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On Frequency Weighted Balanced Truncation: Hankel Singular Values and Error Bounds

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The concept of frequency weighted balancing proposed by Enns is a generalisation of internally balanced model truncation which is simple to apply and additionally attractive because of the existence of an upper H_∞ error bound that is a function of the neglected Hankel singular values. However, a generalisation of this error bound based on the frequency weighted Hankel singular values has not been reported. In this paper, it is shown that there does not exist a frequency weighted upper error bound that depends only on the neglected frequency weighted Hankel singular values. Based on this result, it is shown that truncation of the states corresponding to the lowest frequency weighted Hankel singular values does not always yield the lowest approximation error. It is explained that this is due to cross-terms that appear in the frequency weighted error bound and make the discussion on stability of the reduced order model more complex. These cross-terms are inherent in the frequency weighted balancing technique proposed by Enns.

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1. Introduction

Various model reduction methods have been proposed in the last decades. The most popular methods are internally balanced truncation and optimal Hankel norm approximation [4]. Their main advantage, apart from simplicity of application, is that there exists an a priori lower and upper error bound based on the Hankel singular values of the full-order system. The concept of frequency weighted balancing was introduced by Enns [2,3] as a generalisation of internal balancing in order to take the frequency dependence of the admissible model reduction error into account. Frequency weighted balanced model truncation has applications in system identification and controller design, e.g., to tune the approximation error of the identified model in certain frequency ranges [7,11] or to enhance the robustness of the controller [2,14].

However, the generalisation of the a priori balanced model reduction upper error bound in terms of the so-called frequency weighted Hankel singular values has not been found yet, as has been reported several times in the literature [1,2,5,11,14]. In this paper, it is

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shown that certain types of generalisations of the upper bound, which we will call Enns' Conjecture [2] in Section 2, cannot serve as an upper error bound. This is due to a cross term, inherent in frequency weighted balancing, that can become unbounded in terms of the neglected frequency weighted Hankel singular values. Based on this result an example is constructed to illustrate that truncating the states corresponding to the lowest frequency weighted singular values, does not always yield the smallest frequency weighted approximation error. Because of the cross-terms, stability of the reduced order model does not follow from a truncated Lyapunov equation and may not always be guaranteed. When modifying Enns' frequency weighted balancing technique so as to guarantee stability of the reduced order system [6,10,12], an interpretation of the obtained singular values can be given in the sense of internally balanced singular values of a related realisation.

2. Model Reduction

Internally balanced model reduction is reviewed first. Frequency weighted balanced truncation proposed by Enns [2,3] is discussed in Section 2.2, while alternative frequency weighted balancing algorithms [6,12] are reviewed in Section 2.3.

2.1. Internally Balanced Model Reduction

Consider a stable, continuous time linear time-invariant system of order n with transfer matrix $G(s)=C(sI-A)^{-1}B+D$ and corresponding realisation (A, B, C, D). The system is assumed to be minimal which means that the controllability Gramian $P=\lim_{t\to\infty}\int_0^t \exp(A\tau)BB^{\mathrm{T}}\exp(A^{\mathrm{T}}\tau)\,\mathrm{d}\tau$ and the observability Gramian $Q=\lim_{t\to\infty}\int_0^t \exp(A^{\mathrm{T}}\tau)C^{\mathrm{T}}C\exp(A\tau)\,\mathrm{d}\tau$ are positive definite. These Gramians P and Q can be computed by solving the linear matrix equations $AP+PA^{\mathrm{T}}+BB^{\mathrm{T}}=0$ and $A^{\mathrm{T}}Q+QA+C^{\mathrm{T}}C=0$, respectively. Considering the problem of minimizing the input energy $\int_{-T}^{0}u(t)^{\mathrm{T}}u(t)\,\mathrm{d}t$ to the system $\dot{x}(t)=Ax(t)+Bu(t)$ subject to $x(-T)=0$ and $x(0)=x_0$, the minimal energy needed to reach the state x_0 is equal to $x_0^{\mathrm{T}}P^{-1}x_0$ for $T\to\infty$. A similar energy interpretation exists for the output Gramian Q via a dual statement: the energy $\int_0^T y(t)^{\mathrm{T}}y(t)\,\mathrm{d}t$ that we can obtain at the output y from state x_0 is equal to $x_0^{\mathrm{T}}Qx_0$ for $T\to\infty$ [9]. A similarity transformation T on $(A,B,C,D)\xrightarrow{T}(TAT^{-1},TB,CT^{-1},D)$ can be found that simultaneously diagonalises the Gramians via the corresponding contragredient transformation $(P,Q)\xrightarrow{T}(TPT^{\mathrm{T}},T^{-\mathrm{T}}QT^{-1})=(\Sigma_n,\Sigma_n)$, while the eigenvalues of $PQ=\Sigma_n^2$ are preserved. The system $G(s)$ with realization $(TAT^{-1}, TB, CT^{-1}, D)$ is then called *internally balanced* with $P=Q=\Sigma_n=\mathrm{diag}([\sigma_1,\ldots,\sigma_n])$. The diagonal elements $\sigma_1\geq\sigma_2\geq\ldots\geq\sigma_n$ are called the Hankel singular values and are ordered in a non-increasing order. Provided that $\sigma_{r+1}<\sigma_r$, the reduced order model $G_r(s)=C_1\,(sI-A_{11})^{-1}B_1+D$ of order $r<n$ is then obtained by truncating the partitioned balanced system:

$$TAT^{-1}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\quad TB=\begin{bmatrix}B_1\\ B_2\end{bmatrix},\quad CT^{-1}=[C_1\ C_2],$$

with $A_{11}\in\mathbb{R}^{r\times r}$. The reduced order system is stable [8] and there exists an a priori upper bound on the H_∞ error [2,4]:

$$E_\infty=\| G(s)-G_r(s)\|_\infty\leq 2\sum_{k=r+1}^{n}\sigma_k. \tag{1}$$

2.2. Frequency Weighted Balancing in the Sense of Enns

The concept of frequency weighted balanced truncation is a generalisation of internally balanced truncation and was introduced by Enns [2,3]. Given both an input weighting filter $W_\mathrm{i}(s)=C_\mathrm{i}(sI-A_\mathrm{i})^{-1}B_\mathrm{i}+D_\mathrm{i}$ and an output weighting filter $W_\mathrm{o}(s)=C_\mathrm{o}(sI-A_\mathrm{o})^{-1}B_\mathrm{o}+D_\mathrm{o}$, the frequency weighting is obtained by making the series connection $W_\mathrm{o}(s)\cdot G(s)\cdot W_\mathrm{i}(s)$ of the input filter $W_\mathrm{i}(s)$, the original system $G(s)$ and the output filter $W_\mathrm{o}(s)$. By constructing the state–space realisations of the augmented systems $G(s)W_\mathrm{i}(s)$ and $W_\mathrm{o}(s)G(s)$:

$$\bar{A}_\mathrm{i}=\begin{bmatrix}A & BC_\mathrm{i}\\ 0 & A_\mathrm{i}\end{bmatrix},\quad \bar{B}_\mathrm{i}=\begin{bmatrix}BD_\mathrm{i}\\ B_\mathrm{i}\end{bmatrix},\quad \bar{C}_\mathrm{i}=[C\quad 0], \tag{2}$$

$$\bar{A}_\mathrm{o}=\begin{bmatrix}A & 0\\ B_\mathrm{o}C & A_\mathrm{o}\end{bmatrix},\quad \bar{B}_\mathrm{o}=\begin{bmatrix}B\\ 0\end{bmatrix},\quad \bar{C}_\mathrm{o}=[D_\mathrm{o}C\quad C_\mathrm{o}], \tag{3}$$

the extended Gramians

$$\bar{P}_\mathrm{i}=\begin{bmatrix}P & P_{12}\\ P_{12}^{\mathrm{T}} & P_{22}\end{bmatrix}\quad\text{and}\quad \bar{Q}_\mathrm{o}=\begin{bmatrix}Q & Q_{12}\\ Q_{12}^{\mathrm{T}} & Q_{22}\end{bmatrix}$$

are obtained as the solutions to the following Lyapunov equations:

$$\bar{A}_\mathrm{i}\bar{P}_\mathrm{i}+\bar{P}_\mathrm{i}\bar{A}_\mathrm{i}^{\mathrm{T}}+\bar{B}_\mathrm{i}\bar{B}_\mathrm{i}^{\mathrm{T}}=0 \tag{4}$$

$$\bar{A}_\mathrm{o}^{\mathrm{T}}\bar{Q}_\mathrm{o}+\bar{Q}_\mathrm{o}\bar{A}_\mathrm{o}+\bar{C}_\mathrm{o}^{\mathrm{T}}\bar{C}_\mathrm{o}=0, \tag{5}$$

respectively. Expanding the upper left blocks of Eqs (4) and (5), one obtains:

$$AP + PA^{\mathrm{T}} + BC_{\mathrm{i}}P_{12} + P_{12}^{\mathrm{T}}C_{\mathrm{i}}^{\mathrm{T}}B^{\mathrm{T}} + BD_{\mathrm{i}}D_{\mathrm{i}}^{\mathrm{T}}B^{\mathrm{T}} = 0, \tag{6}$$

$$A^{\mathrm{T}}Q + QA + Q_{12}B_{\mathrm{o}}C + C^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}Q_{12}^{\mathrm{T}} + C^{\mathrm{T}}D_{\mathrm{o}}^{\mathrm{T}}D_{\mathrm{o}}C = 0, \tag{7}$$

respectively. Assuming that there are no pole-zero cancellations in $G(s)W_{\mathrm{i}}(s)$ and $W_{\mathrm{o}}(s)G(s)$ the Gramians $\bar{P}_{\mathrm{i}}$ and $\bar{Q}_{\mathrm{i}}$ are positive definite.

Similar energy interpretations as in Section 2.1 can be given for $\bar{P}_{\mathrm{i}}$ and $\bar{Q}_{\mathrm{i}}$ in terms of the augmented systems (2) and (3) and their corresponding augmented states[1] $[x(t); x_{W_{\mathrm{i}}}(t)]$ and $[x(t); x_{W_{\mathrm{o}}}(t)]$, respectively. For example, the minimal input energy needed to reach $[x_0; x_{W_{\mathrm{i},0}}]$ in an optimal way is given by $[x_0^{\mathrm{T}}\ x_{W_{\mathrm{i},0}}^{\mathrm{T}}]\bar{P}_{\mathrm{i}}^{-1}[x_0; x_{W_{\mathrm{i},0}}]$ [10]. The system $G(s)$ with realisation (A, B, C, D) is called *frequency weighted balanced in the sense of Enns* (with respect to the input and output weighting transfer functions $W_{\mathrm{i}}(s)$ and $W_{\mathrm{o}}(s)$) iff the input and output frequency weighted Gramians P and Q are diagonal and equal, i.e.. $P = Q = \Sigma_n = \mathrm{diag}([\sigma_1, \ldots, \sigma_n])$. The values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ are now called the frequency weighted Hankel singular values and are ordered in a non-increasing order. For a given realisation, there exists a similarity transformation T that balances the system in the frequency weighted sense [2]. Motivated by the upper error bound (1) for the internally balanced model truncation error, Enns formulated a conjecture about an upper error bound for the frequency weighted balanced model truncation error [2] (p. 105):

Conjecture 1. *(Enns' Conjecture).* When truncating a frequency weighted balanced system, the infinity norm E_∞ of the weighted difference between the original system $G(s) = C(sI - A)^{-1}B$ of order n and the reduced system $G_r(s) = C_1(sI - A_{11})^{-1}B_1$ of order $r < n$ can be upper bounded by $2(1+\alpha)$ times the sum of the neglected weighted singular values:

$$\begin{aligned} E_\infty &= \|E(j\omega)\|_\infty \\ &= \|W_{\mathrm{o}}(j\omega)(G(j\omega) - G_r(j\omega))W_{\mathrm{i}}(j\omega)\|_\infty \\ &\leq 2(1+\alpha)\sum_{k=r+1}^{n}\sigma_k, \end{aligned} \tag{8}$$

with $\alpha < 1$ when $E_\infty < 1$.

It should be noted that by scaling $G(s)$ with a factor $\gamma \in \mathbb{R}^+$, the model reduction algorithm returns $\gamma G_r(s)$ and $\gamma\sigma_i$ $(i = 1, \ldots, n)$, instead of $G_r(s)$ and σ_i. The error E_∞ is also scaled by a factor γ and accordingly there will be a value of γ for which the error will be less than one. Since α is independent of the scaling process, the conjecture is asserting that α is less than one for all errors. Were α to be zero, the bound would be equal to the bound for internally balanced truncation corresponding to $W_{\mathrm{i}}(s) = I_m$ and $W_{\mathrm{o}}(s) = I_l$, with m and l the number of inputs and outputs, respectively. In other words, $\alpha < 1$ is introduced to extend the result of (1) to the frequency weighted case. However, the conjecture has not been proven and no value for α has been reported in the literature. For the sake of completeness, we mention that a (conservative) upper bound was derived in [5]. This bound is not an a priori error bound and depends on the Hankel singular values, the matrices P_{12} and Q_{12}, the weightings and the system.

Also note that stability is not guaranteed when non-constant input and output weightings are both present [10]. This is due to the cross terms $BC_{\mathrm{i}}P_{12} + P_{12}^{\mathrm{T}}C_{\mathrm{i}}^{\mathrm{T}}B^{\mathrm{T}}$ and $Q_{12}B_{\mathrm{o}}C + C^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}Q_{12}^{\mathrm{T}}$ in (6) and (7), which, when one truncates the equation, may result in $A_{11}\Sigma_r + \Sigma_r A_{11} \not\leq 0$. Of itself, this does not imply that A_{11} is unstable, but simply that, in contrast to the unweighted case, stability does not follow from a truncated Lyapunov equation.

2.3. Alternative Frequency Weighted Balancing Techniques

In [6], Lin and Chiu have proposed an alternative frequency weighted balancing method with guaranteed stability of the reduced order model. Stability is obtained by removing the cross-terms by block diagonalising the extended Gramians $\bar{P}_{\mathrm{i}}$ and $\bar{Q}_{\mathrm{o}}$ in order to obtain a positive (semi-) definite term in the upper left block of the Lyapunov equations. The Gramian $\bar{P}_{\mathrm{i}}$ is block diagonalized via the contragredient transformation $\bar{P} \xrightarrow{\bar{T}} \bar{T}\bar{P}\bar{T}^{\mathrm{T}}$:

$$\begin{bmatrix} P & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix} \xrightarrow{\bar{T}} \begin{bmatrix} I & X_{\mathrm{i}}^{\mathrm{T}} \\ 0 & I \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{\mathrm{i}} & I \end{bmatrix}$$

$$= \begin{bmatrix} P - P_{12}P_{22}^{-1}P_{12}^{\mathrm{T}} & 0 \\ 0 & P_{22} \end{bmatrix}, \tag{9}$$

with $X_{\mathrm{i}} = -P_{22}^{-1}P_{12}^{\mathrm{T}}$. The upper left block (6) of (4) now becomes

$$\begin{aligned} P_{\mathrm{i}}A^{\mathrm{T}} + AP_{\mathrm{i}} + BD_{\mathrm{i}}D_{\mathrm{i}}^{\mathrm{T}}B^{\mathrm{T}} + BD_{\mathrm{i}}B_{\mathrm{i}}^{\mathrm{T}}X_{\mathrm{i}} + X_{\mathrm{i}}^{\mathrm{T}}B_{\mathrm{i}}D_{\mathrm{i}}^{\mathrm{T}}B^{\mathrm{T}} \\ + X_{\mathrm{i}}^{\mathrm{T}}B_{\mathrm{i}}B_{\mathrm{i}}^{\mathrm{T}}X_{\mathrm{i}} = 0, \end{aligned} \tag{10}$$

[1]The Matlab notation $[A; B] = [A^{\mathrm{T}}\quad B^{\mathrm{T}}]^{\mathrm{T}}$ is used.

with $P_i = P - P_{12}P_{22}^{-1}P_{12}^T$. In a similar way, one can apply a contragredient transformation in order to block diagonalise the extended output Gramian $\bar{Q}$ of the output weighted frequency Lyapunov equation (5) resulting into $Q \rightarrow Q - Q_{12}Q_{22}^{-1}Q_{12}^T = Q_o$. The input and output Gramians P_i and Q_o have the following energy interpretation. Considering, e.g., the input energy related to the Gramian $\bar{P}_i$ from Section 2.2 and introducing the additional constraint that the states related to the input weighting are zero, i.e., $x_{W_{i,0}} = 0$, the minimum input energy is $x_0^T P_i x_0$ [10]. The system is now called frequency weighted balanced in the sense of Lin and Chiu iff $P_i = Q_o = \Sigma_n$, with $\Sigma_n = \mathrm{diag}([\sigma_1, \sigma_2, \ldots, \sigma_n])$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Since $BD_iD_i^TB^T + BD_iB_i^TX_i + X_i^TB_iD_i^TB^T + X_i^TB_iB_i^TX_i = (BD_i + X_i^TB_i)(BD_i + X_i^TB_i)^T \geq 0$ [6,10], stability can be proven following [8].

In [12], an *alternative balanced truncation method* has been proposed by replacing the symmetric (but possibly) indefinite expression $M = BC_iP_{12} + P_{12}^TC_i^TB^T + BD_iD_i^TB^T$ in (6) by a semi-definite expression $\bar{B}\bar{B}^T \geq 0$. The positive definite expression $\bar{B}\bar{B}^T$ is obtained by calculating the eigenvalue decomposition $M = U\Lambda U^T$, with $U \in \mathbb{R}^{n\times n}$ and $\Lambda = \mathrm{diag}([\lambda_1, \ldots, \lambda_i, 0, \ldots, 0])$, and taking the absolute values of $\lambda_i \neq 0$: $\bar{B} = U\,\mathrm{diag}([|\lambda_1|^{1/2}, \ldots, |\lambda_i|^{1/2}, 0, \ldots, 0])$. In [12], it is then argued that there exists a $\bar{K}$ such that $B = \bar{B}\bar{K}$. In a similar way, one defines $\bar{C}$ and $\bar{L}$, with $C = \bar{L}\bar{C}$, for the output weighted Lyapunov equation. The alternative balanced truncation method [12] is based on simultaneously diagonalising $\hat{P}$ and $\hat{Q}$, which are obtained from $A\hat{P} + \hat{P}A^T + \bar{B}\bar{B}^T = 0$ and $A^T\hat{Q} + \hat{Q}A + \bar{C}^T\bar{C} = 0$, respectively. One can now use results from internally balanced model truncation [2,4,8] on the system with realisation $(A, \bar{B}, \bar{C}, D)$. For a balanced system in the sense of [12], with $\hat{P} = \hat{Q} = \hat{\Sigma} = \mathrm{diag}([\hat{\sigma}_1, \ldots, \hat{\sigma}_n])$, the following upper error bound holds:

$$E_\infty = \|W_o(s)(G(s) - G_r(s))W_i(s)\|_\infty \leq 2\,\|W_o(s)\bar{L}\|_\infty \|\bar{K}W_i(s)\|_\infty \sum_{k=r+1}^{n} \hat{\sigma}_k, \qquad (11)$$

where $B = \bar{B}\bar{K}$ and $C = \bar{L}\bar{C}$. This bound is obtained by first applying the sub-multiplicative property of the H_∞ norm and then the internally balanced truncation error bound (1).

3. Enns' Conjecture Refuted

In Section 3.1, Enns' Conjecture is refuted by means of a constructive counterexample. It is explained that this is due to the cross-terms in the extended Lyapunov equations. Numerical counterexamples are given, while it is also illustrated that truncating the states *corresponding to the lowest singular values does not* always yield the lowest frequency weighted truncation error. In Section 3.2, it is explained that the cross terms are inherent in frequency weighted balancing in the sense of Enns. In the last Subsection, an upper error bound of a similar form as (11) in [12] is discussed for frequency weighted balanced truncation in the sense of Lin and Chiu.

3.1. A Constructive Counterexample to Enns' Conjecture

In Theorem 1, Enns' Conjecture is disproven; moreover, it is shown that there does not exists an α such that (8) holds for all possible systems and weightings.

Theorem 1. (*Enns' Conjecture disproven*). Let $W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$ and $W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o$ be stable, minimum phase transfer functions for the input and output weighting. Let the asymptotically stable system $G(s) = C(sI - A)^{-1}B + D$ be frequency weighted balanced with respect to the input and output weightings $W_i(s)$ and $W_o(s)$ and let the Gramians be given by $P = Q = \Sigma_n = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n)$. Then, there exists no finite α such that a frequency error bound

$$\|W_o(s)(C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1)W_i(s)\|_\infty \leq 2(1 + \alpha)\sum_{k=r+1}^{n} \sigma_k \qquad (12)$$

holds for all possible weightings and all possible transfer functions.

Proof. For disproving Enns' Conjecture, it suffices to produce a counterexample for each $\alpha \in \mathbb{R}^+$. Such a counterexample will be constructed on the error bound for the full-order model $g(s) = c(s - a)^{-1}b$, which is a strictly proper, stable first-order SISO system.[2] The input weighting $w_i(s) = c_i(s - a_i)^{-1}b_i + d_i$ is a stable, minimum phase first-order SISO system and the output weighting $w_o(s) = 1$ is constant. The reduced order model is $g_r(s) = 0$, since we do not introduce a feed-through term d, following the approach of [2]. Since the proof requires quite some algebra, only the outline of the proof is given here; we refer to Appendix A for the details.

[2] In the sequel of the paper, we will assume $D = 0$, because D does not influence the balanced truncation error.

First, the error E_∞ can be split up in a term $\theta_{\mathrm{noFW}}(j\omega)$ similar to the error formula of internally balanced truncation [2] and an extra term $\theta_{\mathrm{FW}}(j\omega)$, due to the cross-terms introduced by the frequency weighting:

$$E_\infty = \sup_\omega(\theta_{\mathrm{noFW}}(j\omega) + \theta_{\mathrm{FW}}(j\omega))^{1/2}.$$

Since it is sufficient to disprove the conjecture at one frequency, we evaluate $\theta_{\mathrm{noFW}}(j\omega) + \theta_{\mathrm{FW}}(j\omega)$ at frequency zero ($\omega = 0$). This yields $\theta_{\mathrm{noFW}}(0) = 4\sigma^2$, as is expected from (1). It is then shown that $\theta_{\mathrm{FW}}(0)/\sigma^2$ can become arbitrarily large (which effectively disproves the conjecture). This is done by initially choosing a, b and σ (or c) and then choosing a_i, b_i, c_i, d_i such that $g(s)$ is frequency weighted balanced in the sense of Enns with respect to the input weighting $w_i(s)$ and such that $w_i(s)$ is stable and minimum phase. See Appendix A for more details.

One might well imagine that a relaxation or reformulation of the conjecture would be true. For a fairly broad relaxation, as Theorem 2 shows, this is not the case.

Theorem 2. Under the same conditions for $W_i(s)$, $W_o(s)$ and $G(s)$ as in Theorem 1, there exists no upper error bound of the type

$$E_\infty = \|W_o(s)\ (C(sI - A)^{-1}B - C_r(sI - A_r)^{-1}B_r)W_i(s)\|_\infty \leq f(\sigma_{r+1}, \ldots, \sigma_n, C, A, B)\sum_{r+1}^{n}\sigma_i, \tag{13}$$

with $f(\cdot)$ depending only on its arguments.

Proof. The detailed proof of Theorem 2 in Appendix A shows that when σ, a, b and c are fixed, there is still enough freedom for selecting the parameters of the weight $w_i(s)$ such that $E_\infty \geq 2(1+\alpha)\sigma$ for a given α. The proof of this theorem then follows by choosing $\alpha = f(\sigma, a, b, c)/2 - 1$.

Since the constructive counterexamples in the proofs start from stable first-order $g(s)$, it follows that the matrix M defined in [12] is a positive scalar. Hence, these theorems also imply that Enns' conjecture does not hold for the modified frequency weighted balancing algorithm [12]. We will now give some numerical counterexamples to Enns' Conjecture.

Example 1. Consider the system $g(s)$ with frequency weighted balanced realisation $(-1, 1, \sqrt{2}, 0)$ and Hankel singular value $\sigma = 1$. Using the constructive algorithm from the proof of Theorem 1 in Appendix A, some counterexamples, showing that $E_\infty \nleq 2(1+\alpha)\sigma$, are constructed for different values of α. The numerical values for the input weigthing $w_i(s)$ are reported in Table 1. Note, however, that the error E_∞ for the third input weighting $w_i(s)$ with realisation $(-0.0015, 0.1163, 0.1900, 1.2728)$ in Table 1, for $g(s)$ with realisation $(1/100000, 1, \sqrt{2}/100000, 0)$ is $E_\infty = 22.5669$,which is nearly 2σ, with $\sigma = 11.2763$. This illustrates that for a given $w_i(s)$, Enns' Conjecture is not violated for all choices of $g(s)$ as one would expect.

Example 2. Based on the result of Theorem 1, an example is constructed indicating that truncating the states corresponding to the smallest frequency weighted singular values does not always yield the best fit. To begin with, consider an example where the two singular values are the same. Let

$$\begin{aligned}
A &= \mathrm{diag}([-1 \quad -0.1]),\\
B &= \mathrm{diag}([1 \quad 1]),\\
C &= \mathrm{diag}([\sqrt{2} \quad \sqrt{0.2}]),\\
A_i &= \mathrm{diag}[(-0.009 \quad -0.1805]),\\
B_i &= \mathrm{diag}([0.255 \quad 0.6429]),\\
C_i &= \mathrm{diag}([0.190 \quad 0.0190]),\\
D_i &= \mathrm{diag}([1.273 \quad 0.4025]),
\end{aligned}$$

then $G(s) = C(sI - A)^{-1}B$ is frequency weighted balanced with respect to the input weighting $W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$, with $\sigma_1 = \sigma_2 = 1$. The singular values might suggest that either deleting the first or second state yields the same H_∞ error, but in

Table 1. Counterexamples to Enns' Conjecture for different values of α.

α	a_i	b_i	c_i	d_i	E_∞	$2(1+\alpha)\sigma$
0.1	−0.8595	1.3116	0.1900	1.2728	2.2100	2.2000
1	−0.0602	0.5101	0.1900	1.2728	4.0782	4.0000
10	−0.0015	0.1163	0.1900	1.2728	22.566	22.000
100	−1.7696E − 05	0.0135	0.1900	1.2728	207.24	202.00

The counterexamples are generated for $\sigma = 1$ and different values of α in order to illustrate that the upper error bound $E_\infty \leq 2(1+\alpha)\sigma$ of the conjecture does not hold. See Example 1 for details.

fact, the errors are quite different: $\|(G(s)-(C_1(sI-A_{11})^{-1}B_1))\cdot W_i(s)\|_\infty=2.103$ and $\|(G(s)-(C_2(sI-A_{22})^{-1}B_2))\cdot W_i(s)\|_\infty=9.392$. The previous system may appear somewhat artificial since it is a parallel connection of two first-order systems. By introducing a coupling term in the output $C=[\sqrt{2}\,1;0\ \sqrt{0.2}]$, we obtain after balancing $\sigma_1=2.512$ and $\sigma_2=0.830$. However, the best fit is obtained by truncating the (balanced) states corresponding to σ_1: $\|(G(s)-C_1(sI-A_{11})^{-1}B_1)W_i(s)\|_\infty=6.280$ and $\|(G(s)-C_2(sI-A_{22})^{-1}B_2)W_i(s)\|_\infty=5.566$, with $A_{11}=-0.150$, $A_{22}=-0.950$, $B_1=0.373$, $B_2=-0.214$, $C_1=0.823$ and $C_2=-0.116$.

3.2. Influence of the Cross-Term

The "problem" in the frequency weighted error is the cross-term θ_{FW} due to a non-zero P_{12} in (4) and (6). The upper bound [5] is also based on the error bound for the cross-term. Stability of the reduced order model cannot be guaranteed in the case of both input and output weighting, due to a non-zero P_{12} and Q_{12} in (4) and (5), respectively. Hence, it would be nice if conditions could be derived for the weighting such that the cross-term disappears, i.e., such that $P_{12}=0$ in (4).

This non-zero P_{12} is, however, inherent to frequency weighted balancing and, as we now show, requiring $P_{12}=0$ effectively limits the set of weights so much that the concept of frequency weighting is destroyed. From (4), it follows that the class of input frequency weighting filters corresponding to $P_{12}=0$ has to satisfy the following two equations:

$$B(C_iP_{22}+D_iB_i^T)=0, \tag{14}$$

$$A_iP_{22}+P_{22}A_i^T+B_iB_i^T=0. \tag{15}$$

By use of (15), one then obtains[3] $B_iB_i^T=(sI-A_i)P_{22}+P_{22}(-sI-A_i^T)$ and using (14) the following expression is obtained

$$\begin{aligned} &BC_i(sI-A_i)^{-1}B_iB_i^T(-sI-A_i^T)^{-1}C_i^TB^T\\ &=BC_iP_{22}(-sI-A_i^T)^{-1}C_i^TB^T\\ &\quad+BC_i(sI-A_i)^{-1}P_{22}C_i^TB^T\\ &=-BD_iB_i^T(-sI-A_i^T)^{-1}C_i^TB^T\\ &\quad+BC_i(sI-A_i)^{-1}P_{22}B_i^TD_iB^T. \end{aligned}$$

[3]The proof is similar to that of Theorem 5.1 in [4].

Substituting in the above expression in $BW_i(s)W_i^T(-s)B^T$ yields:

$$BW_i(s)W_i^T(-s)B^T=BD_iD_i^TB^T. \tag{16}$$

This means that $W_i(s)=W_{i1}(s)+W_{i2}(s)$ with $W_{i1}(s)\cdot W_{i1}^T(-s)=D_iD_i^T$ and $BW_{i2}(s)=0$. In other words, the input weighting $W_i(s)$ is composed of a first part which is constant on the frequency axis and a second part that is in the kernel of B for all frequencies s. This condition means that the second part $W_{i2}(s)$ does not contribute to the frequency weighting of $G(s)$ in the series connection.

Unsurprisingly, the error bound (1) remains valid when applying input (and output) weighting with a constant matrix.[4] An input weighting with a constant matrix corresponds to taking linear combinations of the columns of the B matrix. Similar conditions can be derived for the output weighting filter.

3.3. Error Bounds after Stability Repair

It has been explained that because of the cross-terms in Lyapunov equations (4) and (5), Enns' Conjecture does not hold and the reduced order model may become unstable. In contrast, whereas the stability problem is solved by frequency weighted balanced truncation in the sense of Lin and Chiu [6], the error formula $\|W_o(s)(G(s)-G_r(s))W_i(s)\|_\infty$ is not simplified by applying the transform (9), see, e.g., [10]. Since the eigenvalues λ_i of PQ satisfy are not larger than the eigenvalues of $(P-P_{12}P_i^{-1}P_{12}^T)(Q-Q_{12}^TQ_o^TQ_{12})$, (Lemma 3.1, [10]), the frequency weighted singular values in the sense of Lin and Chiu are not larger than the frequency weighted Hankel singular values in the sense of Enns. This insight can be used to shown that Enns' Conjecture does not hold when applying frequency weighted balanced truncation in the sense of Lin and Chiu as follows. In the proof of Theorem 1, it is shown that $2(1+\alpha)\sigma$ cannot serve as an upper error bound for the truncation error, where $\sigma=\sqrt{pq}$ and with the scalars p and q from (4) and (5), respectively. Because of the stability repair, p is reduced by $p_{12}^2p_i^{-1}>0$. In other words, σ now equals $\sigma=\sqrt{(p-p_{12}^2p_i^{-1})q}\leq\sqrt{pq}$. Since $g(s)$ is a SISO first order system, balancing corresponds to scaling of c and b and the model truncation error $\|g(s)w_i(s)\|_\infty$ is not changed by the alternative

[4]For the sake of completeness, it is mentioned that the matrix may be multiplied with an all pass transfer function $Z(s)$: $D_iZ(s)Z^T(-s)D_i^T=D_iID_i^T=D_iD_i^T$.

frequency weighted balancing procedure. Hence, Enns' conjecture does not hold for the frequency weighted balanced model truncation error in the sense of Lin and Chiu.

An alternative upper error bound will now be derived in a similar way as in [12]. By defining an input weighting matrix[5] $D_{W_i} = D_i + (B^T B)^{\dagger} B^T X_i^T B_i$ and by defining $B_{\perp} = BD_i + X_i^T B_i - BD_{W_i}$, we have $BD_i + X_i^T B_i = BD_{W_i} + B_{\perp}$ and (10) then becomes:

$$\Sigma_n A^T + A\Sigma_n + \bar{B}_{W_i}\bar{B}_{W_i}^T = 0, \qquad (17)$$

with $\bar{B}_{W_i} = [BD_{W_i} B_{\perp}]$. In the case that B has full rank, $B_{\perp} = 0$ and the input weighting matrix D_{W_i} can be interpreted as follows: the input weighting matrix D_{W_i} takes linear combinations of the columns of B such that the same input weighted Gramian is obtained as the input frequency weighted Gramian.

The extended output Gramian $\bar{Q}$ can be block diagonalized by applying a similar transformation as (9) with $X_o = -Q_{22}^{-1}Q_{12}^T$. By defining the output weighting matrix $D_{W_o} = D_o + C_o X_o C^T (CC^T)^{\dagger}$ and by defining $C_{\perp} = C_o X_o + D_o C - D_{W_o} C$, the upper right block of the block diagonalised extended output Lyapunov equation becomes $\Sigma_n A + A^T \Sigma_n + \bar{C}_{W_o}^T \bar{C}_{W_o} = 0$, with $\bar{C}_{W_o}^T = [C^T D_{W_o}^T \; C_{\perp}^T]$. Hence, frequency weighted balancing (A, B, C, D) in the sense of Lin and Chiu corresponds to internally balancing $(A, \bar{B}_{W_i}, \bar{C}_{W_o}, D)$. Since the infinity norm of the full matrix is not greater than the infinity norm of the submatrix, we have

$$\| D_{W_o}(G(s) - G_r(s)) D_{W_i} \|_{\infty} \leq 2 \sum_{k=r+1}^{n} \sigma_k. \qquad (18)$$

Because the H_{∞} norm is sub-multiplicative, a similar reasoning as in [12] can be applied to obtain the following (conservative) upper error bound for the frequency weighted error in terms of the frequency weighted Hankel singular values when D_{W_i} has full row rank and D_{W_o} has full column rank:

$$\| W_o(s)(G(s) - G_r(s)) W_i(s) \|_{\infty} \leq 2 \, \| W_o(s) D_{W_o}^{\dagger} \|_{\infty} \| D_{W_i}^{\dagger} W_i(s) \|_{\infty} \sum_{k=r+1}^{n} \sigma_k. \qquad (19)$$

Since the infinity norm is sub-multiplicative, the upper error bounds (11) and (19) are conservative, as is illustrated in Example 3.

[5] The pseudo-inverse of a matrix X is denoted by $X^{\dagger}$.

Example 3. We consider the SISO first-order system $g(s) = \sqrt{2}(s+1)^{-1}$ from Example 1 with a second-order input weighting $w_i(s) = [1 + (2\zeta/\omega_n s) + (s/\omega_n)^2]^{-1}$ with damping ratio ζ and natural frequency ω_n. Since $\bar{K}$ and D_{W_i} are scalars, they do not influence the final result and are omitted in this analysis for convenience of notation. Applying the sub-multiplicative property we obtain: $\|g(s)w_i(s)\|_{\infty} \leq \|g(s)\|_{\infty} \|w_i(s)\|_{\infty}$, where the error bound $\|g(j\omega)\|_{\infty} = \sqrt{2}$ obtained at $\omega = 0$ is strict (independent of multiplication with a scalar) since only one state is truncated [2]. Hence, this step will not influence the accuracy of the upper error bound. For a second-order system $w_i(s)$ with $\zeta < \sqrt{2}/2$ we have $\|w_i(j\omega)\|_{\infty} = (2\zeta\sqrt{1-\zeta^2})^{-1}$ which is obtained at the resonant frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$. The maximal amplitudes of the transfer functions of both systems $g(s)$ and $w_i(s)$ are obtained at different frequencies, the H_{∞} norm of their product is less. For example, for $\zeta = 0.1$ and $\omega_n = 5$, we have $\|w_i(s)\|_{\infty} = 5.0252$, while $\|g(s)w_i(s)\|_{\infty} = 1.4142 < \|g(s)\|_{\infty}\|w_i(s)\|_{\infty} = 7.1067$. By decreasing ζ and/or increasing ω_n, one can make the difference between the upper error bounds and the truncation error arbitrarily large.

4. Conclusions

Frequency weighted balanced model truncation is a generalisation of internally balanced model truncation, where an a priori H_{∞} upper error bound on the frequency response exists. This upper error bound is two times the sum of the neglected Hankel singular values [2,4]. Although a conjecture was formulated by Enns about an error bound for the frequency weighted case, no a priori error bound based on the frequency weighted Hankel singular values has been found yet, as mentioned frequently in the literature [1,2,5,11,13,14]. In this paper, Enns' Conjecture is refuted and it is shown that there does not exists an error bound depending only on the sum of the neglected frequency weighted Hankel singular values. It is also illustrated that truncating the states corresponding to the smallest frequency weighted singular values does not always yield the smallest approximation error. This is due to a cross-term, which is inherent to the frequency weighted balancing. By removing the cross-terms in the Lyapunov equations [6], stability of the reduced order model is guaranteed, but the frequency weighted Hankel singular values give no information about the frequency weighted error. An interpretation of the obtained singular values is given in the sense of internally balanced singular values of a related realisation.

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Appendix A. Proof of Theorem 1

The case of a truncating a first-order system $g(s) = c(s-a)^{-1}b$, input weighted with $w_i(s) = c_i(s-a_i)^{-1}b_i + d_i$ is considered ($w_o(s) = 1$). The input and output weighted Lyapunov equations (4) and (5) become:

$$2a\sigma + 2bc_ip_{12} + (bd_i)^2 = 0, \tag{20}$$

$$(a + a_i)p_{12} + bc_ip_{22} + bd_ib_i = 0, \tag{21}$$

$$2a_ip_{22} + b_i^2 = 0, \tag{22}$$

$$2a\sigma + c^2 = 0. \tag{23}$$

Following the approach of [2,5] the H_∞ error

$$E_\infty = \|(g(j\omega) - 0) \cdot w_i(j\omega)\|_\infty$$
$$= \sup_\omega (g(j\omega)w_i(j\omega)w_i(-j\omega)g(-j\omega))^{1/2}$$

can be rewritten as follows:

$$E_\infty = \sup_\omega (\sigma^2(1 + \phi\phi^{-H})(1 + \phi^H\phi^{-1}) + 2\sigma \operatorname{Re}\{bc_i\phi_ip_{12}(1 + \phi\phi^{-H})\})^{1/2},$$

with $\phi = (j\omega - a)^{-1}$ and $\phi_i = (j\omega - a_i)^{-1}$. First, $bw_i(j\omega) w_i(-j\omega)b = b^2d_i^2 + b^2b_i^2c_i^2\phi_i\phi_i^H + b^2b_ic_id_i(\phi_i + \phi_i^H)$ is rewritten by replacing $(b^2d_i^2)$, (bd_ib_i) and (b_i^2) by using (20), (21) and (22), respectively, and applying $a_i\phi_i = -1 + j\omega\phi_i$. This yields $bw_i(j\omega)w_i(-j\omega)b = -2a\sigma + p_{12}bc_i(\phi^{-H}\phi_i + \phi^{-1}\phi_i^H)$. Using now $c^2 = -2a\sigma$ and $2a\phi = -\phi\phi^{-H} - 1$ in the expression for $g(s)w_i(s)w_i(-s)g(-s)$, one obtains

$$E_\infty = \sup_\omega (\theta_{\mathrm{noFW}}(j\omega) + \theta_{\mathrm{FW}}(j\omega))^{1/2}, \tag{24}$$

with

$$\theta_{\mathrm{noFW}}(j\omega) = \sigma^2(1 + \phi\phi^{-H})(1 + \phi^H\phi^{-1}), \tag{25}$$

$$\theta_{\mathrm{FW}}(j\omega) = 2\sigma \operatorname{Re}\{bc_i\phi_ip_{12}(1 + \phi\phi^{-H})\},$$

$$= -4\sigma^2\left(\frac{bc_i p_{12}}{\sigma a_i}\right)\left(\frac{a_i^2 a^2 - a_i a \omega^2}{(\omega^2 + a_i^2)(\omega^2 + a^2)}\right). \quad (26)$$

Since $\phi\phi^{-H}$ is an all pass filter, the maximum value for $\theta_{\text{noFW}}(j\omega)$ is obtained at frequency zero and is equal to $4\sigma^2$. In the case of internally balanced truncation, the error bound (1) was proven this way in [2]. We have to show that for each finite α, there exists a stable $g(s)$ and a stable, minimum phase input weighting filter $w_i(s)$ such that the error $E_\infty > 2(1+\alpha)\sigma$. This boils down to finding a system such that

$$\begin{aligned} E_\infty^2 &= \sup_\omega(\theta_{\text{noFW}}(j\omega) + \theta_{\text{FW}}(j\omega)) \\ &> 4(1 + 2\alpha + \alpha^2)\sigma^2. \end{aligned} \quad (27)$$

Since it is sufficient to show that (27) is satisfied for one frequency ω, we evaluate (27) at $\omega = 0$. Using (25) and (26), the inequality (27) becomes:

$$-4\left(\frac{bc_i p_{12}}{\sigma a_i}\right) > 8\alpha + 4\alpha^2. \quad (28)$$

We start by choosing b, a stable a and $\sigma > 0$ and calculate c from (23). Substituting $bc_i p_{12}$ by using (20) in (28), yields the following inequality:

$$\frac{1}{a_i}\left(\frac{b^2 d_i^2}{2\sigma} + a\right) > 2\alpha + \alpha^2. \quad (29)$$

By choosing

$$d_i^2 < -\frac{2a\sigma}{b^2}, \quad (30)$$

a stable a_i can be found such that (29) holds. We now have to show that there exists a c_i, b_i, p_{12} and p_{22} such that $W_i(s) = c_i(s - a_i)^{-1}b_i + d_i$ is minimum phase and $g(s) = c(s - a)^{-1}b$ is frequency weighted balanced with $P = Q = \sigma$. The last condition corresponds to (20)–(23). The last equation is already satisfied by the choice of c and σ. Solving (20) for $c_i p_{12}$ and putting $p_{12} = 1$, we obtain c_i. Notice that it is a key point that $p_{12} \neq 0$. Now, we have to solve (21) and (22) for b_i and p_{22}. Substituting p_{22} in (21) gives a quadratic equation in b_i. The solutions are as follows:

$$\begin{aligned} b_i &= \frac{bd_i \pm \sqrt{(bd_i)^2 + 2bc_i p_{12}(a + a_i)/a_i}}{bc_i/a_i} \\ &= \frac{bd_i \pm \sqrt{-a/a_i\left((bd_i)^2 + 2\sigma(a + a_i)\right)}}{bc_i/a_i} \end{aligned} \quad (31).$$

By substituting $bc_i p_{12}$ in (31) by use of (20), it is easily seen that the solutions b_i will be real, if (30) is satisfied. Therefore, p_{22} will be positive.

The zero z_0 of $w_i(s)$ is given by:

$$\begin{aligned} z_0 &= a_i - \frac{c_i b_i}{d_i} \\ &= \not{a}_i - \not{a}_i \mp \frac{a_i}{d_i b}\sqrt{(bd_i)^2 + 2\frac{bc_i p_{12}(a + a_i)}{a_i}}. \end{aligned} \quad (32)$$

Hence, there exists always a minimum phase filter $w_i(s)$, by an appropriate choice of the sign in (31).

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Discussion on: 'On Frequency Weighted Balanced Truncation: Hankel Singular Values and Error Bounds' by T. Van Gestel, B. De Moor, B.D.O. Anderson and P. Van Overschee

1. Discussion by Z. Hurak[1]

The main result of the paper – formal refutation of Enn's conjecture – is no doubt an important achievement and will be mentioned in coming books on model reduction. By means of a constructive example the authors show that there does not exist a simple error bound depending only on Hankel singular values of the neglected part and the system matrices.

Even though a rigorous proof like the one presented in the paper was needed, one might argue that the outcome was fairly intuitive and obvious. The thing is that the input and output weighting gives enough freedom that is not captured by the type of error bound presented, i.e., error bound of type $f(\sigma_{r+1}, \ldots, \sigma_n, C, A, B) \sum_{r+1}^{n} \sigma_i$. Oversimplifying, adding arbitrary new dynamics into the system (input and output weighting filters), one can obtain an arbitrary frequency domain characteristics of the modified system.

The impact of this fact is rather pessimistic for simple frequency weighted model truncation as formulated by Enn. Neither error bounds can be given nor stability is guaranteed to be preserved after truncation. A trial-and-error approach is then necessary. This constitutes an incentive for developing other frequency weighted balanced truncation schemes.

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2. Discussion by V. Sreeram[2] and G. Wang[3]

The main contribution of the paper is presenting counter examples to disprove two theorems (Theorems 1 and 2). The first theorem is Enns' Conjecture which gives an upper error bound for the frequency weighted balanced model truncation error. The second theorem is a relaxed version of the Theorem 1. A proof of these theorems is presented in Appendix A where in a method to construct a counter example to Theorems 1 and 2 is given. Here, we present a much simpler counter example which disproves both the theorems.

Example. Let the original system and the input weight be given by

$$G(s) = (s+a)^{-1} = \left[\begin{array}{c|c} -a & 1 \\ \hline 1 & 0 \end{array}\right]$$

and

$$W_i(s) = \left(s + \frac{2}{a}\right)^{-1} = \left[\begin{array}{c|c} -\frac{2}{a} & 1 \\ \hline 1 & 0 \end{array}\right],$$

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where $0 < a < \infty$. The infinity norm of the frequency weighted error is given by

$$\begin{aligned}\|E(j\omega)\|_\infty &= \|(G(j\omega) - G_r(j\omega))W_i(j\omega)\|_\infty \\ &= \left\|(s+a)^{-1}\left(s+\frac{2}{a}\right)^{-1}\right\|_\infty \\ &= \frac{1}{2} < 1.\end{aligned}$$

The frequency weighted controllability and observability Gramians are

$$P = \frac{a}{4(2+a^2)} \quad \text{and} \quad Q = \frac{1}{2a}.$$

The weighted singular value is given by

$$\sigma = \sqrt{PQ} = \frac{1}{2\sqrt{(4+2a^2)}}.$$

When $a \to \infty$, we have $2(1+\alpha)\sigma \to 0$ no matter how large α is. Therefore, the error bound $2(1+\alpha)\sigma$ is less than the weighted error.

In Section 3.2, the influence of the cross-terms, P_{12} and Q_{12} on frequency weighted error is discussed and conditions on the weights when these terms are zero are also derived. A special case of these conditions correspond to co-inner and inner input and output weightings as reported in [6]. Under these conditions, three frequency weighted balanced truncation techniques [1,7,9] are equivalent to the unweighted balanced truncation technique [5].

In Section 3.3 of the paper, the authors present a modification to Lin and Chiu's technique [4,7] which results in error bounds similar to the one presented in Wang et al. [9]. There is an error in the derivation of error bounds. Authors claim that if the matrix B is full rank then $B_\perp = 0$ (in the first sentence after Eq. (17)). This is not correct but it can be easily fixed by assuming B to be square and nonsingular. However, such an assumption, limits the applicability of the error bounds.

3. Final Comments by the Authors T. Van Gestel[4], B. De Moor[5], B.D.O. Anderson[6] and P. Van Overschee[7]

Frequency weighted balancing [2] is an important technique with interpretations and applications in system identification and controller design. Unfortunately, the result of this paper may seem pessimistic as properties, that made internally balanced truncation popular, are formally disproven for the frequency weighted balanced case. By means of a constructive algorithm [8], counterexamples can be generated which allow to refute the upper error bound $E_\infty \leq 2(1+\alpha)\sum_{k=r+1}^{n}\sigma_k$ for any first-order system $g(s)$ (that is being reduced) and for any value of $\alpha \in \mathbb{R}^+$. Simpler counterexamples are obtained when also the system $g(s)$ can be chosen, as is seen from the discussion above. Although this result may be intuitively clear, it is still somewhat surprising that the frequency weighted balanced singular values do not contain the necessary information for an upper error bound. Moreover, Example 2 shows that the relative value of the frequency weighted singular values of the same system give no information on which state truncation yields the best approximation.

A closer look at the frequency weighted balancing formulation reveals that the nice properties of internally balancing are lost because of the cross-terms in the expressions for the frequency weighted balancing case. However, these cross-terms are essential to have frequency weightings. Alternative frequency weighted balanced truncation methods have been proposed [4,9], where our method yields a $B_\perp = 0$ when B has full row rank. These methods basically aim at reformulating the problem in such a way that results from internally balanced truncation [2,3] can be used.

As endorsed by Dr Hurak, we have pointed out problems in Enns' frequency weighted balancing formulation and refuted the conjecture on the upper error bound. As he does, we hope that this result may contribute to the development of alternative frequency weighted balanced truncation techniques that have more interesting properties. We agree that the Sreeram and Wang example now provides a simpler way to obtain a counter example than as illustrated in our paper. We endorse their other comments.

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