmatrix} \frac{s-z_k}{s+\overline{z_k}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{l}_k^* \\ \tilde{L}_k^* \end{bmatrix}.$$

Here $\begin{bmatrix} \tilde{l}_k & \tilde{L}_k \end{bmatrix}$ is a constant unitary matrix (in general, not equal to $\begin{bmatrix} l_k & L_k \end{bmatrix}$) and again, $\tilde{\Theta}_{Z/k}(s)$ is a rational (though, again, not necessarily real rational) stable matrix.

(b) Furthermore, if $z_j = \overline{z_k}$ *for complex* z_k *it holds that*

$$\tilde{\Theta}_{zj}(s) = \begin{bmatrix} \overline{\tilde{l}_j} & \overline{\tilde{L}_k} \end{bmatrix} \begin{bmatrix} \frac{s-\overline{z_k}}{s+z_k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{l}_k^T \\ \tilde{L}_k^T \end{bmatrix},$$

that is, $\tilde{l}_j = \overline{\tilde{l}_k}$ *and* $\tilde{L}_j = \overline{\tilde{L}_k}$.

4. *Finally let* G *be a* $p \times m$ *complex matrix and choose* $z_k \in \mathcal{Z}$ *not real, with* j *as the index corresponding to the complex conjugate of* z_k*, that is* $z_j = \overline{z_k} \in \mathcal{Z}$*. Define*

$$T_{kj}(s) = \tilde{\Theta}_{Z/k}(s)\tilde{\Theta}^{-1}_{Z/k}(z_k)G + \tilde{\Theta}_{Z/j}(s)\tilde{\Theta}^{-1}_{Z/j}(\overline{z_k})\overline{G}.$$

Then T_{kj} *is real rational.*

Proof. A procedure for constructing an all-pass matrix with the first three properties is given in Chapter 6 of [27], see also [5] or pages 329-331 of [4]. See Appendix A.2 for proof both of the fact that $\Theta_Z(s)$ can be chosen as real rational, and of Property 4. □

We are now able to present the following corollary of Lemma 5.2 and Lemma 5.1.

COROLLARY 5.3. *Let a set of distinct complex numbers* z_k *for* $k = 1 \dots K$*, with* *Re*$(z_k) > 0$ *be given, as well as a set of interpolation conditions on an unknown* $p \times m$ *transfer matrix* $F(s)$ *such that* $l_k^* F(z_k) = g_k^T$ *for prescribed* $1 \times m$ *complex vectors* g_k^T *and* $p \times 1$ *unit norm direction vectors* l_k*, where* $p \geq m$ *and all such conditions occur in complex conjugate pairs or are real.*

If there exists $F(s) \in \mathcal{RH}_\infty$ *satisfying the interpolation conditions with* $\|F(s)\|_\infty \leq 1$ *then there exists an inner matrix* $\Pi(s) \in \mathcal{RH}_\infty$ *with* $\|\Pi(s)\|_\infty = 1$ *which also satisfies the interpolation conditions* $l_k^* \Pi(z_k) = g_k^T$.

Proof. Here we exploit the equivalence of $\mathcal{H}_\infty$ model matching and the Nevanlinna interpolation problem (see [6] for the scalar case). Let Θ_Z be a stable real-rational all-pass (square) transfer matrix of McMillan degree [19] K such that $l_k^* \Theta_Z(z_k) = 0$ for all z_k, that is, with zeros $z_k, k = 1 \dots K$ with output zero directions l_k^*, as in Lemma 5.2. We then define

$$\begin{aligned} T_{11}(s)^{p\times m} &= \sum_{k=1}^{K} \left[\tilde{\Theta}_{Z/k}(s)\tilde{\Theta}^{-1}_{Z/k}(z_k) l_k g_k^T\right], \\ T_{21}(s) &= I^{m\times m}, \\ \text{and } T_{12}(s) &= \Theta_Z(s)^{p\times p}. \end{aligned}$$

From the last claim of Lemma 5.2 we can deduce that T_{11} is real rational. If z_k is real, then by the lemma hypothesis, the corresponding $\tilde{l}_k$ and g_k^T are real. If z_k is complex then there is a corresponding $z_j = \overline{z_k} \in \mathcal{Z}$ with corresponding $\tilde{l}_j = \overline{\tilde{l}_k}$ and $g_j^T = \overline{g_k^T} = g_k^*$. We now claim that any $F \in \mathcal{H}_\infty$ satisfying the interpolation conditions is necessarily of the form

$$F = T_{11} - T_{12} Q T_{21} \tag{5.1}$$

for some $Q \in \mathcal{H}_\infty^{p\times m}$, and conversely. This is established as follows.

Since $l_k^*\Theta_Z(z_k) = 0$, it is readily seen that $l_j^*\tilde{\Theta}_{Z/k}(z_j) = 0$ for $j = 1,\dots,k-1,k+1,\dots,K$. This is clear since $\Theta_Z(z_j) = \tilde{\Theta}_{Z/k}(z_j)\tilde{\Theta}_{zk}(z_j)$ and the second term in the product is nonsingular for $k \neq j$. It is obvious that the condition that F may be expressed as (5.1) with Q stable is sufficient for F to satisfy the interpolation conditions. For necessity, consider $\hat{Q} \stackrel{\text{def}}{:=} F - T_{11} \in \mathcal{RH}_\infty$ and note that $l_k^*\hat{Q}(z_k) = l_k^*[F(z_k - T_{11}(z_k)] = 0$ for each k. Since $l_k^*\hat{Q}(z_k) = 0$ for $\hat{Q} \in \mathcal{RH}_\infty$ it follows [27] that it may be factorised as an inner-outer factorisation $\hat{Q}(s) = \Theta_Z(s)Q(s)$ for some $Q \in \mathcal{RH}_\infty$. The proof now follows immediately upon application of Lemma 5.1. □

In the proof of Lemma 5.1 it is shown that the $\mathcal{H}_\infty$ model matching problem can be converted to the Nehari problem of approximating an $\mathcal{RL}_\infty$ function $\mathbf{G}$ by an $\mathcal{RH}_\infty$ matrix $\bar{\mathbf{Q}}$. By Lemma 5.1, given any $\gamma \geq \gamma^\star$ it is possible to find, using the all-pass embedding method in [17] (see also [15]), a $\bar{\mathbf{Q}} \in \mathcal{RH}_\infty$ which results in $\mathbf{E}^*\mathbf{E} = \gamma^2\mathbf{I}$. In order to prove the continuity property for the homotopy that we will construct, we will also need to know that for $\gamma^\star < 1$, if we take $\gamma = 1$, then this all-pass embedding algorithm gives a result (the all-pass transfer matrix $\mathbf{E}$) which is $\mathcal{H}_\infty$-norm continuous in the input data (the plant data $\mathbf{G}$.)

THEOREM 5.4. *Let an $\mathcal{L}_\infty$ norm homotopy ${\mathbf{G}_\lambda}^{p\times m} \in \mathcal{RL}_\infty$ (with dimensions $p\times m$, with $p \geq m$) be given as a function of $\lambda \in [0,1]$. Provided that for each λ the following conditions hold:*

- *The (real rational) $\mathbf{G}_\lambda$ has a finite dimensional state-space representation, with state-space parameters bounded and continuous in λ;*
- *The Hankel norm $\gamma_\lambda^\star$ of $\mathbf{G}_\lambda$ is strictly less than unity;*

then there exists an $\mathcal{L}_\infty$ homotopy of paraunitary error functions $\mathbf{E}_\lambda = \mathbf{G}_\lambda - \bar{\mathbf{Q}}_\lambda$ (that is, one which satisfies ${\mathbf{E}_\lambda}^*\mathbf{E}_\lambda = I$), where $\bar{\mathbf{Q}}_\lambda \in \mathcal{RH}_\infty$ is stable for each λ, which may be obtained by the all-pass embedding method in [17].

Furthermore, suppose that there is a finite number J of λ interpolation conditions on the homotopy, that is, there are given J specific values of the $\mathbf{G}_\lambda$ homotopy parameter λ, namely $\hat{\lambda}_j$ for $j = 1\dots J$ as well as corresponding paraunitary error functions $\hat{\mathbf{E}}_j$, and the homotopy $\mathbf{E}_\lambda$ of paraunitary error functions is required to interpolate $\hat{\mathbf{E}}_j$ at each $\lambda = \hat{\lambda}_j$.

This is possible provided the following conditions are true

- *For each j there exists some $\bar{\mathbf{Q}}_\lambda \in \mathcal{RH}_\infty$ such that $\hat{\mathbf{E}}_j = \mathbf{G}_\lambda - \bar{\mathbf{Q}}_\lambda$ when $\lambda = \hat{\lambda}_j$.*
- *Either the transfer matrix $\mathbf{G}_\lambda$ is strictly tall ($p > m$), or, in the case that $\mathbf{G}_\lambda$ is square, both the winding numbers of the determinants of $\hat{\mathbf{E}}_j$ and the signs of $\det_{s\to\infty}[\hat{\mathbf{E}}_j(s)]$ are equal for each j.*

Proof. See Appendix A.3. □

An immediate consequence of the above theorem is that the solution to the model matching problem in Lemma 5.1 is continuous, provided the problem data are. This is formalised in Corollary 5.5 following. A similar conclusion follows in Corollary 5.6

for the Nevanlinna interpolation problem, which was introduced in Corollary 5.3.

COROLLARY 5.5. *Assume there is given a set, parametrised by* λ*, of model matching problems as in Lemma 5.1, where* $T_{11}(\lambda)$*,* $T_{12}(\lambda)$ *and* $T_{21}(\lambda)$ *are each parametrised by* λ*. Suppose that each of* $T_{11}(\lambda)$*,* $T_{12}(\lambda)$ *and* $T_{21}(\lambda)$ *have state-space realisations with state-space parameters that are continuous in* λ*. Suppose the value of the optimisation*

$$\gamma^\star(\lambda) = \inf_{Q \in \mathcal{H}_\infty} T_{11}(\lambda) - T_{12}(\lambda) Q T_{21}(\lambda)$$

is strictly less than unity for all λ *of interest, that is* $\gamma^\star(\lambda) < 1$*. Then there exists a paraunitary* $E_\lambda = T_{11}(\lambda) - T_{12}(\lambda)Q(\lambda)T_{21}(\lambda)$ *(that is* $E_\lambda^* E_\lambda = I$*) where* $Q(\lambda) \in \mathcal{H}_\infty$ *and* E_λ *is an* $\mathcal{H}_\infty$ *homotopy.*

Furthermore, suppose that there is given a finite number J *of* λ *interpolation conditions on the homotopy. That is, there are given* J *specific values of the homotopy parameter* λ*, namely* $\hat{\lambda}_j$ *for* $j = 1 \ldots J$ *as well as corresponding paraunitary error functions* $\hat{E}_j$*, and the homotopy* E_λ *of paraunitary error functions is required to interpolate* $\hat{E}_j$ *at each* $\lambda = \hat{\lambda}_j$*.*

This is possible provided the following conditions are true

- *For each* j *there exists some* $Q(\hat{\lambda}_j) \in \mathcal{RH}_\infty$ *such that* $\hat{E}_j = T_{11}(\hat{\lambda}_j) + T_{12}(\hat{\lambda}_j)Q(\hat{\lambda}_j)T_{21}(\hat{\lambda}_j)$*.*
- *Either the transfer matrix* T_{11} *is strictly tall (*$p > m$*), or, in the case that* T_{11} *is square, both the winding numbers of the determinants of* $\hat{E}_j$ *and the signs of* $\det_{s \to \infty}[\hat{E}_j(s)]$ *are equal for each* j*.*

Proof. The equivalence of the model matching problem of Lemma 5.1 to the Nehari problem of Theorem 5.4 is demonstrated in the proof of Lemma 5.1. The requirement that $E_\lambda^* E_\lambda = \gamma^2 I$ corresponds choosing $\gamma = 1$ and the condition that $\gamma^\star(\lambda) < 1$ ensures that an appropriate $Q(\lambda) \in \mathcal{RH}_\infty$ exists. □

COROLLARY 5.6. *Assume that there is given a set of interpolation problems, parametrised by* λ*, as in Corollary 5.3, where* $g_i(\lambda)^{1\times m}$*,* $l_i(\lambda)^{p\times 1}$ *and the* s*-domain interpolation points* $z_i(\lambda)$ *are each parametrised by* λ*. Then provided each of* $g_i(\lambda)$*,* $l_i(\lambda)$ *and* $z_i(\lambda)$ *is continuous in* λ*, it is possible to solve the interpolation problem to give a (continuous) paraunitary* $U_\lambda(s)^{p\times m}$ *homotopy.*

Furthermore, suppose that there is given a finite number J *of* λ *interpolation conditions on the homotopy. That is, there are given* J *specific values of the homotopy parameter* λ*, namely* $\hat{\lambda}_j$ *for* $j = 1 \ldots J$ *as well as corresponding paraunitary functions* $\hat{U}_j$*, and the homotopy* $U_\lambda(s)$ *of paraunitary error functions is required to interpolate* $\hat{U}_j(s)$ *at each* $\lambda = \hat{\lambda}_j$*.*

This is possible provided the following conditions are true

- *For each* j *the paraunitary transfer matrix* $\hat{U}_j$ *satisfies the* s*-domain interpolation conditions* $l_i(\hat{\lambda}_j)^* \hat{U}_j[z_i(\hat{\lambda}_j)] = g_i(\hat{\lambda}_j)$*.*
- *Either* $p > m$*, or, in the case that* $p = m$*, then both the winding numbers of the determinants of* $\hat{U}_j$ *and the signs of* $\det_{s \to \infty}[\hat{U}_j(s)]$ *are equal for each* j*.*

Proof. The equivalence of the interpolation problem to the model matching problem is shown in Corollary 5.3. Because each of the interpolation conditions is continuous in λ, the transfer functions $T_{11}(\lambda)$, $T_{12}(\lambda)$ and $T_{22}(\lambda)$ corresponding to the resultant model matching problem have representations that are continuous in the state-space parameters. The equivalence of the model matching problem to the Nehari problem of Theorem 5.4 is demonstrated in the proof of Lemma 5.1. □

6. Homotopy Construction. We prove the existence of a homotopy that satisfies the required conditions in Theorem 4.1 by construction. We first describe a construction, and then we show that it satisfies the required conditions.

Our multivariable homotopy construction procedure is as follows. Without loss of generality we investigate the "tall" plant case and take plants $P_\alpha^{p\times m}$ and $P_\beta^{p\times m}$ with $p \geq m$ and $\delta_\nu(P_\alpha, P_\beta) < 1$. We make the simplifying assumptions that $P_\alpha - P_\beta$ is full rank and has no zeros on the imaginary axis or at infinity. We may also assume that the set of poles in the normalised coprime fraction representation $G_\alpha = \begin{bmatrix} N_\alpha^T & D_\alpha^T \end{bmatrix}^T$ of $P_\alpha = N_\alpha M_\alpha^{-1}$ are distinct.

If either or both of these simplifying assumptions is not true, then we can make an abitrarily small perturbation of P_α, of magnitude no more than some positive scalar ϵ such that both $\epsilon < 1 - \delta_\nu(P_\alpha, P_\beta)$ and $\epsilon < \frac{1}{2}\eta$, where η is the monotonicity perturbation upper bound parameter of Theorem 4.1, to give P'_α such that $P'_\alpha - P_\beta$ is full rank and has no zeros on the imaginary axis or at infinity, and the normalised coprime representation of P'_α has distinct poles. We then construct a homotopy P_λ from P'_α to P_β using the methods presented in this section and complete with a homotopy from P_α to P'_α. By the triangle inequality it follows that $\delta_\nu(P'_\alpha, P_\lambda) \leq \delta_\nu(P_\alpha, P'_\alpha) + \delta_\nu(P'_\alpha, P_\beta)$ will remain strictly subunitary, and will not violate the desired property of the perturbed homotopy being arbitrarily close to monotonic. Note that this is just a perturbation of the end point P_α, and not a perturbation of the P_λ homotopy *en route*.

We proceed to define the P_λ homotopy as follows. First we define $R^{m\times m} = G_\alpha^* G_\beta$ and $W^{p\times m} = \tilde{G}_\alpha G_\beta$. Because $\delta_\nu(P_\alpha, P_\beta) < 1$ it follows that for every ω, $\det[R(j\omega)] \neq 0$, and $\mathrm{wno}[\det(R)] = 0$; moreover R has no poles on the imaginary axis. Furthermore, we know that $W = \tilde{M}_\alpha(P_\alpha - P_\beta)M_\beta$ is stable and full-rank, with $\|W\|_\infty = \delta_\nu(P_\alpha, P_\beta) < 1$. We also know that $R^*R + W^*W = I$.

We now proceed to construct homotopies R_λ and W_λ and define a homotopy G_λ as

$$G_\lambda = G_\alpha R_\lambda + \tilde{G}_\alpha^* W_\lambda. \tag{6.1}$$

We later show that G_λ defines a homotopy for a normalised coprime factorisation of a plant P_λ satisfying the requirements of the theorem statement.

As in equation (2.1) of Section 2, perform an inner-outer factorisation to factorise $W \in \mathcal{RH}_\infty$ as $\Pi_{Wz}\hat{W}$ where Π_{Wz} is an inner transfer matrix and $\hat{W}$ is square and outer. Also, factorise the square transfer matrix R as $R = \Theta_{Rp}^* \Theta_{Rz} \hat{R}$ (see equation

(2.6)), where Θ_{Rp} and Θ_{Rz} are stable all-pass transfer matrices (chosen so that the signs of their determinants at infinity is the same), and $\hat{R}$ is outer. Note that $R^*R + W^*W = \hat{R}^*\hat{R} + \hat{W}^*\hat{W} = I$. We will be able to apply the results of Corollary 5.3 to interpolation conditions at the right half-plane poles of Θ^*_{Rp}, since we have assumed that the poles of G_α and hence of G^*_α, are distinct. Note that each unstable pole of Θ^*_{Rp} is also an unstable pole of G^*_α, that is, the set of unstable poles of G^*_α forms a superset of those of Θ^*_{Rp}.

We will define the homotopies for R_λ and W_λ as products $R_\lambda = \Theta^*_{Rp}\Theta_{R\lambda}\hat{R}_\lambda$ and $W_\lambda = \Pi_{W\lambda}\hat{W}_\lambda$. We do this in two distinct stages, one from $\lambda \in [0,1]$ and one for $\lambda \in [1,2]$, so that $\lambda = 0$ will correspond to P_α and $\lambda = 2$ will correspond to P_β. For $\lambda \in [0,1]$ the stable minimum-phase factor $\hat{R}_\lambda$ is transformed from I to $\hat{R}$, while the stable minimum-phase factor $\hat{W}_\lambda$ is transformed from 0 to $\hat{W}$, with $\Theta_{R\lambda}$ and $\Pi_{W\lambda}$ fixed respectively at Θ_{Rp} and $\Pi_{\alpha p}\Theta_{Vp}$ (where $\Pi_{\alpha p}$ and Θ_{Vp} are to be defined below). Then, for $\lambda \in [1,2]$ we keep the factors $\hat{R}_\lambda$ and $\hat{W}_\lambda$ fixed at $\hat{R}$ and $\hat{W}$, while $\Theta_{R\lambda}$ is transformed from Θ_{Rp} to Θ_{Rz}, and $\Pi_{W\lambda}$ is transformed from $\Pi_{\alpha p}\Theta_{Vp}$ to Π_{Wz}. All the time it has to be ensured that the resultant G_λ defined by (6.1), is stable, coprime and normalised, the winding number condition $\operatorname{wno}[\det(G^*_\alpha G_\lambda)] = 0$ is satisfied, and the monotonicity property holds.

6.1. Homotopy for $\lambda \in [0,1]$.

6.1.1. Homotopy for the Minimum-Phase Factors $\hat{R}$ and $\hat{W}$ for $\lambda \in [0,1]$.

For $\lambda \in [0,1]$ we specifically define the homotopy for a stable and minimum-phase $\hat{R}^{m\times m}_\lambda$ via a spectral factorisation

$$\hat{R}^*_\lambda \hat{R}_\lambda = \lambda^2 \hat{R}^*\hat{R} + (1-\lambda^2)I, \tag{6.2}$$

and note that the right hand side of the above equation has no imaginary axis poles, so that $\hat{R}_\lambda$ is bounded in the $\mathcal{H}_\infty$ norm. Although the $\hat{R}_\lambda$ so defined is unique only up to a constant orthogonal factor, it is a trivial matter to select this factor to find a $\hat{R}_\lambda$ such that it is $\mathcal{H}_\infty$ continuous in λ. In fact if $\hat{R}$ has a minimal state space realisation $[A_{\hat{R}}, B_{\hat{R}}, C_{\hat{R}}, D_{\hat{R}}]$, then $\hat{R}_\lambda$ can be found with realisation $[A_{\hat{R}}, B_{\hat{R}}, C_{\hat{R}\lambda}, D_{\hat{R}\lambda}]$ (note that the $A_{\hat{R}}$ and $B_{\hat{R}}$ state space parameters may be taken as fixed). The spectral factorisation and construction of $C_{\hat{R}\lambda}, D_{\hat{R}\lambda}$ depends on solving a Riccati equation [27]. The coefficients in the Riccati equation are analytic in λ and the resultant $C_{\hat{R}\lambda}$ and $D_{\hat{R}\lambda}$ are also analytic in λ.

It then follows that if we define

$$\hat{W}^{m\times m}_\lambda = \lambda \hat{W}, \tag{6.3}$$

then the following holds:

$$\begin{aligned} \hat{W}^*_\lambda \hat{W}_\lambda &= \lambda^2 (I - \hat{R}^*\hat{R}) \\ &= I - \hat{R}^*_\lambda \hat{R}_\lambda. \end{aligned} \tag{6.4}$$

6.1.2. The Inner Factors $\Theta_{R\lambda}$ and $\Pi_{W\lambda}$ Held Fixed for $\lambda \in [0,1]$. Still for $\lambda \in [0,1]$, we define an all-pass homotopy for $\Theta_{R\lambda}$ by simply holding it fixed at Θ_{Rp}. In order to reveal its right half-plane pole and zero structure, we also perform a factorisation of $\tilde{G}^*_\alpha$ as in equation (2.6) as

$$\begin{aligned} \tilde{G}^*_\alpha &= \begin{bmatrix} -\tilde{M}^*_\alpha \\ \tilde{N}^*_\alpha \end{bmatrix} \\ &= \begin{bmatrix} -\bar{M}_\alpha \\ \bar{N}_\alpha \end{bmatrix} \tilde{\Theta}_{\alpha z} \tilde{\Theta}^*_{\alpha p} \\ &= \bar{G}_\alpha \tilde{\Theta}_{\alpha z} \tilde{\Theta}^*_{\alpha p}. \end{aligned} \tag{6.5}$$

In the above, $\tilde{\Theta}_{\alpha z}$ and $\tilde{\Theta}_{\alpha p}$ are stable and all-pass, and $\bar{G}_\alpha$ is co-outer. The zeros of $\tilde{\Theta}_{\alpha p}$ are the right half-plane poles of $\tilde{G}^*_\alpha$, the same as the poles of G^*_α: these poles form a superset of the unstable poles of R which are the zeros of Θ_{Rp}. These facts will be exploited later in proving particular stability properties of the homotopy.

In order to define a fixed $\Pi_{W\lambda}$ for $\lambda \in [0,1]$, we introduce several ancillary quantities. This will result in a choice of fixed $\Pi_{W\lambda}$ for $\lambda \in [0,1]$ such that the resultant G_λ in equation (6.1) is stable *and* so that the resulting $\Pi_{W\lambda}$ can be continuously connected to the homotopy that will be defined for $\lambda = [1,2]$. First we introduce $X^{m\times p}, Y^{p\times p}, \tilde{X}^{m\times p}$ and $\tilde{Y}^{m\times m}$, which are $\mathcal{RH}_\infty$ transfer matrices, satisfying the double Bezout identity

$$\begin{bmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{M}_\alpha & \tilde{N}_\alpha \end{bmatrix} \begin{bmatrix} N_\alpha & -Y \\ M_\alpha & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6.6}$$

Such transfer matrices exist because the fractional descriptions $M_\alpha^{-1} N_\alpha = \tilde{N}_\alpha \tilde{M}_\alpha^{-1}$ of P_α are coprime. We define a transfer matrix $Z^{m\times p}$ as

$$Z = \tilde{Y}\tilde{N}^*_\alpha - \tilde{X}\tilde{M}^*_\alpha \tag{6.7}$$

and as in equation (2.13) of Section 2, we factorise Z as

$$Z = \bar{Z}^{m\times m} \Theta^{m\times m}_{\alpha z} (\Pi^{p\times m}_{\alpha p})^* \tag{6.8}$$

where $\Pi_{\alpha p}$ is inner, $\bar{Z}$ is stable, minimum-phase, and $\Theta_{\alpha z}$ is all-pass and stable (inner). For the purposes of the following construction, we must assume that $\bar{Z}$ is also full rank and possesses no imaginary axis or infinite zeroes and $\bar{Z}^{-1} \in \mathcal{RH}_\infty$. Since Z is uniquely defined by the homotopy end-point P_α, if this is not case for the originally given P_α, we can find an arbitrarily small perturbation P'_α such that it is true, in the same way that it is possible to ensure the perturbed end-point has normalised coprime fraction representation with distinct poles, and such that $P'_\alpha - P_\beta$ is full rank and has no imaginary axis zeros. (See the third paragraph of Section 6.)

We can now define a homotopy for $\Pi_{W\lambda}$ by holding it fixed at $\Pi_{W\lambda} = \Pi_{\alpha p}\Theta_{Vp}$ for $\lambda \in [0,1]$, where Θ_{Vp} is a stable, all-pass matrix to be defined later in the development in Lemma 6.2. With these assignments, that is, $R_\lambda = \hat{R}_\lambda$, $W_\lambda = \lambda \cdot \Pi_{\alpha p}\Theta_{Vp}\hat{W}$ and G_λ defined in equation (6.1), we will later show that G_λ is stable.

6.2. Homotopy for $\lambda \in [1,2]$. We now construct the homotopy for $\lambda \in [1,2]$. In this part of the homotopy, we hold $\hat{R}_\lambda$ fixed at $\hat{R}$ and allow $\Theta_{R\lambda}$ to vary from Θ_{Rp} to Θ_{Rz}. Note that this homotopy must be such as to preserve the winding number of R_λ, that is, $\Theta_{R\lambda}$ must have the same number of non-minimum-phase zeros as Θ_{Rp} for all λ. Note that Θ_{Rz} has the same number of non-minimum phase zeros as Θ_{Rp}, because R has zero winding number. In the construction of the homotopy we also hold $\hat{W}_\lambda$ fixed at $\hat{W}$ and allow $\Pi_{W\lambda}$ to vary from $\Pi_{\alpha p}\Theta_{Vp}$ to Π_{Wz}.

6.2.1. Homotopy for $\Theta_{R\lambda}$ for $\lambda \in [1,2]$. In order to efficaciously construct $\Pi_{W\lambda}$ below and in order to satisfy the winding number condition for R_λ, we make the $\Theta_{R\lambda}$ homotopy in a special way. In fact, for each λ, we make $\Theta_{R\lambda}$ such that $l_{pk}^*\Theta_{R\lambda}(p_k) = [\lambda - 1] \cdot l_{pk}^*\Theta_{Rz}(p_k)$ for each unstable zero p_k of Θ_{Rp} with normalised (output) zero direction [19] l_{pk}.

Such a stable all-pass $\Theta_{R\lambda}$ exists by Corollary 5.3 because $[\lambda - 1]$ never exceeds unity, that is, a stable function with less than unity norm and that satisfies the interpolation constraints exists (namely $[\lambda - 1] \cdot \Theta_{Rz}$) and hence there exists an stable all-pass (that is, with exactly unity norm) transfer matrix, that also satisfies the interpolation constraints.

We can see that $\Theta_{R\lambda} = \Theta_{Rz}$ at $\lambda = 2$ and that $\Theta_{R\lambda} = \Theta_{Rp}$ at $\lambda = 1$ are solutions for the above (s-plane) interpolation constraints. It is also clear, since $\det(R)$ has zero winding number that $\det(\Theta_{Rz})$ and $\det(\Theta_{Rp})$ share equal winding numbers, and recall that Θ_{Rz} and Θ_{Rp} have been chosen so that the signs of their determinants at infinity are the same. We note that by the Corollary 5.6, it follows that $\Theta_{R\lambda}$ can be defined in such a way is an $\mathcal{H}_\infty$-norm homotopy, and such that it is equal to Θ_{Rp} and Θ_{Rz} at $\lambda = 1$ and $\lambda = 2$ respectively.

6.2.2. Homotopy for $\Pi_{W\lambda}$ for $\lambda \in [1,2]$. Having defined the homotopy for R_λ for the interval $\lambda \in [1,2]$ using $\hat{R}$ and $\Theta_{R\lambda}$, we now define the homotopy for $\Pi_{W\lambda}$, guided by the need to find an inner transfer function which will make G_λ in (6.1) stable. We can now recast the requirement that G_λ be stable in terms of the following equivalence result.

LEMMA 6.1. *Let G_α correspond to a normalised right coprime factorisation $N_\alpha M_\alpha^{-1}$ of P_α and let $\tilde{G}_\alpha$ correspond to a normalised left coprime factorisation $\tilde{M}_\alpha^{-1}\tilde{N}_\alpha$. Define Z by equation (6.7) above where $X, Y, \tilde{X}$ and $\tilde{Y}$ satisfy the Bezout identity (6.6). Suppose that a transfer matrix $R_\lambda \in \mathcal{RL}_\infty$ has been constructed and transfer matrix W_λ is sought. Make the definition*

$$T^{m\times m} = R_\lambda^{m\times m} + Z^{m\times p}W_\lambda^{p\times m}.$$

Then the pair $W_\lambda, T \in \mathcal{RH}_\infty$ if and only if $G_\lambda = G_\alpha R_\lambda + \tilde{G}_\alpha^* W_\lambda \in \mathcal{RH}_\infty$.

Proof. The lemma is demonstrated by the observation that

$$\begin{aligned}\begin{bmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{M}_\alpha & \tilde{N}_\alpha \end{bmatrix} G_\lambda &= \begin{bmatrix} R_\lambda + (\tilde{Y}\tilde{N}_\alpha^* - \tilde{X}\tilde{M}_\alpha^*)W_\lambda \\ W_\lambda \end{bmatrix} \\ &= \begin{bmatrix} T \\ W_\lambda \end{bmatrix}.\end{aligned}$$

We also see that

$$\begin{aligned}G_\lambda &= \begin{bmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{M}_\alpha & \tilde{N}_\alpha \end{bmatrix}^{-1} \begin{bmatrix} T \\ W_\lambda \end{bmatrix} \\ &= \begin{bmatrix} N_\alpha & -Y \\ M_\alpha & X \end{bmatrix} \begin{bmatrix} T \\ W_\lambda \end{bmatrix}.\end{aligned}$$

The expression for the inverse matrix is a consequence of the Bezout identity (6.6). Because the product of two stable transfer matrices is also stable, the lemma statement holds. □

In the light of the above lemma, and the fact that for $\lambda \in [1,2]$ we require that $W_\lambda = \Pi_{W\lambda}\hat{W}$ for some $p \times m$ paraunitary $\Pi_{W\lambda}$, we shall work with T and seek to ensure, through appropriate choice of $\Pi_{W\lambda}$, that $T \in \mathcal{RH}_\infty$. The unknown stable matrix T is a free quantity at our disposal.

Let us for the moment define $T(U_{W\lambda})$ for an arbitrary (that is, not necessarily paraunitary) $U_{W\lambda}^{p\times m} \in \mathcal{RH}_\infty$ as

$$\begin{aligned}T(U_{W\lambda})^{m\times m} &= R_\lambda + ZU_{W\lambda}\hat{W} \\ &= \Theta_{Rp}^*\Theta_{R\lambda}\hat{R} + \bar{Z}^{m\times m}\Theta_{\alpha z}^{m\times m}(\Pi_{\alpha p}^{p\times m})^* U_{W\lambda}^{m\times m}\hat{W}^{m\times m}.\end{aligned}$$

Our task is to show that there is an inner $U_{W\lambda}$, (continuous in λ), that will ensure that $T(U_{W\lambda})$ (and therefore G_λ,) is in $\mathcal{RH}_\infty$. We can then identify $U_{W\lambda}$ with $\Pi_{W\lambda}$. The quantity $\hat{\Pi}_{W\lambda}$, defined and existence proven in the following lemma, defines the required $\Pi_{W\lambda}$ homotopy for $\lambda \in [1,2]$. Observe also that the following lemma defines the quantity Θ_{Vp} that is used in the definition of the homotopy for inner $\Pi_{W\lambda}$ for $\lambda \in [0,1]$.

LEMMA 6.2. *Let $\Theta_{Rp}^{m\times m}$, $\hat{R}^{m\times m}$ and $\Theta_{R\lambda} \in \mathcal{RH}_\infty$ be as defined at the beginning of Section 6.2 and in Section 6.2.1. Also let $\Pi_{\alpha p}^{p\times m}$, $\Theta_{\alpha z}^{m\times m}$ and $\bar{Z}^{m\times m} \in \mathcal{RH}_\infty$ be defined according to the factorisation of Z following equation (6.7). Finally let $\hat{W}^{m\times m} \in \mathcal{RH}_\infty$ be as defined from the inner-outer factorisation $W = \Pi_{Wz}\hat{W} = \tilde{G}_\alpha G_\beta$. Recall that $\Pi_{\alpha p}$ is inner and Θ_{Rp}, $\Theta_{\alpha z}$ are stable and all-pass and $\hat{W}$ is of full rank and possesses a stable inverse. Define*

$$T(U_{W\lambda}) = \Theta_{Rp}^*\Theta_{R\lambda}\hat{R} + \bar{Z}^{m\times m}\Theta_{\alpha z}^{m\times m}(\Pi_{\alpha p}^{p\times m})^* U_{W\lambda}^{p\times m}\hat{W}^{m\times m} \tag{6.9}$$

where $U_{W\lambda} \in \mathcal{RH}_\infty^{p\times m}$.

Then

- *If* $U_{W\lambda}$ *is set to* $U_{W\lambda} = (\lambda - 1)\Pi_{Wz}$ *then this ensures that* $T(U_{W\lambda}) \in \mathcal{RH}_\infty^{p\times m}$*, and*
- *there exists an inner* $\hat{\Pi}_{W\lambda}^{p\times m}$ *such that* $U_{W\lambda} = \hat{\Pi}_{W\lambda}$ *ensures that* $T(U_{W\lambda}) \in \mathcal{RH}_\infty$*.*
- *Furthermore, such an inner* $\hat{\Pi}_{W\lambda}$ *can be chosen such that it is an* $\mathcal{H}_\infty$ *homotopy in* λ*, which interpolates* $\Pi_{\alpha p}\Theta_{Vp}$ *at* $\lambda = 1$*, for some stable, all-pass transfer matrix* Θ_{Vp}*, and* Π_{Wz} *at* $\lambda = 2$*.*

Proof. We define $\hat{T}_\lambda$ as being the T that results from the choice of $U_{W\lambda} = [\lambda - 1]\cdot \Pi_{Wz}$ in equation (6.9). We claim that the $\hat{T}_\lambda \in \mathcal{RH}_\infty$. To see this note that the only possibility of instability in equation (6.9) arises from the unstable poles of Θ_{Rp}^* and $\Pi_{\alpha p}^*$. Each unstable pole of Θ_{Rp}^* and $\Pi_{\alpha p}^*$ is also an unstable pole of $\tilde{\Theta}_{\alpha p}^*$ (defined in equation (6.5)).

In order to demonstrate that $\hat{T}_\lambda$ is stable, we evaluate the residuals of $\hat{T}_\lambda$ in equation (6.9) at each potential unstable pole p_k. This can be done pole by pole, since by assumption all the unstable poles of $G_\alpha*$ and thus zeros of Θ_{Rp} are distinct.

First recall that for each k, there exists a factorisation of Θ_{Rp} as

$$\Theta_{Rp} = \begin{bmatrix} l_k & L_k \end{bmatrix} \begin{bmatrix} \frac{s-p_k}{s+\overline{p_k}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} l_k^* \\ L_k^* \end{bmatrix} \Theta_{Rp/k}$$

where $\Theta_{Rp/k}$ has no zeroes at p_k. This results in the following equality. Recall that have defined $\Theta_{R\lambda}$ so that $l_{pk}^*\Theta_{R\lambda}(p_k) = [\lambda - 1]\cdot l_{pk}^*\Theta_{Rz}(p_k)$.

$$\begin{aligned}
&\lim_{s\to p_k}\{[s-p_k][\Theta_{Rp}(s)^{-1}\Theta_{R\lambda}(s)\hat{R}(s)\} \\
&= \lim_{s\to p_k}\left\{\begin{bmatrix} l_k & L_k \end{bmatrix}\begin{bmatrix} s+\overline{p_k} & 0 \\ 0 & (s-p_k)I \end{bmatrix}\begin{bmatrix} l_k^* \\ L_k^* \end{bmatrix}\Theta_{Rp/k}(s)^{-1}\Theta_{R\lambda}(s)\hat{R}(s)\right\} \\
&= \begin{bmatrix} l_k & L_k \end{bmatrix}\begin{bmatrix} 2\mathrm{Re}(p_k) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} l_k^* \\ L_k^* \end{bmatrix}\Theta_{Rp/k}^{-1}(p_k)\Theta_{R\lambda}(p_k)\hat{R}(p_k) \\
&= [\lambda-1]\Theta_{Rp/k}^{-1}(p_k)\begin{bmatrix} l_k & L_k \end{bmatrix}\begin{bmatrix} 2\mathrm{Re}(p_k) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} l_k^* \\ L_k^* \end{bmatrix}\Theta_{Rz}(p_k)\hat{R}(p_k) \\
&= [\lambda-1]\lim_{s\to\infty}\{[s-p_k]\cdot\Theta_{Rp}(s)^{-1}\Theta_{Rz}(s)\hat{R}(s)\}.
\end{aligned}$$

We can thus determine residuals of $\hat{T}_\lambda(s)$ at p_k for each k as follows.

$$\begin{aligned}
\lim_{s\to p_k}\{[s-p_k]\hat{T}_\lambda(s)\} &= \lim_{s\to p_k}\{[s-p_k][\Theta_{Rp}(s)^{-1}\Theta_{R\lambda}(s)\hat{R}(s) + Z(s)U(s)\hat{W}(s)]\} \\
&= [\lambda-1]\lim_{s\to\infty}\{[s-p_k][\Theta_{Rp}(s)^*\Theta_{Rz}(s)\hat{R}(s) + Z(s)\Pi_{Wz}\hat{W}(s)]\} \\
&= [\lambda-1]\lim_{s\to\infty}\{[s-p_k][R(s) + Z(s)W(s)]\} \\
&= 0.
\end{aligned}$$

The last equality is due to the fact that $R(s) + Z(s)W(s)$ is stable due to the stability of $G_\beta(s)$ and Lemma 6.1. This proves the first lemma statement.

In order to prove the second lemma statement we shall find an expression for the set of $U_{W\lambda}$ which give rise to a particular T. The first part of the lemma in addition to Theorem 5.4 and its corollaries, will then allow us to identify an all-pass $U_{W\lambda}$ that will guarantee that $T \in \mathcal{RH}_\infty$.

Since $\Pi_{\alpha p}$ is inner we can find an inner $\Pi_{\alpha pc}$ which is an all-pass completion of the columns of $\Pi_{\alpha p}$, that is, such that $\begin{bmatrix} \Pi_{\alpha p} & \Pi_{\alpha pc} \end{bmatrix}$ is all-pass. It is then immediate from (6.9) that we can write

$$\begin{bmatrix} T \\ \hat{Q}\hat{W} \end{bmatrix} = \begin{bmatrix} \Theta^*_{Rp}\Theta_{R\lambda}\hat{R} \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{Z}\Theta_{\alpha z}\Pi^*_{\alpha p} \\ \Pi^*_{\alpha pc} \end{bmatrix} U_{W\lambda}\hat{W}. \tag{6.10}$$

where $\hat{Q}^{(p-m)\times m}$, is not necessarily stable. In the above T and $\hat{Q}$ are given as functions of the variable $U_{W\lambda}$, with the other terms in the equation known. We can now rearrange this to make $U_{W\lambda}$ the subject– a function of T and $\hat{Q}$.

$$\begin{aligned} U_{W\lambda} &= \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1}\left(T - \Theta^*_{Rp}\Theta_{R\lambda}\hat{R}\right)\hat{W}^{-1} + \Pi_{\alpha pc}\hat{Q} \\ &= \begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix}^{p\times p} \begin{bmatrix} T \\ \hat{Q}\hat{W} \end{bmatrix}^{p\times m} \\ &\quad (\hat{W}^{m\times m})^{-1} - \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1}\Theta^*_{Rp}\Theta_{R\lambda}\hat{R}\hat{W}^{-1} \end{aligned} \tag{6.11}$$

Our goal is to find $U_{W\lambda}$ that satisfies the conditions in the second Lemma statement. In order to do so, note that $T = \hat{T}_\lambda \in \mathcal{RH}_\infty$ and $\hat{Q}\hat{W} = [\lambda - 1]\cdot\Pi^*_{\alpha pc}\Pi_{Wz} \in \mathcal{RL}_\infty$ substituted in equation (6.11) gives $U_{W\lambda} = [\lambda - 1]\cdot\Pi_{Wz} \in \mathcal{RH}_\infty$. We now introduce a change of variables in equation (6.11), as $T = \hat{T}_\lambda + Q_1$ and $\hat{Q}\hat{W} = [\lambda - 1]\cdot\Pi^*_{\alpha pc}\Pi_{Wz} + Q_2$ to give

$$\begin{aligned} U_{W\lambda} &= (\lambda - 1)\Pi_{Wz} + \begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \hat{W}^{-1} \\ &= (\lambda - 1)\Pi_{Wz} + \begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix} Q\hat{W}^{-1} \end{aligned} \tag{6.12}$$

where $Q^{p\times m} \stackrel{\text{def}}{:=} \begin{bmatrix} Q_1^T & Q_2^T \end{bmatrix}^T$ and $Q_1 \in \mathcal{RH}_\infty$ is equivalent to $T \in \mathcal{RH}_\infty$.

Equation (6.12) gives us an expression for $U_{W\lambda}$ in terms of unknown variables Q_1 and Q_2 where it is sufficient that Q_1 is stable for $U_{W\lambda}$ substituted in (6.10) to give a stable T. (That is, it is not necessary for Q_2 to be stable to give a stable T.) Consequently, defining $U_{W\lambda}$ as in (6.12) by choosing a particular stable Q is sufficient to ensure that T in (6.10) is stable.

One way to choose a particular (stable) Q is to temporarily ignore the inner-constraint on U_λ and to consider the following $\mathcal{H}_\infty$ minimisation of U_λ (thus retaining the stability constraint on U_λ) over stable Q.

$$\mu_\lambda \stackrel{\text{def}}{:=} \inf_{Q\in\mathcal{H}_\infty} \left\| \begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix} Q\hat{W}^{-1} + (\lambda - 1)\Pi_{Wz} \right\|_{\mathcal{H}_\infty}. \tag{6.13}$$

By considering the case where we set $Q = 0$ to give $U_{W\lambda} = [\lambda - 1] \cdot \Pi_{Wz}$ it is obvious that (6.13) admits a finite value with $\mu_\lambda \le (\lambda - 1) \le 1$. By Lemma 5.1 there will exist a $Q^\star \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix} Q^\star \hat{W}^{-1} + (\lambda - 1)\Pi_{Wz} = \mu_\lambda \Pi^\star_{W\lambda}$$

where $\Pi^\star_{W\lambda}$ is inner. Also, by Lemma 5.1, it is possible to choose a stable $\bar{Q}$ such that

$$U_{W\lambda} = \hat{\Pi}_{W\lambda} = \begin{bmatrix} \Pi_{\alpha p}\Theta^*_{\alpha z}\bar{Z}^{-1} & \Pi_{\alpha pc} \end{bmatrix} \bar{Q}\hat{W}^{-1} + (\lambda - 1)\Pi_{Wz} \tag{6.14}$$

where $\hat{\Pi}_{W\lambda}$ is inner and hence has unity norm as required for the second lemma statement.

For the third lemma statement we check the λ-interpolation conditions of $\hat{\Pi}_{W\lambda}$ at the end points of $\lambda \in [1, 2]$. Rearranging (6.12) in terms of Q gives

$$Q = \begin{bmatrix} \bar{Z}\Theta_{\alpha z}\Pi^*_{\alpha p} \\ \Pi^*_{\alpha pc} \end{bmatrix} \{U_{W\lambda} - (\lambda - 1)\Pi_{Wz}\}\hat{W}. \tag{6.15}$$

At $\lambda = 2$, we can set $U_{W\lambda}$ to Π_{Wz} in equation (6.15) to yield a stable $Q = \tilde{Q}(= 0)$ and hence $\bar{Q} = \tilde{Q}(= 0)$ in (6.14) gives back $\hat{\Pi}_{W\lambda} = \Pi_{Wz}$. Hence Π_{Wz} is a potential solution for the inner dilation (6.14) when $\lambda = 2$. It can also be confirmed that for $\lambda = 1$, setting $U_{W\lambda} = \Pi_{\alpha p}\Theta_{Vp}$ makes Q stable in (6.15) for any all-pass Θ_{Vp}. Hence $U_{W\lambda} = \Pi_{\alpha p}\Theta_{Vp}$ is also potential solution candidate for (6.14).

Note that $(\lambda - 1)\Pi_{Wz}$ in the $\mathcal{H}_\infty$ model matching problem (6.13) is a λ-homotopy, and the other transfer matrices in the model matching problem are fixed. By Corollary 5.5 of Theorem 5.4, it follows that we may construct $\hat{\Pi}_{W\lambda}$ to form a homotopy *as well as* to interpolate $\Pi_{\alpha p}\Theta_{Vp}$ for any Θ_{Vp} at $\lambda = 1$ and Π_{Wz} at $\lambda = 2$, provided that Π_{Wz} is strictly tall.

Also for the the case where Π_{Wz} is square it is possible to find a $\Pi_{W\lambda}$ *homotopy* via the minimisation problem (6.13) for $\lambda \in [1, 2]$ by Corollary 5.5. In order to prove the third lemma statement, we will now demonstrate that, at $\lambda = 1$, all inner solutions of (6.14) can be written as $\Pi_{W\lambda} = \Pi_{p\alpha}\Theta_{Vp}$ for some stable all-pass Θ_{Vp}. Consider once again, equation (6.14) for the case that $\lambda = 1$ and note that stable $\bar{Q}$ ensures that $T = \hat{T}_\lambda + Q_1$ is stable. If we observe equation (6.9) and note that for $\lambda = 1$ we have $\Theta_{R\lambda} = \Theta_{Rp}$ so that $R_\lambda = \hat{R}$ is stable, we conclude that the term $\bar{Z}\Theta_{\alpha z}\Pi^*_{\alpha p}\Pi_{W\lambda}\hat{W}$ is also stable. Since the inner transfer functions $\Theta_{\alpha z}$ and $\Pi_{\alpha p}$ are coprime, it follows that any cancellations of unstable poles in $\Pi^*_{\alpha p}$ must occur in the product $\Pi^*_{\alpha p}\Pi_{W\lambda}$ so that $\Pi_{W\lambda} = \Pi_{\alpha p}\Theta_{Vp}$ for some particular (square) stable inner Θ_{Vp}. □

6.2.3. Summary: Homotopy for $\lambda \in [0, 2]$. For our $\Pi_{W\lambda}$ homotopy for $\lambda \in [1, 2]$, recall that we simply set $\Pi_{W\lambda} = \hat{\Pi}_{W\lambda}$, where $\hat{\Pi}_{W\lambda}$ is defined by the previous lemma, Lemma 6.2. It then follows by Lemma 6.1 that stability of $\Pi_{W\lambda}$ and the stability of Q_1 (the submatrix of $\bar{Q}$ used in the construction of $\Pi_{W\lambda}$), ensures the stability of $T = \hat{T}_\lambda + Q_1$ and hence the stability of G_λ. This specifies the homotopy for $\lambda \in [1,2]$. A particular Θ_{V_p} is chosen that corresponds to the second half of the $\Pi_{W\lambda}$ homotopy, that is, for $\lambda \in [1,2]$. It is at this particular value that we hold $\Pi_{W\lambda}$ fixed for the first half of the homotopy, that is, for $\lambda \in [0,1]$, and hence we can ensure continuity of the $\Pi_{W\lambda}$ homotopy over the entire interval $\lambda \in [0,2]$. Recall also that $R_\lambda = \Theta_{R_p}^* \Theta_{R\lambda} \hat{R}$ and $W_\lambda = \Pi_{W\lambda} \hat{W}$, for $\lambda \in [1,2]$, and that this clearly matches the homotopies R_λ and W_λ for $\lambda \in [0,1]$ defined in Section 6.1. Substitution of these component homotopies back in equation (6.1) gives G_λ.

Now we can show that an arbitrarily small perturbation of a homotopy produced by the the above construction enjoys the three properties specified in the statement of Theorem 4.1 for the entire interval $\lambda \in [0,2]$, by identifying λ_α with $\lambda = 0$ and λ_β with $\lambda = 2$.

6.3. Coprimeness Recoverable by Perturbation. The following lemma demonstrates that for normalised stable transfer matrices $G_\lambda = \begin{bmatrix} N_\lambda^T & M_\lambda^T \end{bmatrix}^T$ where M_λ is square and the number of rows in N_λ is greater than unity, even if G_λ itself is not coprime, there exists arbitarily small perturbations G'_λ of G_λ, that are coprime.

LEMMA 6.3. *Suppose that*

$$G_\lambda(s) = \begin{bmatrix} N_\lambda(s) \\ M_\lambda(s) \end{bmatrix}$$

is a homotopy with N_λ, M_λ *stable rational transfer function matrices where the coefficients of the transfer function elements of the transfer matrices have a piecewise rational dependence on* λ. *Assume that* G_λ *is normalised, that is,* $G_\lambda^* G_\lambda = I$, *that* N_λ *either has more than one row and that* M_λ *is square.*

Then, even if G_λ *is not coprime, given any perturbation bound* η, *there exists a* **coprime** *normalised homotopy*

$$G'_\lambda(s) = \begin{bmatrix} N'_\lambda(s) \\ M'_\lambda(s) \end{bmatrix}$$

such that $\|G_\lambda - G'_\lambda\|_\infty < \eta$ *for all* λ.

Proof. See Appendix A.4. □

6.4. Proof of Sufficiency. *Proof.* [The Sufficiency part of the statement of Theorem 4.1.]

If $\delta_\nu(P_\alpha, P_\beta) < 1$, then construct a family of transfer matrices G_λ, parametrised by λ as in Section 6. If at, at most, a finite number of intervals in the λ interval the transfer matrix G_λ is not coprime, we produce a perturbation G'_λ which is normalised and coprime, and differs from G_λ by no more than $\frac{1}{2}\eta$, by appeal to Lemma 6.3. The construction in Section 6 ensures that G_λ fulfills the conditions of the Lemma, namely that the coefficients of the transfer function elements of the transfer matrices have a piecewise rational dependence on λ. We now show that P'_λ corresponding to G'_λ is a homotopy enjoying the three properties specified in the statement of Theorem 4.1.

Normalisation Property: Firstly, we check that $G_\lambda = \begin{bmatrix} N_\lambda^T & M_\lambda^T \end{bmatrix}^T$ defined by (6.1) is normalised.

First note that

$$\begin{aligned} G_\lambda^* G_\lambda &= \left(R_\lambda^* G_\alpha^* + W_\lambda^* \tilde{G}_\alpha\right)\left(G_\alpha R_\lambda + \tilde{G}_\alpha^* W_\lambda\right) \\ &= R_\lambda^* R_\lambda + W_\lambda^* W_\lambda \\ &= \hat{R}_\lambda^* \hat{R}_\lambda + \hat{W}_\lambda^* \hat{W}_\lambda \\ &= I. \end{aligned}$$

This is due to the properties of the right and left normalised coprime factorisations G_α and $\tilde{G}_\alpha$ and equation (6.4). That G_λ is stable follows from care taken in its construction. In particular, when $\lambda \in [0,1]$, $R_\lambda = \hat{R}_\lambda \in \mathcal{H}_\infty$ and $W_\lambda = \lambda \tilde{\Pi}_{\alpha p} \bar{\Theta}_{Vp} \hat{W}$ so that $\tilde{G}_\alpha W_\lambda \in \mathcal{H}_\infty$ and hence by equation (6.9), T is stable. When $\lambda \in [1,2]$ we have that $\hat{W}_\lambda = \Pi_{W_\lambda} \hat{W} \in \mathcal{H}_\infty$ where $\Pi_{W\lambda}$ also corresponds to $T = \bar{T} \in \mathcal{H}_\infty$. Hence by Lemma 6.1, it follows that G_λ is stable for all $\lambda \in [0,2]$.

Coprime Factorisation Property: While the numerator-denominator pair N_λ, G_λ may not be coprime at some points in homotopy, our appeal to Lemma 6.3 shows that, for multivariable plants, that is, with either more than one input or output, there exist the desired perturbations N'_λ and M'_λ within $\frac{1}{2}\eta$ of N_λ and M_λ in the $\mathcal{H}_\infty$-norm, which ensures coprimeness whilst preserving the normalisation property.

Endpoint Property: When $\lambda = 0$, then by equations (6.2) and (6.3) we get $R_\lambda = I$ and $W_\lambda = 0$. This gives $G_\lambda = G_\alpha$ as required. Furthermore, when $\lambda = 2$ we get $R_\lambda = \Theta_{Rp}^* \Theta_{Rz} \hat{R} = G_\alpha^* G_\beta$ and $W_\lambda = \Theta_W \hat{W} = W = \tilde{G}_\alpha G_\beta$. Hence $G_\lambda = G_\alpha G_\alpha^* G_\beta + \tilde{G}_\alpha^* \tilde{G}_\alpha G_\beta = G_\beta$.

Subunitary and Monotonicity Properties: For any P_λ where N_λ and M_λ are coprime, we have that $\bar{\kappa}(P_\alpha, P_\lambda) = \|\tilde{G}_\alpha G_\lambda\|_\infty = \|\tilde{G}_\alpha(G_\alpha R_\lambda + \tilde{G}_\alpha^* W_\lambda)\|_\infty = \|W_\lambda\|_\infty$. For the *unperturbed* homotopy, this is may be evaluated as $\lambda \cdot \|W\|_\infty = \lambda \cdot \delta_\nu(P_\alpha, P_\beta) < 1$ for $\lambda \in [0,1]$ and $\|W\|_\infty = \delta_\nu(P_\alpha, P_\beta) < 1$ for $\lambda \in [1,2]$. Except at values of λ where the coprimeness of N_λ, M_λ, the monotonicity property is also obvious from the previous expressions. Since we recover coprimeness by arbitrarily small perturbations N'_λ, M'_λ, of magnitude at most $\frac{1}{2}\eta$, we hence ensuring that the deviation $\bar{\kappa}(P_\alpha, P'_\lambda)$ from $\bar{\kappa}(P_\alpha, P_\lambda)$ is abitrarily small, and hence that $\bar{\kappa}(P_\alpha, P'_\lambda)$ is arbitrarily close to monotonic in the sense made precise in the theorem statement. Since $\delta_\nu(P_\alpha, P_\beta) < 1$ and $\delta_\nu(P_\alpha, P'_\lambda)$ can be made arbitrarily close to monotonic, it follows that it can also be made to fulfil the Subunitary Property.

Vinnicombe Continuity Property: We first determine that $R_\lambda = \Theta_{R\lambda} \hat{R}_\lambda$ and $W_\lambda = \Pi_{W\lambda} \hat{W}_\lambda$ are continuous in the $\mathcal{H}_\infty$ norm, because the products of $\mathcal{H}_\infty$ norm homotopies are also homotopies in $\mathcal{H}_\infty$ and each of $\Theta_{R\lambda}$, $\hat{R}_\lambda$, $\Pi_{W\lambda}$ and $\hat{W}_\lambda$ is continuous in the $\mathcal{H}_\infty$ norm. The transfer matrix $\hat{R}_\lambda$ is derived from (6.2) as a spectral factorisation of a rational parahermitian transfer matrix depending smoothly in λ and which has no imaginary axis poles or zeros, and W_λ is defined as $W_\lambda = \lambda \hat{W}$ for $\lambda \in [0,1]$

in equation (6.3), and as $W_\lambda = \hat{W}$ for $\lambda \in [1,2]$. The all-pass matrix $\Theta_{R\lambda}$ and the inner matrix $\Pi_{W\lambda}$ are constructed using the inner homotopy construction method in [17] with continuous input data. Hence, by Theorem 5.4 (and its corollaries), they are also continuous. This implies that G_λ is continuous, that is, an $\mathcal{H}_\infty$-homotopy. Furthermore, if any perturbation G'_λ is required in order to recover coprimeness, then the perturbation can be made to ensure that G'_λ is also an $\mathcal{H}_\infty$-homotopy.

For $P_{\hat{\lambda}}$ and P_λ with $\hat{\lambda}, \lambda \in [0,2]$ we have that

$$\begin{aligned} G^*_{\hat{\lambda}} G_\lambda &= R^*_{\hat{\lambda}} R_\lambda + W^*_{\hat{\lambda}} W_\lambda \\ &= I + R^*_{\hat{\lambda}}(R_\lambda - R_{\hat{\lambda}}) + W^*_{\hat{\lambda}}(W_\lambda - W_{\hat{\lambda}}) \end{aligned} \tag{6.16}$$

From the above expression, it can clearly be seen, by the continuity of W_λ and Q_λ that $\text{wno}(G^*_{\hat{\lambda}} G_\lambda) = 0$ for $\hat{\lambda}$ and λ sufficiently close. Since the winding number condition is satisfied for sufficiently close $\hat{\lambda}$ and λ, it then follows that

$$\delta_\nu(P_{\hat{\lambda}}, P_\lambda) = \|\tilde{G}_{\hat{\lambda}} G_\lambda\|_\infty .$$

Note however that from equation (6.16) we have

$$\begin{aligned} \underline{\sigma}^2(G^*_{\hat{\lambda}} G_\lambda) &\geq 1 - 2\bar{\sigma}\left[R^*_{\hat{\lambda}}(R_\lambda - R_{\hat{\lambda}}) + W^*_{\hat{\lambda}}(W_\lambda - W_{\hat{\lambda}})\right] \\ &\geq 1 - 2\bar{\sigma}\,(R_\lambda - R_{\hat{\lambda}}) - 2\bar{\sigma}\,(W_\lambda - W_{\hat{\lambda}}) . \end{aligned}$$

The second inequality in the above follows from the fact that both $\|R_{\hat{\lambda}}\|_\infty \leq 1$ and $\|W_{\hat{\lambda}}\|_\infty \leq 1$. Furthermore, since

$$\begin{aligned} &\delta_\nu(P_{\hat{\lambda}}, P_\lambda)^2 + \underline{\sigma}^2(G^*_{\hat{\lambda}} G_\lambda) = 1, \\ &\qquad \text{then } \delta_\nu(P_{\hat{\lambda}}, P_\lambda)^2 \leq 2\|R_\lambda - R_{\hat{\lambda}}\|_\infty + 2\|W_\lambda - W_{\hat{\lambda}}\|_\infty . \end{aligned}$$

Therefore $\delta_\nu(P_{\hat{\lambda}}, P_\lambda)$ is continuous in λ at $\lambda = \hat{\lambda}$ and hence by the triangle inequality $\delta_\nu(P_{\hat{\lambda}}, P_\lambda)$ is continuous in λ for all $\hat{\lambda}, \lambda \in [0,2]$. □

REMARK 6.4. *For case of scalar plants with the homotopy constrained to also be scalar, it remains to demonstrate that we can draw a similar conclusion provided that* $\mathcal{I}^{\infty}_{-\infty}[p_\alpha] = \mathcal{I}^{\infty}_{-\infty}[p_\beta]$. *This issue is not addressed in this paper, and is left as an open question.*

7. Conclusion. We have proven a conjecture of [1], also stated in [25], by showing that the winding number condition on two linear operators is equivalent to the existence of a homotopy in the Vinnicombe metric between them, although in the case of scalar plants, a *multivariable* homotopy must be permitted. We recall that a Cauchy index condition being fulfilled is necessary for the existance of a scalar homotopy between two given scalar plants, even if they are separated by a Vinnicombe metric distance of less than unity. Note that the Cauchy index is not well defined for general multivariable transfer matrices, but only for square symmetric multivariable transfer matrices.

This equivalence result enables us to give a characterisation of the winding number condition that is not specifically dependent on the fact that the operators are linear and enable the finalisation of the definition of a metric for nonlinear operators, originally given in [1], that reduces to the standard Vinnicombe metric for the case of linear operators. Further research in this area involves a more complete treatment of the scalar case. In particularly, it remains to deal with the case of scalar plants with differing Cauchy indices, and also to investigate whether equality of Cauchy index is sufficient to recover the equivalence of the winding number condition with scalar homotopy existence.

It remains to further investigate the properties of the nonlinear operator metric of [1], in order to investigate whether its robustness properties are as similarly non-conservative as they are in the linear case. It also remains to develop approximations and error bounds on the nonlinear Vinnicombe distance between nonlinear plants and their linear approximations.

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Appendix A. Proofs.

A.1. All-Pass Embedding: Proof of Lemma 5.1.

Proof. [Proof of Lemma 5.1] First factorise T_{12} into inner-outer factors $T_{12i}T_{12o}$ (as in equation (2.1) for the square case) and similarly T_{21} into outer-inner factors $T_{21o}T_{21i}$ (as in (2.2): again for the square case, and using the fact that for square transfer matrices, co-inner and inner are equivalent, as are outer and co-outer.) Since multiplication by inner factors preserves the $\mathcal{H}_\infty$ norm the optimisation problem is equivalent to

$$\inf_{Q\in\mathcal{H}_\infty} \|T_{12i}^* T_{11} T_{21i}^* - T_{12o} Q T_{21o}\|_{\mathcal{H}_\infty} \tag{A.1}$$

Decompose $T_{12i}^* T_{11} T_{21i}^* = G + H$ into a strictly antistable part G and a strictly stable part H and then reparametrise $T_{12o}QT_{21o} - H = \bar{Q}$. Neither T_{12o} nor T_{21o} has imaginary axis zeros and it follows that $\bar{Q} \in \mathcal{RH}_\infty \Leftrightarrow Q \in \mathcal{RH}_\infty$. The above problem (A.1) can be seen to be equivalent to

$$\inf_{\bar{Q}\in\mathcal{H}_\infty} \|G - \bar{Q}\|_{\mathcal{H}_\infty}$$

where $G = T_{12i}^* T_{11} T_{21i}^* - H \in \mathcal{RH}_\infty^-$. This is equivalent to a Nehari problem of approximating an $\mathcal{RH}_\infty$ function G by an $\mathcal{H}_\infty^-$ function $\bar{Q}$.

Using the all pass embedding method in [17], Section 10.3.1, given *any* γ (not necessarily greater than $\gamma^\star$), it is possible to find a $\bar{Q}_a^{(p+m)\times(p+m)}$ (not necessarily antistable, that is, not necessarily in $\mathcal{RH}_\infty^-$) which results in $\bar{E}_a = G_a^{(p+m)\times(p+m)} - \bar{Q}_a^{(p+m)\times(p+m)}$ where $\bar{E}^*\bar{E} = \gamma^2 I$. In the previous expression, G_a is an augmented version of $G^{p\times m}$ defined by

$$G_a^{(p+m)\times(p+m)} = \begin{bmatrix} G^{p\times m} & 0^{p\times p} \\ 0^{m\times m} & 0^{m\times p} \end{bmatrix}.$$

All $\bar{Q}^{p\times m} \in \mathcal{RH}_\infty^-$ such that $\bar{E} = G - \bar{Q}$ has the property $\bar{E}^*\bar{E} = \gamma^2 I$ are then given by $\bar{Q}^{p\times m} = F_l(\bar{Q}_a^{(p+m)\times(p+m)}, \gamma^{-1}U^{p\times m})$ where $U \in \mathcal{RL}_\infty$ is paraunitary, that is $U^*U = I$ (see Theorem 10.3.2 in [17] and its proof.)

In the set-up of [17], G is taken to be stable, that is, in $\mathcal{RH}_\infty$, and for $\gamma \geq \gamma^\star$, this results in antistable $\bar{Q} \in \mathcal{RH}_\infty^-$. In our problem, by looking at the conjugate systems the same method can be used for antistable $G \in \mathcal{RH}_\infty^-$ to result, for $\gamma \geq \gamma^\star$, in stable $\bar{Q} \in \mathcal{RH}_\infty$. We define the optimal $\bar{Q}^\star$ as the $\bar{Q} \in \mathcal{RH}_\infty$ which results from $\gamma = \gamma^\star$ and the optimal $Q^\star$ in the Lemma statement can be defined as $T_{12o}^{-1}(\bar{Q}^\star + H)T_{21o}^{-1}$. Substituting $Q = T_{12o}^{-1}(\bar{Q} + H)T_{21o}^{-1}$ back into the original system gives

$$\begin{aligned} E &= T_{11} - T_{12}QT_{21} \\ &= T_{12i}\left(G - \bar{Q}\right)T_{21i} \\ &= T_{12i}\bar{E}T_{21i} \end{aligned}$$

Since T_{12i}, T_{21i} and $\gamma^{-1}\bar{E}$ are paraunitary functions, then E has the properties described in the Lemma statement. □

A.2. All-Pass Transfer Matrices with Prescribed Zeros: Proof of Lemma 5.2.

Proof. [Proof of Lemma 5.2] A procedure for constructing an all-pass matrix with presribed zeros and zero directions appears in Chapter 6 of [27], see also [5] or pages 329-331 of [4]. The recursive procedure in [27] results in a rational $\Theta_Z(s)$ that satisfies the first three properties in the lemma statement.

We now argue that $\Theta_Z(s)$ so constructed can be chosen as real rational. That the factors of $\Theta_Z(s)$ corresponding to simple real zeros are real rational is fairly obvious.

To understand the case of complex z_k pairs we first consider the comparatively simple situation where we have only a single pair of complex numbers z and $\bar{z}$ and corresponding unit norm zero input direction vectors $\tilde{l}$ and $\bar{\tilde{l}}$. Decompose z as $\alpha + j\beta$ with $\alpha > 0$ and α and β real so that $\bar{z} = \alpha - j\beta \in \mathcal{Z}$.

We can easily find a stable all-pass real rational matrix $\Theta_{z\bar{z}}(s)$ with only two zeros such that both $\Theta_{z\bar{z}}(z)\tilde{l} = 0$ and $\Theta_{z\bar{z}}(\bar{z})\bar{\tilde{l}} = 0$. In [4] (equations (14)-(16), Chapter 11) this is achieved as follows.

Define

$$\begin{aligned} \eta &= \frac{\alpha}{z}(\tilde{l}^T\tilde{l}), \\ t &= \frac{1}{\sqrt{1 - |\eta|^2}}, \\ M(s) &= 2\alpha t^2\left[-(s+\alpha)(\bar{\tilde{l}}\tilde{l}^T + \tilde{l}\tilde{l}^* - \eta\tilde{l}\tilde{l}^T - \eta\bar{\tilde{l}}\tilde{l}^*) + j\beta(\bar{\tilde{l}}\tilde{l}^T - \tilde{l}\tilde{l}^* + \eta\bar{\tilde{l}}\tilde{l}^* - \eta\tilde{l}\tilde{l}^T)\right], \\ \Theta_{z\bar{z}}(s) &= I + \frac{1}{(s+\alpha)^2 + \beta^2}M(s). \end{aligned}$$

Then $\Theta_{z\bar{z}}(s)$ satisfies the desired properties $\Theta_{z\bar{z}}(z)\tilde{l} = \Theta_{z\bar{z}}(\bar{z})\bar{\tilde{l}} = 0$. In addition the polynomial matrix $M(s)$ is linear (a polynomial) in s, with real coefficients. Because

$\Theta_Z(s)$ is recursively constructed as a product of factors which are real rational, due to the fact that either each factor has a simple real zeros, or has a pair of complex conjugate zeroes, it follows that $\Theta_Z(s)$ itself is a real rational transfer matrix.

We can now show that $\Theta_Z(s)$ enjoys Property 4 considering the case of complex z_k pairs. Note that if z_k and z_j are complex conjugate pairs then $\tilde{\Theta}_{zk}(z_j) = \tilde{\Theta}_{zk}(\overline{z_k}) = \overline{\tilde{\Theta}_{zj}(z_k)}$. Note that $\tilde{\Theta}(s) = \tilde{\Theta}_{Z/k}(s)\tilde{\Theta}_{zk}(s) = \tilde{\Theta}_{Z/j}(s)\tilde{\Theta}_{zj}(s)$ is real rational. This implies that $\tilde{\Theta}(s) = \overline{\tilde{\Theta}(\overline{s})}$ and hence $\tilde{\Theta}_{Z/k}(s)\tilde{\Theta}_{zk}(s) = \overline{\tilde{\Theta}_{Z/j}(\overline{s})\tilde{\Theta}_{zj}(\overline{s})}$. Substitution of $s = z_j$ in the previous equation gives $\tilde{\Theta}_{Z/k}(z_j)\tilde{\Theta}_{zk}(z_j) = \overline{\tilde{\Theta}_{Z/j}(z_k)\tilde{\Theta}_{zj}(z_k)}$. It then follows that $\tilde{\Theta}_{Z/k}(z_k) = \overline{\tilde{\Theta}_{Z/j}(\overline{z_k})}$. We then have

$$\begin{aligned}
T_{kj}(s) &= \tilde{\Theta}_{Z/k}(s)\tilde{\Theta}_{Z/k}(z_k)^{-1}G + \tilde{\Theta}_{Z/j}(s)\tilde{\Theta}_{Z/j}(\overline{z_k})^{-1}\overline{G} \\
\overline{T_{kj}(s)} &= \overline{\tilde{\Theta}_{Z/k}(s)}\ \overline{\tilde{\Theta}_{Z/k}(z_k)^{-1}}\overline{G} + \overline{\tilde{\Theta}_{Z/j}(s)}\ \overline{\tilde{\Theta}_{Z/j}(\overline{z_k})^{-1}}G \in \mathbb{R}^{p\times m} \\
&= \tilde{\Theta}_{Z/k}(\overline{s})\tilde{\Theta}_{Z/j}(z_j)^{-1}\overline{G} + \tilde{\Theta}_{Z/k}(\overline{s})\tilde{\Theta}_{Z/k}(z_k)^{-1}G \\
&= T(\overline{s}).
\end{aligned}$$

It then follows that $T_{kj}(s) \in \mathbb{R}^{p\times m}$ is a real rational transfer matrix of s as required.

□

A.3. All-Pass Homotopy: Proof of Theorem 5.4. Before proving Theorem 5.4, we need several lemmas. One that we need is a lemma about the existence of a paraunitary homotopy connecting arbitrary compatibly dimensioned paraunitary transfer matrices. After proving some results regarding the existence of paraunitary homotopies in Section A.3.1, we then derive some properties of lower fractional transformation homotopies, recall a useful lemma from Reference [27] and finally prove a result showing that transfer matrix homotopies can be decomposed into stable and antistable homotopies, before finally presenting the proof of Theorem 5.4 in Section A.3.5.

A.3.1. Existence of Paraunitary Homotopies. We will eventually build up to some results regarding the existence of paraunitary homotopies by first presenting a homotopy result for constant orthogonal matrices and then extending that result to square and inner transfer matrices.

LEMMA A.1. *Let D_α and D_β be square and constant* **real orthogonal** *matrices. A* **real orthogonal** *homotopy D_λ from D_α and D_β exists if and only if the determinants of D_α and D_β are equal (of the same sign).*

Proof. We first prove that a homotopy D_λ exists for the case where the determinants of D_α and D_β are both positive. Given an arbitrary real orthogonal matrix D_ξ with positive determinant, all the eigenvalues have unit magnitude and there exists a (real) orthogonal transformation matrix V, such that $V_\xi^T D_\xi V_\xi = \Lambda_\xi$ where Λ_ξ is a block diagonal matrix with diagonal elements consisting of matrices of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and $\begin{bmatrix} +1 \end{bmatrix}$ and $\begin{bmatrix} -1 \end{bmatrix}$. Because $\det[D_\xi] = \det[\Lambda_\xi] = 1$ there are an even number of $\begin{bmatrix} -1 \end{bmatrix}$ blocks which may be expressed as

$$\begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix}$$

Noting that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \exp\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 \end{bmatrix} = \exp\begin{bmatrix} 0 \end{bmatrix}$$

it follows that we can express Λ as $\Lambda = e^S$ where S is a real skew matrix ($S + S^T = 0$). It follows that $D_\xi = V_\xi e^{S_\xi} V_\xi^T$. If the determinants of D_α to D_β are both positive, we may then construct a homotopy from D_α to D_β for $\lambda \in [0, 2]$ as follows. Define

$$D_\lambda = \begin{cases} V_\alpha e^{S_\alpha(1-\lambda)} V_\alpha^T & \text{for } \lambda \in [0,1] \\ V_\beta e^{S_\beta(\lambda-1)} V_\beta^T & \text{for } \lambda \in (1,2] \end{cases} \tag{A.2}$$

Note that at $\lambda = 0, 1$ and 2 we have $D_\lambda = D_\alpha, I$ and D_β respectively, and for $\lambda \in [0, 2]$ the matrix D_λ is real orthogonal with positive determinant. Thus we have shown that a real orthogonal homotopy D_λ exists for the case where $\det[D_\alpha] = \det[D_\beta] = 1$.

For the case where $\det[D_\alpha] = \det[D_\beta] = -1$ define a diagonal matrix $J = J^{-1} = \mathrm{diag}\{I, -1\}$ so that $\bar{D}_\xi = JD_\xi$ is orthogonal with determinant $+1$. We can use equation (A.2) to give a homotopy $\bar{D}_\lambda$ from $\bar{D}_\alpha$ to $\bar{D}_\beta$ and form a homotopy $D_\lambda = J\bar{D}_\lambda$.

To prove that for a real orthogonal homotopy D_λ to exist it must be that $\det[D_\alpha] = \det[D_\beta]$, note that the determinant of a constant unitary matrix, not necessarily real, has unit magnitude and the determinant of a constant real matrix is also real. Therefore a real orthogonal matrix has determinant either $+1$ or -1. If D_λ is a homotopy, then so is $\det[D_\lambda]$, and so it must be constant. □

We now use the above result to show that there exists a homotopy connecting arbitrary stable all-pass (square) transfer matrices provided they possess the same number of non-minimum-phase zeros. We first present a slightly reworded version Theorem 8.4 of [27] as a lemma about the properties of state-space representations of all-pass transfer matrices.

LEMMA A.2 (Properties of all-pass transfer matrices). *If $\Omega(s) \in \mathcal{RL}_\infty$ is an all-pass transfer matrix, not necessarily stable, then the state-space parameters of a minimal state-space realisation given by*

$$\Omega = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

satisfy the following. There exist $P = P^T$ *and* $Q = Q^T$ *such that*

$$\begin{aligned} D^T D = DD^T &= I, \\ AP + PA^T + BB^T &= 0, \\ A^T Q + QA + C^T C &= 0, \\ PQ &= I, \\ D^T C + B^T Q &= 0, \\ DB^T + CP &= 0. \end{aligned} \tag{A.3}$$

Conversely, if the state-space parameters of a particular realisation of a square transfer matrix satisfy equations (A.3), then the transfer matrix is all-pass, though not necessarily minimal. For minimal balanced realisations of stable systems $P = Q = I$.

Proof. See Theorem 8.4 of [27], and note that for a balanced realisation the controllability and observability Gramians are equal, and so for a balanced minimal realisation of an all-pass matrix they are equal to the identity. □

LEMMA A.3. *Let* $\Theta_\alpha(s)$ *and* $\Theta_\beta(s)$ *be two stable all-pass (square) transfer matrices with the same number* n *of non-minimum-phase zeros, and where* $\det[\lim_{s\to\infty} \Theta_\alpha(s)] = \det[\lim_{s\to\infty} \Theta_\beta(s)]$. *Then there exists an* $\mathcal{RH}_\infty$ *norm homotopy* $\Theta_\lambda(s)$ *from* $\Theta_\alpha(s)$ *to* $\Theta_\beta(s)$, *where* $\Theta_\lambda(s)$ *is stable and all-pass, with* n *non-minimum-phase zeros and* $\det[\lim_{s\to\infty} \Theta_\lambda(s)] = \det[\lim_{s\to\infty} \Theta_\alpha(s)] = \det[\lim_{s\to\infty} \Theta_\beta(s)]$.

Proof. Minimal state-space representations of Θ_α and Θ_β will be of equal state-space dimension n, the number of non-minimum-phase zeros, where $n = 2m + 1$ or $n = 2m$ for some integer m. Let *balanced* and *minimal* realisations of Θ_α and Θ_β be given by

$$\Theta_\xi = \left[\begin{array}{c|c} A_\xi & B_\xi \\ \hline C_\xi & D_\xi \end{array}\right],$$

with $\xi = \{\alpha, \beta\}$. By Lemma A.2 since Θ_ξ are all-pass, and because the realisation is balanced, then A_ξ, B_ξ, C_ξ and D_ξ satisfy equations (A.3) with $P = Q = I$.

We will form a state-space homotopy which satisfies the equations in Lemma A.2 and is also minimal. However we first investigate the observability properties of particular skew transition matrices. Let S be an n-dimensional skew matrix with distinct eigenvalues, and of the form

$$S = \operatorname{diag}\left\{ \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_m \\ -\omega_m & 0 \end{bmatrix} \right\}. \tag{A.4}$$

The zero block may be absent but all the eigenvalues $\pm j\omega_i$ lie on the imaginary axis and are ordered with $0 < \omega_1 < \omega_2 < \ldots < \omega_m$. To determine whether or not $[S, C]$ is observable, partition C compatibly with S as

$$C = \begin{bmatrix} C_0 & C_1 & C_2 & \cdots & C_m \end{bmatrix},$$

where C_0 may be absent or consists of a single column and for $i = 1 \dots m$ each C_i consists of two columns. Then it is easy to show that $[S, C]$ is an observable pair if and only if each C_i (with, of course, the possible exception of nonexistent C_0) contains at least one non-zero entry.

Note that an orthogonal change of coordinate basis preserves equations (A.3), so without loss of generality we may assume that the skew part of A_ξ, namely $\frac{1}{2}(A_\xi - A_\xi^T)$, is in the form of (A.4) where the zero matrix may be absent or of any size and $\omega_1 \le \omega_2 \le \dots \le \omega_m$ (there may also be other repeated imaginary eigenvalues).

If the number n, of non-minimum phase zeros of Θ_α, is odd, so that there is necessarily at least one zero eigenvalue, assume that the first non-zero entry in the first column of C_0 is positive (this may be ensured by the appropriate choice of orthogonal basis transformation).

Now form a homotopy for the skew part of A_λ as follows.

$$\text{(A.5)} \qquad \frac{1}{2}(A_\lambda - A_\lambda^T) = \operatorname{diag}\left\{ \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_{\lambda 1} \\ -\omega_{\lambda 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_{\lambda 2} \\ -\omega_{\lambda 2} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_{\lambda m} \\ -\omega_{\lambda m} & 0 \end{bmatrix} \right\}.$$

In the above, for $\lambda \neq 0$, $\lambda \neq 1$, we set $0 < \omega_{\lambda 1} < \omega_{\lambda 2} < \cdots < \omega_{\lambda m}$ so that the eigenvalues $j\omega_i$ are distinct and so that the zero matrix is absent in the case that the dimension n of A_λ is even. Also may we choose $A_\lambda - A_\lambda^T$ to correctly interpolate $A_\alpha - A_\alpha^T$ or $A_\beta - A_\beta^T$ at $\lambda = 0$ and $\lambda = 1$ respectively.

We may also form a homotopy $\tilde{C}_\lambda$ as

$$\tilde{C}_\lambda = (1-\lambda)C_\alpha + \lambda C_\beta.$$

It follows that $[\frac{1}{2}(A_\lambda - A_\lambda^T), \tilde{C}_\lambda]$ is observable provided that with $\tilde{C}_\lambda$ partitioned compatibly with (A.6) as

$$\tilde{C}_\lambda = \begin{bmatrix} \tilde{C}_{\lambda 0} & \tilde{C}_{\lambda 1} & \tilde{C}_{\lambda 2} & \cdots & \tilde{C}_{\lambda m} \end{bmatrix}$$

each $\tilde{C}_{\lambda i}$ contains at least one non-zero entry. Even if this is not the case, there will exist a perturbation C_λ of the above $\tilde{C}_\lambda$, which will preserve the homotopy property but ensure the required non-zero property for each $C_{\lambda i}$. This is obvious for each of the two column $C_{\lambda i}$ for $1 \le i \le m$. For $C_{\lambda 0}$, the requirement that the first non-zero entry of the first column of C_α and C_β is positive, is sufficient to ensure that an appropriate perturbation C_λ exists. We then define our homotopy C_λ as a (homotopic) perturbation of $\tilde{C}_\lambda$ such that $[\frac{1}{2}(A_\lambda - A_\lambda^T), C_\lambda]$ is observable.

At this point we have established that for $\lambda \in (0,1)$, every pair $[\frac{1}{2}(A_\lambda - A_\lambda^T), C_\lambda]$ is observable. Now define the symmetric part of A_λ as

$$A_\lambda + A_\lambda^T = -C_\lambda^T C_\lambda.$$

Since for $\xi = \alpha, \beta$ the feedthrough terms $D_\xi = \lim_{s\to\infty} \Theta_\xi(s)$ share the same sign (equal) determinants, it is possible, by Lemma A.1 to find a orthogonal homotopy D_λ connecting D_α to D_β. We may then define

$$B_\lambda = -C_\lambda^T D_\lambda$$

so that $A_\lambda, B_\lambda, C_\lambda, D_\lambda$ satisfy the properties (A.3) with $P = Q = I$, that is, they form a balanced realisation of a stable, all-pass transfer matrix Θ_λ, which is a homotopy connecting Θ_α to Θ_β.

We now demonstrate that the realisation is minimal. Since $[A_\lambda - A_\lambda^T, C_\lambda]$ is observable then, because output feedback preserves observability [19], so is $[\frac{1}{2}(A_\lambda - A_\lambda^T - C_\lambda^T C_\lambda), C_\lambda]$, that is $[A_\lambda, C_\lambda]$. Since $[A_\lambda, C_\lambda]$ is observable, by duality we have that $[A_\lambda^T, C_\lambda^T] = [A_\lambda^T, B_\lambda D_\lambda]$ and $[A_\lambda^T, B_\lambda]$ is controllable. Because state feedback preserves controllability so too is $[A_\lambda^T + B_\lambda B_\lambda^T, B_\lambda]$ controllable and by equation (A.3) also $[-A_\lambda, B_\lambda]$ and so $[A_\lambda, B_\lambda]$. It follows that the state-space parameters $A_\lambda, B_\lambda, C_\lambda$ and D_λ form a minimal realisation. Since the realisation is minimal and all-pass, the poles of A_λ can never cross the imaginary axis and therefore Θ_λ with state-space dimension n possesses n non-minimum-phase zeros. □

We now extend Lemma A.3 to prove the existence of homotopies for more general paraunitary transfer matrices, which may or may not be square.

LEMMA A.4. *Assume that we are given two $p \times m$ dimensioned ($p \geq m$) real rational paraunitary matrices $U_\alpha(s)$ and $U_\beta(s)$. Then if $p > m$ then there exists an $\mathcal{RL}_\infty$ homotopy $U_\lambda(s)^{p\times m}$ such that $U_\lambda(s)$ is paraunitary for each λ. If $p = m$ then there exists an $\mathcal{RL}_\infty$ homotopy $U_\lambda(s)^{p\times p}$ such that $U_\lambda(s)$ is paraunitary for each λ if and only if the winding numbers of* $\det[U_\alpha(s)]$ *and* $\det[U_\beta(s)]$ *are equal and* $\det[\lim_{s\to\infty} U_\alpha(s)] = \det[\lim_{s\to\infty} U_\beta(s)]$.

Proof. For $p > m$, we can express a $p \times m$ paraunitary matrix U_ξ as a product

$$\begin{aligned} U_\xi(s) &= \begin{bmatrix} U_\xi(s) & U_{\xi c}(s) \end{bmatrix} \begin{bmatrix} I^{m\times m} \\ 0 \end{bmatrix} \\ &= \Omega_\xi(s)^{p\times p} \begin{bmatrix} I \\ 0 \end{bmatrix}, \end{aligned}$$

where $U_{\xi c}(s)$ is a paraunitary completion of $U_\xi(s)$ so that $\Omega_\xi(s)$ is all-pass. For the case where $p > m$ it is possible to choose all-pass completions Ω_α and Ω_β of U_α and U_β such that both the winding numbers and signs of the determinant at infinity for each are equal. To see this, observe that

$$\Omega_\xi^{p\times p}(s) \begin{bmatrix} I \\ 0 \end{bmatrix}^{p\times m} = \Omega_\xi(s)^{p\times p} \begin{bmatrix} I^{m\times m} & 0 \\ 0 & \tilde{\Omega}_\xi(s)^{(p-m)\times(p-m)} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}^{p\times m}$$

where $\tilde{\Omega}_\xi(s)$ can be chosen as all-pass and both wno$\{\det[\tilde{\Omega}_\xi]\}$and $\det[\lim_{s\to\infty} \tilde{\Omega}_\xi(s)]$ can be chosen arbitrarily.

Hence, it will be sufficient to show that arbitrary all-pass $\Omega_\xi(s)$ are homotopically equivalent, provided the winding numbers of their determinants are equal and that the signs of $\det[\lim_{s\to\infty} \Omega_\xi(s)]$ are also equal. Since $\Omega_\xi(s)$ is real rational, then $\lim_{s\to\infty} \Omega_\xi(s)$ is (real) orthonormal, so that $\det[\lim_{s\to\infty} \Omega_\xi(s)]$ is either $+1$ or -1. Without loss of generality, we will assume that it is $+1$.

As in equation (2.10) factorise Ω_α and Ω_β, with equal winding numbers, as $\Theta_{z\alpha}\Theta^*_{p\alpha}$ and $\Theta_{z\beta}\Theta^*_{p\beta}$ respectively, where $\Theta_{z\alpha}, \Theta_{p\alpha}, \Theta_{z\beta}$ and $\Theta_{p\beta}$ is each stable all-pass. By exploiting the freedom to adjust $\Theta_{z\xi}$ and $\Theta_{p\xi}$ by a constant orthogonal matrix we can assume that the determinants at infinity are each $+1$. Without loss of generality assume that the number of zeros in $\Theta_{z\alpha}$ is less than or equal to that in $\Theta_{z\beta}$ and form an augmented stable all-pass $\Theta_{z\alpha'} = \Theta_{z\alpha}\Theta_n$ where Θ_n is a stable all-pass transfer matrix with $n = \text{wno}\{\det[\Theta_{z\alpha}]\} - \text{wno}\{\det[\Theta_{z\beta}]\}$ non-minimum-phase zeros and so that $\Theta_{z\alpha'}$ and $\Theta_{z\beta}$ have the same number of non-minimum phase zeroes and sign of the determinant at infinity. Similarly define $\Theta_{p\alpha'} = \Theta_{p\alpha}\Theta_n$. It can easily be seen that $\Theta_{p\alpha'}$ and $\Theta_{p\beta}$ also have the same number of non-minimum-phase zeros and sign of determinant at infinity.

We can then construct a homotopy $\Theta_{z\lambda}$ from $\Theta_{z\alpha'}$ to $\Theta_{z\beta}$ and a homotopy $\Theta_{p\lambda}$ from $\Theta_{p\alpha'}$ to $\Theta_{p\beta}$. Such a homotopy exists by the previous lemma, Lemma A.3. The product $\Theta_\lambda = \Theta_{z\lambda}\Theta^*_{p\lambda}$ is a homotopy from Θ_α to Θ_β which preserves the winding number at each λ.

So we have shown that if the winding numbers and signs of the determinants of two all-pass transfer matrices are equal, then there exists an all-pass $\mathcal{H}_\infty$-norm continuous homotopy. We complete the proof by arguing that these conditions are necessary.

Suppose that we have an all-pass homotopy $\Omega_\lambda(s)$, from Ω_α to Ω_β, then $|\det[\Omega_\lambda(s)]| = 1$ for all s on the imaginary axis, for each λ. Since for fixed $j\omega$ we have that $\det[\Omega_\lambda(j\omega)]$ is a continuous function of λ it then follows that $\text{wno}\{\det[\Omega_\lambda]\}$ is constant for all λ. A similar continuity argument applies for $\det[\lim_{s\to\infty} \Omega_\lambda(s)]$ since $\lim_{s\to\infty} \Omega_\lambda(s)$ is real. $\square$

A.3.2. Properties of Lower Fractional Transformation Homotopies. We now present a result regarding the preservation of winding numbers and determinants at infinity, when a paraunitary transfer matrix undergoes a lower fractional transformation.

LEMMA A.5. *Assume that we are given a real-rational square* $2p \times 2p$ *all-pass matrix*

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{G}_{11}{}^{p\times p} & \mathbf{G}_{12}{}^{p\times p} \\ \mathbf{G}_{21}{}^{p\times p} & \mathbf{G}_{22}{}^{p\times p} \end{bmatrix}$$

and a square $p \times p$ *all-pass matrix* $\mathbf{Q}$. *Assume that the maximum singular values of the block diagonal elements* $\mathbf{G}_{11}$ *and* $\mathbf{G}_{22}$ *are strictly less than unity (that is* $\bar{\sigma}(\mathbf{G}_{11}) = \bar{\sigma}(\mathbf{G}_{22}) < 1$*).*

Then the winding number of the determinant of the feedback interconnection

$$\bar{\mathbf{E}} = F_l(\mathbf{G}, \mathbf{Q}) = \mathbf{G}_{11} + \mathbf{G}_{12}\mathbf{Q}(\mathbf{I} - \mathbf{G}_{22}\mathbf{Q})^{-1}\mathbf{G}_{21}$$

obeys

$$\operatorname{wno}\left\{\det[\bar{\mathbf{E}}]\right\} = \operatorname{wno}\{\det[\mathbf{G}]\} + \operatorname{wno}\{\det[-\mathbf{Q}]\}.$$

Furthermore $\det[\lim_{s\to\infty} \bar{\mathbf{E}}(s)] = \det[\lim_{s\to\infty} \mathbf{G}(s)] \cdot \det[\lim_{s\to\infty} -\mathbf{Q}(s)]$.

REMARK A.6. *Note that* $\det[\lim_{s\to\infty} -\mathbf{Q}(s)] = \det[\lim_{s\to\infty} \mathbf{Q}(s)](-1)^p$, *where* p *is the size of the matrix* $\mathbf{Q}(s)$. *Note also that* $\operatorname{wno}\{\det[-\mathbf{Q}]\} = \operatorname{wno}\{\det[\mathbf{Q}]\}$, *but that in order to be consistent with the determinant at infinity condition we use* $-\mathbf{Q}$.

Proof. We first note that

$$F_l\left(\begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix}, \mathbf{Q}\right)) = F_l\left(\begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12}\mathbf{Q} \\ \mathbf{G}_{21} & \mathbf{G}_{22}\mathbf{Q} \end{bmatrix}, \mathbf{I}\right),$$

and let a minimal state-space realisation of

$$\begin{aligned} \mathbf{E} :&\stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12}\mathbf{Q} \\ \mathbf{G}_{21} & \mathbf{G}_{22}\mathbf{Q} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{Q} \end{bmatrix} \end{aligned}$$

be given by

$$\left[\begin{array}{c|cc} A_E & B_{E1} & B_{E2} \\ \hline C_{E1} & D_{E11} & D_{E12} \\ C_{E2} & D_{E21} & D_{E22} \end{array}\right].$$

We can see that $\operatorname{wno}[\det(\mathbf{E})] = \operatorname{wno}[\det(\mathbf{G})] + \operatorname{wno}[\det(-\mathbf{Q})]$ and $\det[\lim_{s\to\infty} \mathbf{E}(s)] = \det[\lim_{s\to\infty} \mathbf{G}(s)] \cdot \det[\lim_{s\to\infty} \mathbf{Q}(s)]$. Because $\mathbf{E}$ is the product of two paraunitary transfer matrices, it is itself paraunitary. Hence the minimal state-space realisation above obeys the properties in equation (A.3), with controllability and observability Gramians P_E and Q_E, with $P_E = Q_E^{-1}$. A state space realisation of

$$\begin{aligned} \bar{\mathbf{E}} &= F_l(\mathbf{G}, \mathbf{Q}) = F_l(\mathbf{E}, \mathbf{I}) \\ &= \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \end{aligned}$$

is given in Chapter 10 of [27] as

$$\begin{aligned} \bar{A} &= A_E + B_{E2}(I - D_{E22})^{-1}C_{E2}, \\ \bar{B} &= B_{E1} + B_{E2}(I - D_{E22})^{-1}D_{E21}, \\ \bar{C} &= C_{E1} + D_{E12}(I - D_{E22})^{-1}C_{E2}, \\ \bar{D} &= D_{E11} + D_{E12}(I - D_{E22})^{-1}D_{E21}. \end{aligned} \tag{A.6}$$

As in Lemma 4.3.4 of [17] we consider the zeros of $\det(\bar{A}_{cl}(\epsilon) - sI)$ where

$$\begin{aligned}
&\bar{A}_{cl} \stackrel{\text{def}}{:=} A_E + \epsilon B_{E2}(I - \epsilon D_{E22})^{-1} C_{E2},\\
&\det[\bar{A}_{cl}(\epsilon) - sI] = \det[A_E - sI + \epsilon B_{E2}(I - \epsilon D_{E22})^{-1} C_{E2}]\\
&= \det[A_E - sI]\det[I - \epsilon(sI - A_E)^{-1} B_{E2}(I - \epsilon D_{E22})^{-1} C_{E2}]\\
&= \det[A_E - sI]\det[I - \epsilon C_{E2}(sI - A_E)^{-1} B_{E2}(I - \epsilon D_{E22})^{-1}]\\
&= \det[A_E - sI]\det[(I - \epsilon D_{E22})^{-1}]\det[I - \epsilon\{C_{E2}(sI - A_E)^{-1} B_{E2} + D_{22}\}]\\
&= \det[A_E - sI]\det[(I - \epsilon D_{E22})^{-1}]\det[I - \epsilon E_{22}(s)].
\end{aligned}$$

Using similar arguments as in Lemma 4.3.4 of [17] we can deduce that for small enough ϵ, the polynomial $\det(\bar{A}_{cl}(\epsilon) - sI)$ has the same number of right half-plane zeros as $\det[A_E - sI]$. Furthermore, we can deform $\det(\bar{A}_{cl}(\epsilon) - sI)$ continuously as ϵ moves in the interval $[0, 1]$, and because $\bar{\sigma}[E_{22}(s)] < 1$, the polynomial $\det(\bar{A}_{cl}(1) - sI)$ also has the same number of right half-plane zeroes as $\det[A_E - sI]$.

Because the inverse $\bar{\mathbf{E}}^{-1}$ has a state-space realisation (Lemma 3.15 [27])

$$\bar{\mathbf{E}} = \left[\begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & -\bar{B}\bar{D}^{-1} \\ \hline \bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right],$$

we also investigate the eigenvalues of the matrix $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$. From the fact that $D_E^{-1} = D_E^T$ it is straightforward, though tedious, to show that

$$\bar{D}^{-1} = D_{E11}^T + D_{E21}^T(I - D_{E22}^T)^{-1} D_{E12}^T.$$

We can then determine, by using equations (A.6), the fact that $D_E^{-1} = D_E^T$ and after much algebraic manipulation, that

$$\begin{aligned}
\bar{A} - \bar{B}\bar{D}^{-1}\bar{C} \; &= A_E - \begin{bmatrix} B_{E1} & B_{E2} \end{bmatrix}\\
&\begin{bmatrix} D_{E11}^T + D_{E21}^T(I - D_{E22}^T)^{-1} D_{E12}^T & D_{E21}^T(I - D_{E22}^T)^{-1} \\ (I - D_{E22}^T)^{-1} D_{E12}^T & D_{E22}^T(I - D_{E22}^T)^{-1} \end{bmatrix} \begin{bmatrix} C_{E1} \\ C_{E2} \end{bmatrix}.
\end{aligned} \tag{A.7}$$

In proving the above assertion, it is useful to note that

$$\begin{aligned}
&(I - D_{E22})^{-1} D_{E21}\left[D_{E11}^T + D_{E21}^T(I - D_{E22}^T)^{-1} D_{E12}^T\right] = (I - D_{E22}^T)^{-1} D_{E12}^T,\\
&\left[D_{E11}^T + D_{E21}^T(I - D_{E22}^T)^{-1} D_{E12}^T\right] D_{E12}(I - D_{E22})^{-1} = D_{E21}^T(I - D_{E22}^T)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
&(I - D_{E22})^{-1} D_{E21}\left[D_{E11}^T + D_{E21}^T(I - D_{E22}^T)^{-1} D_{E12}^T\right] D_{E12}(I - D_{E22})^{-1} - I\\
&= D_{E22}^T(I - D_{E22}^T)^{-1},\\
&= (I - D_{E22}^T)^{-1} D_{E22}^T.
\end{aligned}$$

From equation (A.3) we can also establish that

$$\begin{bmatrix} B_{E1} & B_{E2} \end{bmatrix} = -P_E \begin{bmatrix} C_{E1}^T & C_{E2}^T \end{bmatrix} D_E,$$

and substitution of the above into (A.7) gives

$$
\begin{aligned}
\bar{A} - \bar{B}\bar{D}^{-1}\bar{C} &= A_E + P_E\left[C_{E1}^T C_{E1} + C_{E2}^T (I - D_{E22}^T)^{-1}(D_{E12}^T C_{E1} + C_{E2})\right] \\
&= A_E + P_E\left[C_{E1}^T C_{E1} + C_{E2}^T C_{E2} + C_{E2}^T (I - D_{E22}^T)^{-1}(D_{E12}^T C_{E1} + D_{E22}^T C_{E2})\right] \\
&= A_E + P_E\left[C_E^T C_E - C_{E2}^T (I - D_{E22}^T)^{-1} B_{E2}^T Q_E\right] \\
&= P_E(Q_E A_E + C_E^T C_E - C_{E2}^T (I - D_{E22}^T)^{-1} B_{E2}^T Q_E) \\
&= P_E(-A_E^T - C_{E2}^T (I - D_{E22}^T)^{-1} B_{E2}^T) Q_E \\
&= -P_E \bar{A}^T P_E^{-1}
\end{aligned}
$$

where the last string of equalities is also due to (A.3). It then follows that the number of right half-plane eigenvalues of $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ is the same as the number of left half-plane eigenvalues of $\bar{A}$. By the homotopy argument involving $\bar{A}_{cl}$ appearing in the previous paragraph, this is the same as the number of left half-plane eigenvalues in A_E, that is, the number of stable poles of $\mathbf{E}$. For an arbitrary real rational paraunitary transfer matrix $\mathbf{E}(s)$ the poles and zeroes occur in complementary pairs, reflected in the imaginary axis. The number of right half-plane eigenvalues of $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ is therefore equal to the number of right half-plane zeros of $\mathbf{E}$.

Hence there exists a state space realisation of $\bar{\mathbf{E}}$ where the state-transition matrix $\bar{A}$ has the same number of right half-plane eigenvalues as that of A_E (the number of unstable poles of $\mathbf{E}$) and the state transition matrix $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ of the inverse $\bar{\mathbf{E}}^{-1}$ has the same number of right half-plane eigenvalues as the number of non-minimum phase zeros of $\mathbf{E}$. Now, not necessarily all eigenvalues of $\bar{A}$ and $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ are poles and zeros respectively of $\bar{\mathbf{E}}$, but even if there are some pole-zero cancellations in this particular state-space realisation of $\bar{\mathbf{E}}$, it would still hold true that wno{det[$\bar{\mathbf{E}}$]} is equal to the difference between the number of right hand plane eigenvalues of $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ and $\bar{A}$. By the immediately preceding arguments, this is exactly the difference between the number of right half-plane zeros and poles of $\mathbf{E}(s)$ which is wno[det($\mathbf{E}$)] = wno[det($\mathbf{G}$)] + wno[det($-\mathbf{Q}$)]. Hence the winding number result is proved.

The proof for det[$\lim_{s\to\infty} \bar{\mathbf{E}}(s)$] follows similarly as follows. Note that the determinant of the following matrix is equal to det[D_E].

$$
\begin{aligned}
&\begin{bmatrix} \bar{D} & 0 \\ D_{E21} & D_{E22} - D_{E21}\bar{D}^{-1}D_{E12}(I - D_{E22})^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I & D_{E12}(I - D_{E22})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{E11} & D_{E12} \\ D_{E21} & D_{E22} \end{bmatrix} \begin{bmatrix} I & -\bar{D}^{-1}D_{E12}(I - D_{E22})^{-1} \\ 0 & I \end{bmatrix}.
\end{aligned}
$$

Using the fact that $\bar{D}^{-1} = \bar{D}^T$, it is possible to show that the matrix in the lower right block of the above evaluates to

$$
D_{E22} - D_{E21}\bar{D}^{-1}D_{E12}(I - D_{E22}^{-1})^{-1} = (D_{E22} - I)(I - D_{E22}^T)^{-1}.
$$

So that

$$\begin{aligned}\det[D_E] &= \det[\bar{D}]\det[D_{E22} - D_{E21}\bar{D}^{-1}D_{E12}(I - D_{E22})^{-1}] \\ &= \det[\bar{D}]\det[D_{E22} - I]\det[I - D_{E22}^T]^{-1}, \\ \det[\bar{D}] &= \frac{\det[I - D_{E22}^T]}{\det[D_{E22} - I]} \cdot \det[D_E] \\ &= (-1)^p \det[D_E].\end{aligned}$$

Hence,

$$\begin{aligned}\det[\lim_{s\to\infty} \bar{\mathbf{E}}(s)] &= (-1)^p \det[\lim_{s\to\infty} \mathbf{E}(s)] \\ &= (-1)^p \det[\lim_{s\to\infty} \mathbf{G}(s)] \cdot \det[\lim_{s\to\infty} \mathbf{Q}(s)] \\ &= \det[\lim_{s\to\infty} \mathbf{G}(s)] \cdot \det[\lim_{s\to\infty} -\mathbf{Q}(s)]\end{aligned}$$

as required. □

A.3.3. Inertia Properties: Lyapunov Equation Solution. The reference [27] establishes the following useful result on the inertia of the solutions to the Lyapunov equation.

LEMMA A.7. *Consider the Lyapunov equation*

$$AP + PA^T + BB^T = 0$$

where P is constrained as $P = P^T$. Let $\pi(X), \nu(X), \zeta(X)$ denote the number of positive real part, negative real part and zero real part eigenvalues of a matrix X. Then:

- $\zeta(P) = 0$ *implies that* $\pi(A) \leq \nu(P)$ *and* $\nu(A) \leq \pi(P)$ *and*
- $\zeta(A) = 0$ *implies that* $\pi(P) \leq \nu(A)$ *and* $\nu(P) \leq \pi(A)$.
- *As a trivial consequence we see that* $\zeta(P) = \zeta(A) = 0$ *implies that* $\pi(A) = \nu(P)$ *and* $\nu(A) = \pi(P)$.

Proof. See reference [27]. □

A.3.4. Stable-Unstable Decomposition of Transfer Matrix Homotopies. We will soon introduce a lemma showing that a real-rational transfer matrix homotopy, with continuous state-space parameters, can be decomposed into a strictly stable and an unstable part, each also with continuous state-space parameters. In order to do so, we first demonstrate that a particular operator, that arises in [16, 22] in discussion there of the Schur decomposition factorisation of matrices, has a non-zero lower bound, given certain conditions on its arguments.

LEMMA A.8. *Assume that we are given two equally dimensioned square matrices R and S, such that real parts of the eigenvalues of each of the matrices are bounded away from each other, that is, there exist real numbers r and δ such that* $\mathrm{Re}(\lambda_i(R)) < r - \delta$ *and* $\mathrm{Re}(\lambda_i(R)) > r + \delta$. *Suppose also that the matrices are norm bounded by M so that $\|R\|_F < M$ and $\|S\|_F < M$. It then follows that the separation [16, 22], between the two matrices defined by*

$$\mathrm{sep}(R, S) = \inf_X \frac{\|RX - XS\|_F}{\|X\|_F}$$

is bounded below by a non-zero bound.

Proof. Consider the optimisation problem

$$\begin{aligned}
\mu &= \inf_{R\in\bar{\mathcal{R}}, S\in\bar{\mathcal{S}}} \operatorname{sep}(R,S) \\
&= \inf_{R\in\bar{\mathcal{R}}, S\in\bar{\mathcal{S}}, \|X\|_F=1} \|RX - XS\|_F \\
\mathcal{R} &= \{R|\operatorname{Re}(\lambda_i(R)) < r-\delta, \|R\|_F < M\} \\
\mathcal{S} &= \{S|\operatorname{Re}(\lambda_i(S)) > r+\delta, \|S\|_F < M\} \\
\bar{\mathcal{R}} &= \operatorname{clos}(\mathcal{R}) \\
\bar{\mathcal{S}} &= \operatorname{clos}(\mathcal{S})
\end{aligned}$$

Because the sets $\bar{\mathcal{R}}$ and $\bar{\mathcal{S}}$ are compact sets, it follows by Theorem 2.5.18 [21] that the infimum is actually a minimum, that is, it is attained by some $R^\star \in \bar{\mathcal{R}}, S^\star \in \bar{\mathcal{S}}, \|X^\star\|_F = 1$. Let $Y^\star$ be defined by the Sylvester equation

$$Y^\star = R^\star X^\star - X^\star S^\star.$$

It is clear that $\|Y^\star\|_F \neq 0$ since by Lemma 2.7 [27], since $R^\star$ and $S^\star$ have no common eigenvalues, the unique solution in X to

$$0 = R^\star X - X S^\star$$

is $X = 0 \neq X^\star$ where $\|X^\star\|_F = 1$. Therefore, for all $R \in \mathcal{R} \subset \bar{\mathcal{R}}, S \in \mathcal{S} \subset \bar{\mathcal{S}}$ we have

$$0 < \|Y^\star\|_F = \operatorname{sep}(R^\star, S^\star) \stackrel{\text{def}}{:=} \eta \leq \operatorname{sep}(R,S).$$

□

The above property of the sep operator is needed to prove the following lemma.

LEMMA A.9. *Assume that we are given finite dimensional homotopy* $\mathbf{G}_\lambda \in \mathcal{RL}_\infty$ *where the poles of* $\mathbf{G}_\lambda$ *are bounded away from the imaginary axis, and with a continuous and bounded state-space parameter realisation* $\mathbf{G}_\lambda(s) = C_\lambda(sI - A_\lambda)^{-1}B_\lambda + D_\lambda$. *Then we can decompose* $\mathbf{G}_\lambda = \mathbf{G}_{\lambda\mathbf{u}} + \mathbf{G}_{\lambda\mathbf{s}}$ *where* $\mathbf{G}_{\lambda\mathbf{s}} \in \mathcal{RH}_\infty$ *and* $\mathbf{G}_{\lambda\mathbf{u}} \in \mathcal{RH}_\infty^-$, *each with continuous state-space parameters.*

Proof. Since the eigenvalues of A_λ are bounded away from the imaginary axis. Then there exists an orthogonal transformation Q_λ on A_λ such that

$$Q_\lambda^T A_\lambda Q_\lambda = \begin{bmatrix} A_{s\lambda} & A_{12\lambda} \\ 0 & A_{u\lambda} \end{bmatrix},$$

where all the eigenvalues of $A_{s\lambda}$ are in the open left half-plane and all the eigenvalues of $A_{u\lambda}$ are in the open right half-plane. It can be shown that because the eigenvalues of $A_{s\lambda}$ and $A_{u\lambda}$ are bounded away from each other, that Q_λ may be chosen to also be continuous in λ. This can be seen on consideration of Lemma A.8 and the discussion in [16], (pages 341-351) and [22]. (Discussion of the continuity of eigenspaces of continuous (holomorphic) A_λ is also dealt with in [20]. See Chapter 2, Section 5.3 on page 110.) This implies that $A_{s\lambda}$, $A_{u\lambda}$ and $A_{12\lambda}$ are also continuous. Furthermore, we can define a matrix X_λ, also continuous in λ, as the solution to the Lyapunov equation

$$A_{s\lambda}X_\lambda + X_\lambda(-A_{u\lambda}) = A_{21\lambda},$$

where in fact, $X = \int_0^\infty \exp[A_{s\lambda}\tau]A_{21\lambda}\exp[-A_{u\lambda}\tau]d\tau$. The solution to the Lyapunov equation is unique and finite because the none of the real parts of the eigenvalues of $A_{s\lambda}$ and $-A_{u\lambda}$ will sum to zero [27]. Because $A_{s\lambda}$, $A_{u\lambda}$ and $A_{12\lambda}$ is each continuous in λ it follows that X_λ is also continuous in λ. We can now define a (continuous) state-space transformation $T_\lambda = Q_\lambda\hat{X}_\lambda$ where

$$\hat{X}_\lambda = \begin{bmatrix} I & X_\lambda \\ 0 & I \end{bmatrix},$$

so that $T_\lambda^{-1}A_\lambda T_\lambda = \text{diag}\{A_{s\lambda}, A_{u\lambda}\}$. Hence, if we define $C_{s\lambda}$, $C_{u\lambda}$, $B_{s\lambda}$ and $B_{u\lambda}$ by

$$C_\lambda T_\lambda = \begin{bmatrix} C_{s\lambda} & C_{u\lambda} \end{bmatrix}$$
$$\text{and } T_\lambda^{-1}B_\lambda = \begin{bmatrix} B_{s\lambda} \\ B_{u\lambda} \end{bmatrix},$$

then it easily follows that

$$\begin{aligned} \mathbf{G}_\lambda &= C_{s\lambda}(sI - A_{s\lambda})^{-1}B_{s\lambda} + C_{u\lambda}(sI - A_{u\lambda})^{-1}B_{u\lambda} + D_\lambda \\ &= \mathbf{G}_{\lambda\mathbf{s}} + \mathbf{G}_{\lambda\mathbf{u}} \end{aligned}$$

where the state space parameters of $\{A_{s\lambda}, B_{s\lambda}, C_{s\lambda}, 0\}$ of strictly stable $\mathbf{G}_{\mathbf{s}\lambda}$ are continuous in λ, as are the state space parameters $\{A_{u\lambda}, B_{u\lambda}, C_{u\lambda}, D_\lambda\}$ of unstable $\mathbf{G}_{\lambda\mathbf{u}}$. □

A.3.5. Proof of Theorem 5.4. With the four lemmas A.4, A.5, A.7 and A.9 in place, we are now in a position to prove Theorem 5.4.

Proof. [Proof of Theorem 5.4] This proof is based on the all-pass embedding method of [17], which, given a stable real rational $\mathbf{G}^{p\times m} \in \mathcal{RH}_\infty$ determines a corresponding error function $\mathbf{E} = \mathbf{G} - \bar{\mathbf{Q}}$, such that $\mathbf{E}^*\mathbf{E} = I$ with $\bar{\mathbf{Q}} \in \mathcal{RH}_\infty^-$ anti-stable. To prove the theorem, we will first consider the case where $\mathbf{G}_\lambda^{p\times m}$ is stable and show that given an n-dimensional state-space realisation of a stable $\mathbf{G}_\lambda^{p\times m} \in \mathcal{RH}_\infty$, which is continuous in λ its state-space parameters, and has no imaginary axis eigenvalues in the state transition matrix A, then there exists a state-space realisation of an all-pass error function $\mathbf{E}_\lambda$with $\mathbf{E}_\lambda = \mathbf{G}_\lambda - \mathbf{Q}_\lambda$ and $\mathbf{Q}_\lambda \in \mathcal{RH}_\infty^-$ anti-stable, where the state-space parameters of $\mathbf{E}_\lambda$ are also continuous. It will then be a simple matter, at the end of the proof, to extend the result to ${\mathbf{G}_\lambda}^{p\times m} \in \mathcal{RL}_\infty$ and $\mathbf{Q}_\lambda \in \mathcal{RH}_\infty$.

Let a not necessarily minimal realisation of a stable $\mathbf{G}^{p\times m}$ be given by

$$\mathbf{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where all of the eigenvalues of A lie within the *open* right half-plane (this is possible because $\mathbf{G}_\lambda \in \mathcal{RH}_\infty$) and the corresponding non-negative definite controllability and observability Gramians P and Q satisfy

$$\begin{aligned} AP + PA^T + BB^T &= 0, \\ QA + A^T Q + C^T C &= 0. \end{aligned} \tag{A.8}$$

Define an augmented transfer matrix $\mathbf{G_a}^{(p+m)\times(p+m)}$ as

$$\mathbf{G_a} = \begin{bmatrix} \mathbf{G}^{p\times m} & 0^{p\times p} \\ 0^{m\times m} & 0^{m\times p} \end{bmatrix}.$$

This has a state-space realisation (not necessarily minimal) given by

$$\mathbf{G_a} = \left[\begin{array}{c|c} A & B_a \\ \hline C_a & D_a \end{array}\right]$$

where

$$\begin{aligned} B_a &= \begin{bmatrix} B & 0 \end{bmatrix}, \\ C_a &= \begin{bmatrix} C \\ 0 \end{bmatrix} \end{aligned}$$

and

$$D_a = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.$$

As in [17] we define a state space realisation of $\mathbf{Q_a}^{(p+m)\times(p+m)}$ as

$$\mathbf{Q_a} = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right],$$

where A_Q, B_Q, C_Q, D_Q are defined below. We then show that $\mathbf{Q_a} \in \mathcal{RH}_\infty^-$ and that $\mathbf{E_a} = \mathbf{Q_a} - \mathbf{G_a}$ enjoys the property that $\mathbf{E_a}^* \mathbf{E_a} = I$. In [17] this is done for a minimal realisation of $\mathbf{G}_\lambda$. Here we show that the construction is also valid for non-minimal realisation. We first define

$$E = QP - I,$$

We note that the inverse $E^{-1} = (PQ - I)^{-1}$ is guaranteed to exist because the Hankel norm of $\mathbf{G}$ is strictly less than unity so that the corresponding Hankel singular values are given by $\bar{\sigma}(QP) < 1$. We can then define

$$
\begin{aligned}
B_Q &= E^{-1}(QB_a + C_a^T D_e) \\
&= (QP - I)^{-1} \begin{bmatrix} QB & -C^T \end{bmatrix}, \\
C_Q &= D_e B_a^T + C_a P \\
&= \begin{bmatrix} CP \\ -B^T \end{bmatrix}, \\
A_Q &= -A^T - B_Q B_a^T \\
&= -E^{-1}(A^T E + C_a^T C_Q) \\
&= -A^T - (QP - I)^{-1} QBB^T \\
&= -(QP - I)^{-1}(A^T QP - A^T - C^T CP),
\end{aligned} \tag{A.9}
$$

and

$$
D_Q = \begin{bmatrix} D & I \\ I & 0 \end{bmatrix}.
$$

We next show that $\mathbf{E_a} = \mathbf{G_a} - \mathbf{Q_a}$ is paraunitary. We first note that a state-space realisation of $\mathbf{E_a}$ is given by

$$
\mathbf{E_a} = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array} \right]
$$

where

$$
\begin{aligned}
A_e &= \begin{bmatrix} A & 0 \\ 0 & A_Q \end{bmatrix}, \\
B_e &= \begin{bmatrix} B_a \\ B_Q \end{bmatrix}, \\
C_e &= \begin{bmatrix} C_a & -C_Q \end{bmatrix},
\end{aligned} \tag{A.10}
$$

and

$$
\begin{aligned}
D_e &= D_a - D_Q \\
&= \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}.
\end{aligned}
$$

We then note that P_e and Q_e defined by

$$
\begin{aligned}
P_e &= \begin{bmatrix} P & I \\ I & E^{-1}Q \end{bmatrix}, \\
Q_e &= \begin{bmatrix} Q & -E \\ -E^T & PE \end{bmatrix},
\end{aligned}
$$

satisfy the Lyapunov equations

$$
\begin{aligned}
A_e P_e + P_e A_e^T + B_e B_e^T &= 0, \\
Q_e A_e + A_e^T Q_e + C_e^T C_e &= 0,
\end{aligned}
\tag{A.11}
$$

and that furthermore

$$
\begin{aligned}
P_e Q_e &= I, \\
D_e^T C_e + B_e^T Q_e &= 0, \\
D_e^T D_e &= I.
\end{aligned}
\tag{A.12}
$$

By Lemma A.2 (and as noted in [27], without assuming observability or controllability), equations (A.11) and (A.12) are sufficient to establish that $\mathbf{E_a}^* \mathbf{E_a} = I$.

Next we establish that $\mathbf{Q_a}$ has no imaginary axis poles. In fact we prove by contradiction that $\zeta(A_Q) = 0$. As in [17] the (2,2) block of the second equation of (A.11) gives

$$
PEA_Q + A_Q^T PE + C_Q^T C_Q = 0. \tag{A.13}
$$

Temporarily assume that $\zeta(A_Q) \neq 0$ so that A_Q has an eigenvalue $j\omega \neq 0$ with $A_Q x = j\omega x$ for some nonzero eigenvector $x \neq 0$. Pre- and post- multiplication of equation (A.13) by x^T and x implies that $C_Q x = 0$. Consideration of the expression for A_Q in equation (A.9) allows us to conclude that $A^T Ex = -j\omega Ex$. Because A has no imaginary-axis eigenvalues we must have that $Ex = 0$ and because E is nonsingular we have that $x = 0$. This is a contradiction, so $\zeta(A_Q) = 0$ and hence $\mathbf{Q_a}$ has no imaginary axis poles.

We now prove a stronger result on the poles of $\mathbf{Q_a}$ by showing that they are all in the open left half-plane. It is easy to verify that $QE^T = EQ^T$ so that $E^{-1}Q$ and hence P_e is symmetric, and therefore has only real eigenvalues. We note that P_e is nonsingular, since $P_e^{-1} = Q_e$ so we can conclude that P_e has no eigenvalues on the imaginary axis, that is $\zeta(P_e) = 0$. Now consider

$$
\tilde{P}_e = \begin{bmatrix} P & I \\ I & E^{-1}Q \end{bmatrix} + \epsilon^2 I
$$

where ϵ is a small scalar introduced so as to ensure that $P + \epsilon^2 I$ is positive definite. Since $\tilde{P}_e = P_e + \epsilon^2 I$, with both $\tilde{P}_e$ and P_e symmetric, we can conclude that the real eigenvalues of $\bar{P}_e$ must be more negative than those of $\tilde{P}_e$. Because P_e is symmetric and nonsingular, we can choose a sufficiently small non-zero ϵ so that the inertias of P_e and $\tilde{P}_e$ are equal. By using an invertible similarity transformation we define $\bar{P}_e$ as

$$
\begin{aligned}
\bar{P}_e &= \begin{bmatrix} I & 0 \\ -(P+\epsilon^2 I)^{-1} & I \end{bmatrix} P_e \begin{bmatrix} I & -(P+\epsilon^2 I)^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} P + \epsilon^2 I & 0 \\ 0 & (E^{-1}Q + \epsilon^2 I) - (P+\epsilon^2 I)^{-1} \end{bmatrix}.
\end{aligned}
$$

This establishes that $\tilde{P}_e$ and $\bar{P}_e$ have the same inertia. Clearly the eigenvalues of the top-left block of $\bar{P}_e$, given by $P + \epsilon^2 I$, are all positive. Now we consider the lower-left block. We can show that (the symmetric)

$$(E^{-1}Q + \epsilon^2 I) - (P + \epsilon^2 I)^{-1} = (QP - I)^{-1}(I + \epsilon^2(Q + EP) + \epsilon^4 I)(P + \epsilon^2 I)^{-1}.$$

Since $(QP - I) < 0$ we see that that $(E^{-1}Q + \epsilon^2 I) - (P + \epsilon^2 I)^{-1}$ has all negative eigenvalues for some sufficiently small ϵ. It then follows that $\nu(\tilde{P}_e) = \pi(\tilde{P}_e) = n$ and $\zeta(\tilde{P}_e) = 0$ for ϵ sufficiently small. Because of our choice of small ϵ it follows that $\nu(P_e) = \pi(P_e) = n$ and $\zeta(P_e) = 0$ also.

Since $\zeta(P_e) = 0$, Lemma A.7 leads us to conclude that $\nu(P_e) = \pi(A_e) = n$ and hence that $\nu(A_e) = n$. But all the eigenvalues of A are in the open left half-plane so that from (A.10) we see that all the eigenvalues of A_Q are strictly in the right half-plane, so that $\mathbf{Q_a} \in \mathcal{RH}_\infty^-$.

Thus we have shown that even if $\{A, B, C, D\}$ is a non-minimal realisation of $\mathbf{G_a} \in \mathcal{RH}_\infty$, so long as $\mathrm{Re}[\lambda_i(A)] < 0$, the $\mathbf{Q_a}(s)$ constructed by [17] with $\bar{\sigma}(PQ) < I$, has a realisation $\{A_Q, B_Q, C_Q, D_Q\}$. Since $\{A, B, C, D\}$, the state space realisation of $\mathbf{G}_\lambda$ is continuous in λ, we also have that P, Q and E are continuous and hence by equations (A.9) it follows that $\{A_Q, B_Q, C_Q, D_Q\}$ are continuous in λ. Finally, because $\zeta(A_Q) = 0$, we may conclude that $\mathbf{Q_a}$ is $\mathcal{H}_\infty$ norm continuous.

By Theorems 4.3.2 and 10.3.2 in [17], all $\mathbf{Q}^{p\times m} \in \mathcal{RH}_\infty^-$ such that $\mathbf{E} = \mathbf{Q} - \mathbf{G}$ enjoys the property that $\mathbf{E}^*\mathbf{E} = I$, are given by the feedback interconnection

$$\mathbf{Q} = F_l(\mathbf{Q_a}^{(p+m)\times(p+m)}, \mathbf{U}^{p\times m}) \tag{A.14}$$

where the paraunitary operator $\mathbf{U}$ satisfies $\mathbf{U}^*\mathbf{U} = I$. Notice that because $\mathbf{Q_{a12}}$ and $\mathbf{Q_{a21}}$ are each square and have full rank in a half-plane, as functions of s, they are invertible. This means that, given $\mathbf{Q}(s)$ and $\mathbf{Q_a}(s)$, the function $\mathbf{U}(s)$ is uniquely determined by (A.14). In fact, by Lemma 10.4 of [27], $\mathbf{U}(s) = F_u(\mathbf{Q_a}^{-1}, \mathbf{Q})$.

We have shown that $\mathbf{Q_a}$ is $\mathcal{H}_\infty$ continuous in its state-space parameters. As in Section 10.4.3 of [17], we can deduce that $\mathbf{Q_{a12}} = \mathbf{E_{a12}}$ and $\mathbf{Q_{a21}} = \mathbf{E_{a21}}$ have respectively full column and row rank in the closed left-half complex plane. Because $\mathbf{E_a}$ is all-pass, it follows that this is equivalent to $\bar{\sigma}[\mathbf{E_{a11}}(s)] = \bar{\sigma}[\mathbf{E_{a22}}(s)] < 1$. Since $\mathbf{E_{a22}}(s) = \mathbf{Q_{a22}}(s)$ we can deduce that provided that we choose a paraunitary $\mathbf{U}$ which is $\mathcal{H}_\infty$-norm continuous, it follows that $\mathbf{Q} = F_l(\mathbf{Q_a}, \mathbf{U}) = \mathbf{Q_{a11}} + \mathbf{Q_{a12}}\mathbf{U}(I - \mathbf{Q_{a22}}\mathbf{U})^{-1}\mathbf{Q_{a21}} \in \mathcal{RH}_\infty^-$ will also be continuous in λ.

We have considered the case where $\mathbf{G} \in \mathcal{RH}_\infty$ and $\mathbf{Q} \in \mathcal{RH}_\infty^-$. By considering conjugate systems, it is a simple matter to see that the continuity results hold for $\mathbf{G} \in \mathcal{RH}_\infty^-$ with the constraint that $\mathbf{Q} \in \mathcal{RH}_\infty$. For the more general case where $\mathbf{G} \in \mathcal{RL}_\infty$ with a continuous state-space parameter realisation, we can decompose $\mathbf{G}_\lambda = \mathbf{G}_{\lambda u} + \mathbf{G}_{\lambda s}$ where $\mathbf{G}_{\lambda s} \in \mathcal{RH}_\infty$ and $\mathbf{G}_{\lambda u} \in \mathcal{RH}_\infty^-$, each with continuous state-space parameters, by Lemma A.9. We then find continuous $\mathbf{Q}_\lambda \in \mathcal{RH}_\infty$ corresponding to $\mathbf{G}_{\lambda u}$ and use continuous $\mathbf{Q}_\lambda + \mathbf{G}_{\lambda s}$ to give $\mathbf{E}_\lambda = \mathbf{G}_\lambda - (\mathbf{Q}_\lambda + \mathbf{G}_{\lambda s})$.

To prove the assertion that $\mathbf{E}_\lambda$ can be chosen to interpolate given $\hat{\mathbf{E}}_j$ at any finite number J of homotopy parameter values $\hat{\lambda}_j$, note that *all possible* $\mathbf{Q} \in \mathcal{RH}_\infty$ satisfying $\mathbf{G}-\mathbf{Q}$ is paraunitary is given by equation (A.14) for some appropriate choice of paraunitary $\mathbf{U}_\lambda$ so that for each j, we have $\hat{\mathbf{E}}_j = F_l(\mathbf{E}_{ja}, \mathbf{U}_j)$ for some paraunitary U_j. This gives rise to a set of $\mathbf{U}_j$ that the homotopy $\mathbf{U}_\lambda$ must interpolate at each $\lambda = \hat{\lambda}_j$. If $\mathbf{G}_\lambda$ is strictly tall then $\mathbf{U}_\lambda$ is also strictly tall and by Lemma A.4, it is straightforward that such a paraunitary $\mathcal{RL}_\infty$ homotopy may be constructed.

In the case that $\mathbf{G}_\lambda$ is square, note that wno{det[$\mathbf{E}_j$]} = wno{det[$\mathbf{E}_{ja}$]} + wno{det[$-\mathbf{U}_j$]} by Lemma A.5 and since $\mathbf{E}_{\lambda a}$ has continuous state space parameters and is paraunitary, it is $\mathcal{H}_\infty$ norm continuous and its determinant has constant winding number. The winding numbers wno{det[$\mathbf{E}_j$]} are equal for each j by the lemma hypothesis and therefore, so are the winding numbers of the determinants of the required $\mathbf{U}_j$. By Lemma A.5, a similar statement can be made about the determinants of $-\mathbf{U}_j(s)$ at $s = \infty$. From Lemma A.4 it then follows that the required $\mathcal{RL}_\infty$ homotopy $\mathbf{U}_\lambda$ may be constructed. □

A.4. Existence of Coprime Perturbations: Proof of Lemma 6.3.

Proof. Since N_λ has more than one row, we can decompose N_λ as

$$N_\lambda = \begin{bmatrix} n_\lambda \\ \bar{N}_\lambda \end{bmatrix}$$

where n_λ is a row vector of transfer functions. Suppose that G_λ is not coprime for a particular value of $\lambda = \hat{\lambda}$. This means that for some $s = s_j$ in the open right half-plane, the matrix $G_{\hat{\lambda}}(s_j)$ is not full rank. This is equivalent to

$$\ker[N_\lambda(s_j)] \bigcap \ker[D_\lambda(s_j)] \neq \emptyset$$

where ker[·] is the right kernel of a given matrix. Because G_λ is rational with bounded order, we have that $\det[D_{\hat{\lambda}}(s_j)] = 0$ for at most a finite number of points s_j in the open right hand plane. Consider the reduced transfer matrix

$$\begin{bmatrix} \bar{N}_\lambda(s_j) \\ D_\lambda(s_j) \end{bmatrix}.$$

Even if the lack of coprimeness occurs for finite interval of values of λ, and even if $\bar{N}_\lambda(s_j)$ vanishes over a finite λ interval, it is possible to choose an arbitrary real matrix H of dimension compatible with $\bar{N}_\lambda$ such that $\|H\| = 1$ such that for any $\epsilon > 0$ and any fixed $\hat{\lambda}$ in the interval, the columns of H are not in $\ker[\bar{N}(s_j)] \bigcap \ker[D_\lambda(s_j)]$ and hence

$$\ker[N_{\hat{\lambda}}(s_j) + \epsilon H] \bigcap \ker[D_{\hat{\lambda}}(s_j)] = \emptyset$$

Consequently, by invoking the rational dependance of the transfer function coefficients on λ, we can determine that as λ moves from one endpoint to the other, even if

$$\begin{bmatrix} \bar{N}_\lambda(s_j) \\ D_\lambda(s_j) \end{bmatrix}$$

has a loss of rank in the interval, it is the case that

$$\begin{bmatrix} \bar{N}_\lambda(s_j) + \epsilon H \\ D_\lambda(s_j) \end{bmatrix}$$

will have loss of rank for some s_j with $\mathrm{Re}(s) \geq 0$ at most at only a finite number of values of λ. By allowing ϵ to be a function of λ near the end of the interval, we can ensure that for arbitrary $\frac{1}{2}\eta$, that there exists a real constant (in the s-plane) perturbation H_λ of $\bar{N}_\lambda(s)$ such that

$$\begin{bmatrix} \bar{N}_\lambda + H_\lambda \\ D_\lambda \end{bmatrix}$$

is coprime for all but a finite set of $\lambda \in [0, 1]$ and $\|H_\lambda\| < \frac{1}{2}\eta$. Denote the finite set of points λ where we lose coprimeness by $\bar{\Lambda}$. Now consider the full-dimensioned transfer matrix

$$\begin{bmatrix} n_\lambda \\ \bar{N}_\lambda + H_\lambda \\ D_\lambda \end{bmatrix}$$

Let Λ be the set of λ_i such that the above is not coprime so that $\Lambda \subset \bar{\Lambda}$. It is then a simple matter to choose an $\frac{1}{2}\eta$-bounded real perturbation vector h_λ of n_λ in the vicinity of each $\lambda_i \in \Lambda$ such that h_λ is not in the kernel $\ker[n_{\lambda_i}(s_{ij})] \bigcap \ker[\bar{N}_{\lambda_i}(s_{ij}) + H_{\lambda_i}] \bigcap \ker[D_{\lambda_i}(s_{ij})]$ for any s_{ij} and hence

$$\ker[n_\lambda(s_{ij}) + h_\lambda] \bigcap \ker\left(\begin{bmatrix} \bar{N}_\lambda(s_{ij}) + H_\lambda \\ D_\lambda(s_{ij}) \end{bmatrix}\right) = \emptyset$$

for all s_{ij} such that $\mathrm{Re}[s_{ij}] \geq 0$. We have therefore established that there exists a constant (in the s-plane) $\Delta_\lambda = \begin{bmatrix} h_\lambda^T & H_\lambda^T & 0 \end{bmatrix}^T$, such that $\|\Delta_\lambda\| < \eta$ for arbitrarily small η, and that $G_\lambda + \Delta_\lambda$ is a coprime realisation. It is then trivial to construct a normalised coprime realisation as follows. Let R_λ be the stable minimum-phase spectral factor of

$$\begin{aligned} L_\lambda^* L_\lambda &= [G_\lambda^* + H_\lambda^*][G_\lambda + H_\lambda] \\ &= I + H_\lambda^* G_\lambda + G_\lambda^* H_\lambda + H_\lambda^* H_\lambda \end{aligned}$$

where $\|H_\lambda^* G_\lambda + G_\lambda^* H_\lambda + H_\lambda^* H_\lambda\|_\infty \leq 2\eta + \eta^2$.It then follows that we can define our perturbation

$$G'_\lambda = [G_\lambda + H_\lambda] L_\lambda^{-1}$$

and easily show that $\|G_\lambda - G'_\lambda\|_\infty \leq 5\eta$ for $\eta < 1$. $\square$