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A parametrization for closed-loop identification of nonlinear systems based on differentially coprime kernel representations☆

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Abstract

In this paper, we use the notion of a differentially coprime kernel representation to parametrize the set of all nonlinear plants stabilized by a given nonlinear controller using a so-called Youla parameter and to unify understanding of some stability concepts for nonlinear systems. By utilizing the differential kernel representation concept, we are able to convert a closed-loop identification problem into one of open-loop identification. The main advantage of our approach using kernel representations over fractional descriptions is that we address a larger class of nonlinear systems. The idea of a differential kernel representation allows us also to clarify the relationship between three different notions of internal stability available in the literature. The results in the paper thus provide new insights to the stability of nonlinear feedback systems.

Keywords: Nonlinear control system; Parametrization; Coprime factorization; Observers; Operators

1. Introduction

In this paper, we present a parametrization of all plants stabilized by a given controller using kernel representations. We also show how this parametrization can be applied to the identification of nonlinear systems operating under possibly nonlinear feedback.

The development of algorithms for the identification of linear plants operating in a linear closed loop has been a topic of research in the last decade. There are essentially two reasons that explain the interest in closed-loop identification. The first one is that, most often, the data are collected in closed loop, i.e. running the plant in open loop is not possible (unstable plant) or would disturb operating conditions. The second reason is that, as shown in Gevers (1993) and Van den Hof (1997), models identified using closed-loop techniques capture the essential dynamical characteristics that are important for control. We refer to Gevers (1993) and Van den Hof (1997) for a sample of the available linear closed-loop techniques. The abundance of nonlinearities in many problems motivates the need to extend the linear theory to deal with nonlinear issues. The problem of identification of nonlinear models in closed loop is definitely of practical importance for the same reasons as those indicated previously for linear models. The nonlinear problem has essentially been tackled in the literature from two different points of view.

The first point of view is to develop algorithms for identification in closed loop by considering "closed-loop output-error" schemes. A closed-loop output-error-type predictor parametrized in terms of the existing controller and the estimated plant is used, i.e. the parametrization is tailored to the closed-loop configuration. This has led to both off-line and recursive schemes which are reported in De Bruyne, Anderson, Gevers, and Linard (1999) and Linard, Anderson, and De Bruyne (1999).

The second point of view relies on the ability to parameterize the unknown plant in terms of a (known) nominal model and the (known) nonlinear controller along with a so-called Youla–Kucera parameter associated with the plant. The main advantage of this method is that rather than identifying the plant one identifies the Youla–

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Kucera parameter; in the linear case this results (and in the nonlinear case this may result) in the closed-loop identification problem being transformed into one that is open loop in nature. This problem was initially investigated in Hansen (1989) and further investigated in a nonlinear framework in Dasgupta and Anderson (1996) and Linard et al. (1999) based on the left coprime factorizations as developed in e.g. (Hammer, 1987; Tay & Moore, 1989; Paice & Moore, 1990). In Dasgupta and Anderson (1996), using coprime factorizations of the plant and controller, the identification of nonlinear time-varying plants operating under linear, possibly time-varying feedback was investigated. The authors show that there are left and right coprime-based fractional descriptions for the set of all plants stabilized by the linear controller given that the nominal plant model is linear. They use models of the plant that can be based either on the left or right coprime factors of the nominal plant and controller. Linard et al. (1999) extend the theory to enable one to find a left coprime factorization-based description of the set of all plants stabilized by a given controller, given a nominal plant model and a controller that are not necessarily linear. A notion of differential coprimeness is introduced to help characterize the model set of the plant.

The ultimate goal of this paper is to develop these ideas based on kernel representations. Kernel representations were introduced as a generalization of left factorizations in Paice and van der Schaft (1994). Although there is no essential difference for linear systems, for nonlinear systems it was shown in Scherpen and van der Schaft (1994) that state-space realizations of kernel representations are very often computable under mild assumptions whereas those of left factorizations are hard or impossible to obtain (Chen & de Fegueiredo, 1989). Also, it is known that every nonlinear system does not necessarily have a left coprime factorization. For these two reasons, it is important to develop the closed-loop identification procedure based on kernel representations. Indeed, this would increase the applicability of this method to a much larger class of systems.

Results on the parametrization of stabilizing controllers (or their dual) based on kernel representations are available in Paice and van der Schaft (1995, 1996), Fujimoto and Sugie (1998a, b, 2000). However, the parametrization based on kernel representations adopts a different definition of the stability of feedback systems than the usual one used in e.g. Verma (1988), and we cannot use it directly to derive a closed-loop identification procedure. The first step to perform in this paper is to overcome this difficulty, more precisely, to derive the parametrization based on the usual definition of the stability of feedback systems as an extension of the parametrization based on other definitions Fujimoto and Sugie (2000). The closed-loop identification procedure based on kernel representations, which is the final objective, follows this result immediately.

To this end, we employ a special kernel representation, i.e. we assume that (a) the direct feedthrough part is a constant, and that (b) it has a differential coprimeness property. The property (a) is also employed in Linard et al. (1999) and Fujimoto and Sugie (2000) to render the parametrized feedback system well-posed. As for the differential coprimeness assumption in (b), it has been introduced in Linard et al. (1999) in the context of coprime factorizations and amounts usually requiring not just boundedness of the closed-loop operator but Lipschitz continuity as well. Additionally, the Lipschitz continuity is often utilized to investigate internal stability of feedback systems, e.g. Ball and Verma (1994). Kernel representations which obey these two assumptions also play a key role in this paper. It will be shown that these kernel representations connect the stability of feedback systems as employed in the kernel representation approach to the one used in the usual sense, and are therefore ideally suited for a closed-loop identification problem. In addition, the stability equivalence indicated above unifies understanding of different stability concepts of nonlinear systems.

This paper is organized as follows. Section 2 gives notations, definitions, preliminary and basic results used in the paper. Section 3 is devoted to the analysis of the well-posedness and internal stability of feedback systems based on differential coprime kernel representations. Further the parametrization of all internally stabilizing plant and controller pairs are derived. Section 4 provides the procedure of closed-loop identification based on the above parametrization. Moreover, we discuss how to incorporate the disturbances into the closed-loop problem and explain how the identification procedure can be used under a high signal-to-noise ratio assumption.

2. Preliminary background

This section gives definitions and preliminary results borrowed from Vidyasagar (1980), Verma (1988), Paice and van der Schaft (1996), Linard et al. (1999) and Fujimoto and Sugie (2000).

2.1. Signal spaces

τ_T and δ_T are the truncation operator and the delay operator on the vector space of functions mapping from $\mathbb{R}$ to $\mathbb{R}^m$ defined by

$$\tau_T u(t) := \begin{cases} u(t), & t \leqslant T, \\ 0, & t > T, \end{cases} \qquad \delta_T u(t) := \begin{cases} 0, & t \leqslant T, \\ u(t-T), & t > T. \end{cases}$$

$L_2^m[0,\infty)$ denote the vector space of $\mathbb{R}^m$ valued square integrable functions with norm defined by $\|u\|^2 := \int_0^\infty u^T u \, dt$. L_2^m and L_2 are used as shorthand for $L_2^m[0,\infty)$. $L_{2e}^m[0,\infty)$ denotes the vector space of functions u satisfying $\tau_T u(t) \in L_2^m[0,\infty)$ for all $T>0$. L_{2e}^m and L_{2e} are used as shorthand for $L_{2e}^m[0,\infty)$.

2.2. Operator stability

Let $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ denote a system input–output operator with the system initial state $x(0)=x^0 \in \mathscr{X}^0 \subset \mathbb{R}^n$ where $\mathscr{X}^0 \ni 0$ is a connected subset of $\mathbb{R}^n$. The symbol Σ is used as shorthand for Σ^{x^0}.

For such operators, the following properties are defined. To explain Lipschitz continuity, we employ the differential operator $\partial(\cdot)$ borrowed from Linard et al. (1999),

$$\partial \Sigma_{(u)}^{x^0}(v) := \Sigma^{x^0}(u+v) - \Sigma^{x^0}(u).$$

The operator acts on v, and is parametrized by u; in case Σ^{x^0} is a linear operator, the differential operator is independent of u and identical with Σ^0.

Definition 1. Consider an operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$.

- The operator is said to be causal if $\Sigma^{x^0}(u) \in L_{2e}^p$ is uniquely determined for $\forall u \in L_{2e}^m$ and $\forall x^0 \in \mathscr{X}^0$, and $\tau_T \Sigma^{x^0} \tau_T = \tau_T \Sigma^{x^0}$ holds for $\forall T>0$ and $\forall x^0 \in \mathscr{X}^0$.
- The operator is said to be (causally) invertible if it is causal, $m=p$ holds and there exists a causal operator $(\Sigma^{x^0})^{-1}: L_{2e}^m \to L_{2e}^m$ such that $\Sigma^{x^0}(\Sigma^{x^0})^{-1} = (\Sigma^{x^0})^{-1}\Sigma^{x^0} = \mathrm{Id}$ holds for $\forall x^0 \in \mathscr{X}^0$, where Id denotes the identity operator. The operator $(\Sigma^{x^0})^{-1}$ is also denoted by $(\Sigma^{-1})^{x^0}$ or Σ^{-1}.
- The operator is said to be bounded if it is causal and there exists a finite constant γ and a scalar function ϕ satisfying $\phi(0)=0$ such that the following inequality holds for $\forall u \in L_{2e}^m$ and $\forall x^0 \in \mathscr{X}^0$,

  $$\|\Sigma^{x^0}(u)\| \leqslant \gamma \|u\| + \phi(x^0).$$

  The minimum value of γ which satisfies the above inequality is called the gain of Σ and it is denoted by $\|\Sigma\|$.
- The operator is said to be a unit if it is causally invertible, and both Σ^{x^0} and $(\Sigma^{-1})^{x^0}$ are bounded.
- The operator is said to be weakly Lipschitz (or weakly Lipschitz continuous) if it is causal and its Lipschitz semi-norm $\|\tau_T \Sigma^{x^0}\|_L$ is finite for every $T>0$ and $x^0 \in \mathscr{X}^0$, where the Lipschitz semi-norm $\|\cdot\|_L$ is defined as follows.

  $$\|\tau_T \Sigma^{x^0}\|_L := \sup_{u,\, v \in L_{2e}^m,\ \tau_T v \neq 0} \frac{\|\tau_T \partial \Sigma_{(u)}^{x^0}(v)\|}{\|\tau_T v\|}.$$

- The operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ is said to be smoothing if it is weakly Lipschitz and for every $T>0$, $\gamma>0$ and $x^0 \in \mathscr{X}^0$ there exists $t_1 = t_1(T,\gamma,x^0) \in (0,T)$ such that

  $$\|\tau_{t+t_1}(\Sigma^{x^0}\tau_{t+t_1} - \Sigma^{x^0}\tau_t)\|_L \leqslant \gamma$$

  holds for $\forall t \in [0, T-t_1]$. Namely, Σ is smoothing if it has no instantaneous direct feedthrough or if it has zero uniform instantaneous gain.
- The operator is said to be globally Lipschitz (or globally Lipschitz continuous) if there exists a finite constant γ such that the following inequality holds for $\forall u, v \in L_2^m$ and $\forall x^0 \in \mathscr{X}^0$.

  $$\|\partial \Sigma_{(u)}^{x^0}(v)\| \leqslant \gamma \|v\|.$$

The smoothing concept is a powerful tool for establishing the well-posedness property of interconnected systems and it will be used to prove the well-posedness of feedback systems afterwards. We summarize some properties of smoothing operators (Vidyasagar, 1980).

- The sum of two weakly Lipschitz operators is also weakly Lipschitz.
- The sum of two smoothing operators is also smoothing.
- The cascade of two weakly Lipschitz operators is also weakly Lipschitz.
- If the operator Σ is smoothing and the operator Γ is weakly Lipschitz, then $\Sigma\Gamma$ is smoothing; $\Gamma\Sigma$, however, may not necessarily be smoothing.

The following theorem on feedback systems was obtained in Vidyasagar (1980).

Theorem 2 (Vidyasagar, 1980). *Consider a smoothing operator $\Sigma: L_{2e}^m \times L_{2e}^q \to L_{2e}^p$ and a weakly Lipschitz operator $\Gamma: L_{2e}^p \times L_{2e}^r \to L_{2e}^m$. Then the mapping $(z,v) \mapsto (u,y)$ in*

$$y = \Gamma(u,z), \quad u = \Sigma(y,v)$$

have a unique solution $(u,y) \in L_{2e}$ for all $(z,v) \in L_{2e}$ and the mapping $(z,v) \mapsto (u,y)$ is weakly Lipschitz.

In this paper, it is assumed that any operator has the state-space realization

$$\Sigma^{x^0}: \begin{cases} \dot{x} = f(x,u), & x(0) = x^0 \in \mathscr{X}^0, \\ y = h(x,u), \end{cases} \tag{1}$$

where f and h are smooth functions with $f(0,0)=0$ and $h(0,0)=0$ (consequently $\Sigma^0(0)=0$ holds). Systems that have state-space realizations as given in (1) are causal.

We now define the complete controllability concept which is an intrinsic property of state-space realizations of operators. A causal operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ with the state-space realization as in (1) is said to be *completely controllable* if for $\forall x^1, x^2 \in \mathscr{X}^0$ there exist $T>0$ and an input $u \in L_2^m[0,T]$ such that the corresponding trajectory $x(t)$ $(0 \leqslant t \leqslant T)$ of system (1) satisfies $x(0)=x^1$ and $x(T)=x^2$.

Remark 3. If a causal operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ is completely controllable, then for $\forall x^1, x^2 \in \mathscr{X}^0$ there exist

$T > 0$ and $u \in L_2^m$ such that

$$\Sigma^{x^1}(\delta_T(v) + \tau_T(u))(t) = \delta_T \Sigma^{x^2}(v)(t) \quad \forall v \in L_{2e}^m \ \forall t \geqslant T.$$

Furthermore, a completely controllable operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ has a finite gain γ if and only if $\|\Sigma^0(u)\| \leqslant \gamma \|u\|$ for $\forall u \in L_2^m$.

2.3. Kernel representations

This subsection introduces kernel representations (Paice & van der Schaft, 1996; van der Schaft, 1996) as a generalization of left factorizations. A *kernel representation* of a causal operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ is a causal operator $R_\Sigma^{x^0}: L_{2e}^m \times L_{2e}^p \to L_{2e}^p$ such that

$$y = \Sigma^{x^0}(u) \Leftrightarrow R_\Sigma^{x^0}(u, y) = 0 \tag{2}$$

holds for $\forall x^0 \in \mathscr{X}^0$ and $\forall u \in L_{2e}^m$ and $y \in L_{2e}^p$.

A kernel representation $R_\Sigma^{x^0}: L_{2e}^m \times L_{2e}^p \to L_{2e}^p$ is said to be *well defined* if there exists the causal pseudo-inverse operator $(R_\Sigma^{x^0})^\#: L_{2e}^m \times L_{2e}^p \to L_{2e}^p$ such that

$$y = (R_\Sigma^{x^0})^\#(u, z_\Sigma) \Leftrightarrow R_\Sigma^{x^0}(u, y) = z_\Sigma$$

holds for $\forall x^0 \in \mathscr{X}^0$ and $\forall u \in L_{2e}^m$ and $y, z_\Sigma \in L_{2e}^p$.

Kernel representations are natural generalizations of left factorizations, because if an operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ has a left factorization $\Sigma = \tilde{M}^{-1}\tilde{N}$ with $\tilde{N}^{x^0}: L_{2e}^m \to L_{2e}^p$ and $\tilde{M}^{x^0}: L_{2e}^p \to L_{2e}^p$ then a well-defined kernel representation of Σ is given by

$$R_\Sigma^{x^0}(u, y) = -\tilde{N}^{x^0}(u) + \tilde{M}^{x^0}(y). \tag{3}$$

Kernel representations are not equivalent to left factorizations because in general any given kernel representation R_Σ is not "separable", namely, it cannot be divided into two operators $\tilde{N}$ and $\tilde{M}$ as in (3).

Definition 4. A bounded kernel representation $R_\Sigma^{x^0}: L_{2e}^{m+p} \to L_{2e}^p$ is said to be coprime if there exists a bounded operator $X^{x^0}: L_{2e}^p \to L_{2e}^{m+p}$ such that

$$R_\Sigma^{x^0} X^{x^0} = \mathrm{Id} \quad \forall x^0 \in \mathscr{X}^0. \tag{4}$$

When R_Σ specializes to (3), Eq. (4) reduces to

$$-\tilde{N}X_1 + \tilde{M}X_2 = \mathrm{Id}.$$

Therefore, Eq. (4) is a natural generalization of the Bezout identity in linear systems theory.

Definition 5. A globally Lipschitz kernel representation $R_\Sigma^{x^0}: L_{2e}^{m+p} \to L_{2e}^p$ is said to be uniformly differentially coprime if there exists a set of bounded operators $X_{(w)}^{x^0}: L_{2e}^p \to L_{2e}^{m+p}$ which are parametrized by the signal w and have a finite gain uniformly over $w \in L_{2e}^{m+p}$ such that

$$\partial R_{\Sigma(w)}^{x^0} X_{(w)}^{x^0} = \mathrm{Id} \tag{5}$$

holds for $\forall w \in L_{2e}^{m+p}$. Here $X_{(w)}(v)$ is causally dependent on w and v.

If a kernel representation R_Σ is differentially coprime, i.e. there exists a set of bounded operators $X_{(w)}$ such that (5) holds, then the Bezout identity

$$R_\Sigma^{x^0} \underbrace{X_{(0)}^{x^0}(\mathrm{Id} - R_\Sigma^{x^0}(0))}_{\text{bounded}} = \mathrm{Id}$$

also holds by setting $w = 0$ and by multiplying the operator $\mathrm{Id} - R_\Sigma(0)$, i.e. R_Σ is coprime in the usual sense. The definition of differential coprimeness is motivated by the fact that it is implied by a small signal closed-loop stability which is necessary for closed-loop identification (see Linard et al., 1999; Fujimoto, 2000).

Remark 6. In Linard et al. (1999), the following equation is taken as the definition of uniform differential coprimeness of "separable" coprime kernel representations (i.e. left coprime fractional representations),

$$\partial R_{\Sigma(w)} X = W_{(w)} \tag{6}$$

with a bounded operator X and a unit $W_{(w)}$ where $W_{(w)}$ and $W_{(w)}^{-1}$ have finite gains uniformly over $w \in L_{2e}$. Eq. (6) reduces to

$$\partial R_{\Sigma(w)} X W_{(w)}^{-1} = \mathrm{Id}$$

and this equation implies that R_Σ is uniformly differentially coprime in the sense of (5) in Definition 5. However, the converse does not hold.

Remark 7. A trivial sufficient condition for differential coprimeness of R_Σ is the global Lipschitz continuity of Σ. Consider a globally Lipschitz operator $\Sigma: u \mapsto y$. Then its differentially coprime kernel representation given by

$$R_\Sigma(u, y) = y - \Sigma(u).$$

2.4. Internal stability

We now consider the feedback system depicted in Fig. 1. Such a feedback system that interconnects $G^{x_G^0}: L_{2e}^m \to L_{2e}^p$ and $K^{x_K^0}: L_{2e}^p \to L_{2e}^m$ is denoted by $\{G^{x_G^0}, K^{x_K^0}\}$ or just $\{G, K\}$. We use condensed notations: $L_{2e}^{m+p} \ni w := (u, y) \in L_{2e}^m \times L_{2e}^p$, $L_{2e}^{m+p} \ni \bar{w} := (\bar{u}, \bar{y}) \in L_{2e}^m \times L_{2e}^p$, $L_{2e}^{m+p} \ni z_{GK} := (z_K, z_G) \in L_{2e}^m \times L_{2e}^p$ and $L_{2e}^{m+p} \ni e_{12} := (e_1, e_2) \in L_{2e}^m \times L_{2e}^p$, if no confusion arises.

The stability of the feedback system $\{G, K\}$ with additive disturbances as in Fig. 1 is considered. Such a configuration is often treated in the literature on right coprime factorizations, e.g. (Verma, 1988). Let us define a new operator $E_{\{G,K\}}^{(x_G^0, x_K^0)}: L_{2e}^{m+p} \to L_{2e}^{m+p}$ which is a mapping from the external additive signal (e_1, e_2) to the loop signal (u, y)

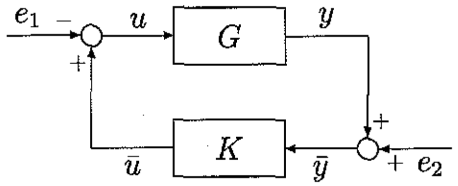

Fig. 1. The feedback system $\{G, K\}$ with additive disturbances.

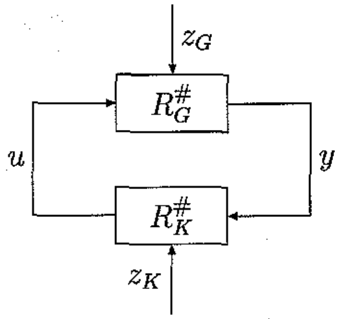

Fig. 2. Null well-posedness and null internal stability of $\{G, K\}$.

in Fig. 1.

$$E_{\{G,K\}}^{(x_G^0,x_K^0)}(e_{12}) := \left(\begin{pmatrix} -\mathrm{Id} & K^{x_K^0} \\ -G^{x_G^0} & \mathrm{Id} \end{pmatrix}^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}\right)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u \\ y \end{pmatrix} = w. \quad (7)$$

Definition 8. A feedback system $\{G, K\}$ is said to be well posed if the operator $E_{\{G,K\}}^{(x_G^0,x_K^0)}$ exists and is weakly Lipschitz. Furthermore, a well-posed feedback system $\{G, K\}$ is said to be internally stable if $E_{\{G,K\}}^{(x_G^0,x_K^0)}$ is bounded.

From Theorem 2, if one of G or K is smoothing and the other is weakly Lipschitz, then the feedback system $\{G, K\}$ is well posed.

2.5. Null internal stability

We state two other stability concepts of feedback systems based on kernel representations, and Section 3 will connect these concepts. The stability of feedback systems as shown in Fig. 2 is discussed here, where R_G and R_K denote the kernel representations of the components G and K of the feedback system $\{G, K\}$, respectively.

$$R_G^{x_G^0} : (u, y) \mapsto z_G, \quad R_K^{x_K^0} : (y, u) \mapsto z_K. \quad (8)$$

Note that the use of $R_G^{\#}$ and $R_K^{\#}$ in Fig. 2 is an abuse of notation because the operators are not well defined in general. Also, if not well defined, then the operators do not necessarily produce a unique open-loop output, in the sense that y is not necessarily uniquely determined by u and z_G, and u is not necessarily uniquely determined by y and z_K.

The kernel representation $R_{\{G,K\}} : L_{2e}^{m+p} \to L_{2e}^{m+p}$ of the feedback system $\{G, K\}$ can be defined by

$$R_{\{G,K\}}^{(x_G^0,x_K^0)}(w) := \begin{pmatrix} R_K^{x_K^0}(y, u) \\ R_G^{x_G^0}(u, y) \end{pmatrix} = \begin{pmatrix} z_K \\ z_G \end{pmatrix} = z_{GK}.$$

Regarding the feedback system $\{G, K\}$ as a null-input system $w = E_{\{G,K\}}(0)$, we obtain the definition of the kernel representation as in (2):

$$w = E_{\{G,K\}}(0) \Leftrightarrow R_{\{G,K\}}(w) = 0.$$

The null well-posedness and null internal stability of the feedback system $\{G, K\}$ is now defined using the kernel representation $R_{\{G,K\}}$.

Definition 9. A feedback system $\{G, K\}$ with a weakly Lipschitz kernel representation $R_{\{G,K\}}^{(x_G^0,x_K^0)}$ is said to be null well posed if the operator $R_{\{G,K\}}^{(x_G^0,x_K^0)}$ is invertible and $R_{\{G,K\}}^{(x_G^0,x_K^0)^{-1}}$ is weakly Lipschitz. Furthermore, a null well-posed feedback system $\{G, K\}$ with a bounded kernel representation $R_{\{G,K\}}^{(x_G^0,x_K^0)}$ is said to be null internally stable if $R_{\{G,K\}}^{(x_G^0,x_K^0)^{-1}}$ is bounded.

Remark 10. The internal stability of a system $\{G, K\}$ with $R_{\{G,K\}}$ is equivalent to the coprimeness of $R_{\{G,K\}}$, because the unimodularity of $R_{\{G,K\}}$ is equivalent to the existence of a bounded operator $X = R_{\{G,K\}}^{-1}$ such that

$$R_{\{G,K\}} X = \mathrm{Id}.$$

This equation is a generalization of the double Bezout identity.

2.6. Strong internal stability

By employing the kernel representations R_G and R_K in (8), a kernel representation of the operator $E_{\{G,K\}}$ can be defined by

$$R_{E_{\{G,K\}}}^{(x_G^0,x_K^0)}(e_{12}, w) := \begin{pmatrix} R_K^{x_K^0}(w + e_{12}) \\ R_G^{x_G^0}(w) \end{pmatrix} = z_{GK}.$$

It can be easily observed that

$$w = E_{\{G,K\}}(e_{12}) \Leftrightarrow R_{E_{\{G,K\}}}(e_{12}, w) = 0,$$

which is the definition of the kernel representation (2). Then strong well-posedness and strong internal stability is defined as follows.

Definition 11. A feedback system $\{G, K\}$ with a weakly Lipschitz kernel representation $R_{\{G,K\}}^{(x_G^0,x_K^0)}$ is said to be strongly well posed if the operator $R_{E_{\{G,K\}}}^{(x_G^0,x_K^0)}$ is well defined and $R_{E_{\{G,K\}}}^{(x_G^0,x_K^0)^{\#}}$ is weakly Lipschitz. Furthermore, a strongly well-posed feedback system $\{G, K\}$ with a bounded kernel representation $R_{\{G,K\}}^{(x_G^0,x_K^0)}$ is said to be strongly internally stable if $R_{E_{\{G,K\}}}^{(x_G^0,x_K^0)^{\#}}$ is bounded.

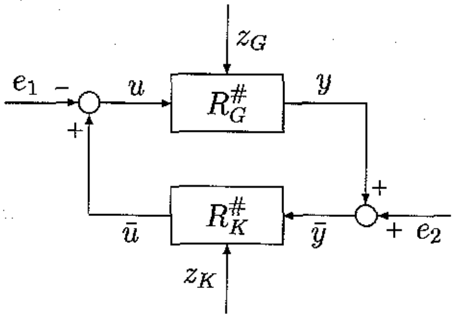

Fig. 3. Strong well-posedness and strong internal stability of $\{G, K\}$.

The concepts of strongly well-posedness and strong internal stability can be shown in Fig. 3. We abuse the notations $R_G^{\#}$ and $R_K^{\#}$ again to denote operators that might not be well defined.

2.7. Parametrization of all strongly internally stabilizing plant-controller pairs

Before stating the results of this subsection, we introduce the concept of strong detectability. Roughly speaking, a kernel representation is strongly detectable if its state-space realization is an asymptotic state-observer of the original system (Fujimoto & Sugie, 1998b, 2000). (See Fujimoto (2000) for a more precise discussion.)

Definition 12. A kernel representation $R_\Sigma^{x^0}: L_{2e}^{m+p} \to L_{2e}^p$ is said to be strongly detectable if there exists a non-negative constant γ and a scalar function ϕ satisfying $\phi(0,0) = 0$ such that

$$\|R_\Sigma^{x^1}(w + v) - R_\Sigma^{x^2}(w)\| \leqslant \gamma\|v\| + \phi(x^1, x^2) \tag{9}$$

holds for $\forall w \in L_{2e}^{m+p}$, $\forall v \in L_2^{m+p}$ and $\forall x^1, x^2 \in \mathscr{X}^0$.

In contrast to Fujimoto and Sugie (1998b), the above definition does not explicitly imply the property of asymptotic estimator because the term ϕ shows no decay with time t. However, setting $v = 0$ in (9) shows that the difference of the outputs of the two kernel representations has a finite L_2 norm and this implies that the difference signal goes to 0 as $t \to \infty$.

By assuming the strong detectability of kernel representations, we can derive the parametrization of all plants which are *strongly* internally stabilized by a given controller K.

Theorem 13 (Fujimoto & Sugie, 2000). *Consider a strongly internally stable feedback system $\{G, K\}$ with weakly Lipschitz bounded kernel representations $R_G: L_{2e}^{m+p} \to L_{2e}^p$ and $R_K: L_{2e}^{m+p} \to L_{2e}^m$. Suppose R_K is strongly detectable. Then the parametrization of all weakly Lipschitz plants which are strongly internally stabilized by K with R_K is given by G_S with a kernel representation*

$$R_{G_S} := R_S R_{\{G,K\}}: L_{2e}^{m+p} \to L_{2e}^p, \tag{10}$$

where $R_S: L_{2e}^{m+p} \to L_{2e}^p$ is any weakly Lipschitz bounded well-defined kernel representation of any weakly Lipschitz bounded operator S such that $R_S^{\#}$ is weakly Lipschitz and bounded.

This parametrization is also used for the parametrization of all strongly internally stabilizing controllers of a given plant G. The next theorem allows both G and K to vary.

Theorem 14 (Fujimoto & Sugie, 2000). *Consider a strongly internally stable feedback system $\{G, K\}$ with bounded kernel representations $R_G: L_{2e}^{m+p} \to L_{2e}^p$ and $R_K: L_{2e}^{m+p} \to L_{2e}^m$. Suppose both $R_{\{G,K\}}$ and $R_{\{G,K\}}^{-1}$ are strongly detectable and $R_{\{G,K\}}$ has a construction as*

$$R_{\{G,K\}} = R_{\{G,K\}}^{\text{smth}}(w) + R_{\{G,K\}}^{\text{cnst}} \cdot w, \tag{11}$$

where $R_{\{G,K\}}^{\text{smth}}$ is a smoothing operator and $R_{\{G,K\}}^{\text{cnst}}$ is a constant matrix. Then the parametrization of all strongly internally stabilizing plant and controller pairs are given by the plants G_S and controllers K_Q with kernel representations

$$R_{K_Q} := R_Q R_{\{G,K\}}: L_{2e}^{m+p} \to L_{2e}^m,$$
$$R_{G_S} := R_S R_{\{G,K\}}: L_{2e}^{m+p} \to L_{2e}^p, \tag{12}$$

where the feedback system $\{S, Q\}$ with the kernel representations $R_S: L_{2e}^{m+p} \to L_{2e}^p$ and $R_Q: L_{2e}^{m+p} \to L_{2e}^m$ is any strongly internally stable one.

Note that Eq. (11) implies that the state-space realization of the operator $R_{\{G,K\}}$ has a constant (linear) direct feedthrough. Similar parametrizations based on *null* internal stability are also obtained in Fujimoto and Sugie (2000). For closed-loop identification, we need the parametrization of all plants which are internally stabilized by a given controller. This is because we cannot inject the external signal z_G into the plant G as in Figs. 2 and 3 in the procedure of closed-loop identification, and consequently cannot check whether the given controller *strongly* (or *null*) internally stabilizes the plant or not. Therefore, Theorem 13 (or Theorem 14) is not applicable to closed-loop identification, and we will investigate the parametrization based on internal stability in Section 3.

3. Equivalence of internal stabilities and parametrization

3.1. Stability analysis

In this subsection, the relationship between the three different well-posedness and stability definitions for feedback systems are discussed. First, we give a basic

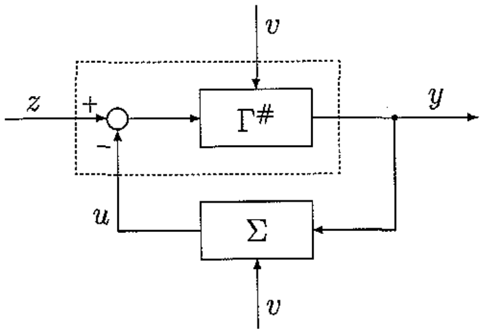

Fig. 4. Configuration of $(\Sigma + \Gamma)^{\#}$.

property about the pseudo-invertibility of weakly Lipschitz operators.

Lemma 15. *Consider two weakly Lipschitz operators $\Sigma : L_{2e}^m \times L_{2e}^p \to L_{2e}^p$ and $\Gamma : L_{2e}^m \times L_{2e}^p \to L_{2e}^p$. Suppose Σ is smoothing. Then Γ has a weakly Lipschitz pseudo-inverse operator $\Gamma^{\#}$ such that*

$$z = \Gamma(v, y) \Leftrightarrow y = \Gamma^{\#}(v, z)$$

holds if and only if this also holds for $(\Sigma + \Gamma)$. Furthermore, if Γ is a constant matrix and $\Gamma^{\#}$ exists, then $(\Sigma + \Gamma)^{\#} - \Gamma^{\#}$ is smoothing.

Proof. Necessity of the former part will be shown first. Suppose $\Gamma^{\#}$ exists and is weakly Lipschitz. Then $y = (\Gamma + \Sigma)^{\#}(v, z)$ can be depicted explicitly by a feedback system of $\Gamma^{\#}$ and Σ

$$y = \Gamma^{\#}(v, z - u), \quad u = \Sigma(v, y)$$

with u as shown in Fig. 4. This is because

$$z = (\Gamma + \Sigma)(v, y) \Leftrightarrow \Gamma(v, y)$$
$$= z - \Sigma(v, y) \Leftrightarrow y = \Gamma^{\#}(v, z - \Sigma(v, y)).$$

Using Theorem 2, the weak Lipschitz continuity of the mapping of $(v, z) \mapsto (u, y)$ can be proved by the smoothing property of Σ and the weak Lipschitz continuity of the operator shown in the dashed box of Fig. 4 (because of that of $\Gamma^{\#}$). Sufficiency will be proved by the same arguments by substituting Γ by $(\Sigma + \Gamma)$ and Σ by $-\Sigma$, respectively.

The latter part is now proved. Suppose Γ is a constant matrix and $\Gamma^{\#}$ exists. Then $(\Sigma + \Gamma)^{\#}$ exists and is weakly Lipschitz using the first part of the lemma. Using the operator $(\Sigma + \Gamma)^{\#}$, we obtain a smoothing operator

$$u = \Sigma(v, (\Sigma + \Gamma)^{\#}(v, z))$$

since Σ is smoothing. We can obtain another description of $(\Sigma + \Gamma)^{\#}$ as

$$(\Sigma + \Gamma)^{\#}(v, z) = \Gamma^{\#}(v, z - u)$$
$$= \Gamma^{\#}(v, z) - \Gamma^{\#}(0, \Sigma(v, (\Sigma + \Gamma)^{\#}(v, z))),$$

where we use the fact that $\Gamma^{\#}$ is a constant matrix. Therefore,

$$((\Sigma + \Gamma)^{\#} - \Gamma^{\#})(v, z) \equiv -\Gamma^{\#}(0, \Sigma(v, (\Sigma + \Gamma)^{\#}(v, z)))$$

holds and this is smoothing. This completes the proof. □

Lemma 15 implies that (pseudo) invertibility of an operator depends only on its non-smoothing part, and further that if the non-smoothing part of an operator is a constant matrix then the non-smoothing part of the (pseudo) inverse operator is also a constant matrix. This property will be intensively used in the rest of the paper. Indeed, we can prove the equivalence between null well-posedness and strong well-posedness using Lemma 15.

Lemma 16. *Consider a well-posed feedback system $\{G, K\}$ with kernel representations R_G and R_K. Suppose either R_G or R_K has a construction as in* (11). *Then the feedback system $\{G, K\}$ with the kernel representation $R_{\{G,K\}}$ is strongly well posed if and only if it is null well posed.*

Proof. Consider the feedback system $\{G, K\}$ in Fig. 3. Suppose R_K has the construction as in (11) without loss of generality, i.e.

$$R_K(w) = R_K^{\text{smth}}(w) + R_K^{\text{cnst}} \cdot w \tag{13}$$

with a smoothing operator R_K^{smth} and a constant matrix R_K^{cnst}. Then the kernel representation $z_{GK} = R_{E_{\{G,K\}}}(e_{12}, w)$ can be described by

$$z_{GK} = \underbrace{\begin{pmatrix} R_K^{\text{smth}}(w + e_{12}) \\ 0 \end{pmatrix}}_{\text{smoothing}} + \underbrace{\begin{pmatrix} R_K^{\text{cnst}} \\ R_G \end{pmatrix}(w) + \begin{pmatrix} R_K^{\text{cnst}} \\ 0 \end{pmatrix} e_{12}}_{\text{non-smoothing}}. \tag{14}$$

It follows from Lemma 15 that

$$\begin{matrix}\text{null} \\ \text{well-posedness}\end{matrix} \Leftrightarrow \begin{pmatrix} R_K^{\text{cnst}} \\ R_G \end{pmatrix}^{-1} : \begin{matrix}\text{weakly} \\ \text{Lipschitz}\end{matrix}$$
$$\Leftrightarrow \begin{matrix}\text{strong} \\ \text{well-posedness}\end{matrix} \tag{15}$$

holds. The first relation can be obtained by substituting $e_{12} = 0$ and neglecting the smoothing part. The second relation holds because of the following fact. Now we write down the "non-smoothing part" on the right-hand side of (14)

$$\hat{z}_{GK} := \begin{pmatrix} R_K^{\text{cnst}} \\ R_G \end{pmatrix}(w) + \begin{pmatrix} R_K^{\text{cnst}} \\ 0 \end{pmatrix} e_{12}. \tag{16}$$

It is obvious that the operator on the right-hand side is pseudo-invertible with respect to w if and only if the operator in the middle of (15) is weakly Lipschitz and

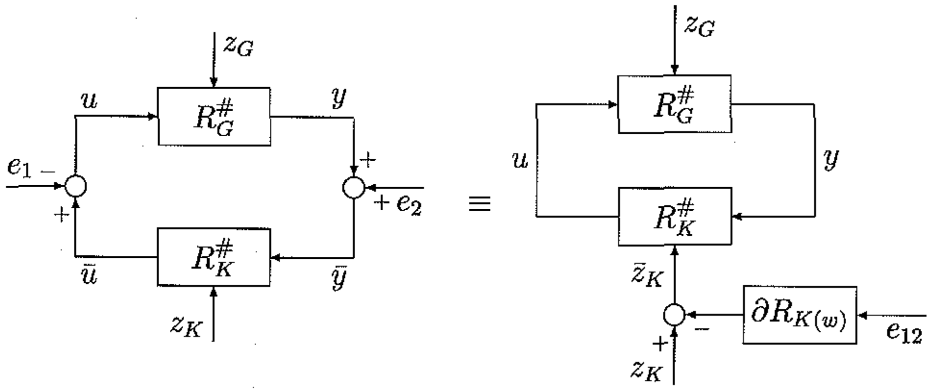

Fig. 5. Equivalent configuration of the feedback system in Fig. 3.

that the pseudo-inverse of (16) is obtained by

$$w = \begin{pmatrix} R_K^{\text{cnst}} \\ R_G \end{pmatrix}^{-1} \left(\hat{z}_{GK} - \begin{pmatrix} R_K^{\text{cnst}} \\ 0 \end{pmatrix} e_{12} \right).$$

It follows from Lemma 15 that $z_{GK} = R_{E_{\{G,K\}}}(e_{12}, w)$ is pseudo-invertible if and only if the operator in (15) is weakly Lipschitz, i.e. the second relation holds. This completes the proof. □

Further, we can show the equivalence between null internal stability and strong internal stability when one of the kernel representations is globally Lipschitz.

Theorem 17. *Consider a well-posed feedback system $\{G, K\}$ with kernel representations $R_G : L_{2e}^{m+p} \to L_{2e}^p$ of G and $R_K : L_{2e}^{m+p} \to L_{2e}^m$ of K. Suppose either R_G or R_K is globally Lipschitz and has a construction as in* (11). *Then the feedback system $\{G, K\}$ with the kernel representation $R_{\{G,K\}}$ is strongly internally stable if and only if it is null internally stable.*

Proof. Necessity is obvious and only sufficiency is proved. Consider the configuration in Fig. 3. Suppose the feedback system is null internally stable and R_K is globally Lipschitz and has a construction as in (13) without loss of generality. Then it follows from Lemma 16 that the feedback system is strongly well posed. We now construct an equivalent feedback system as shown on the right-hand side in Fig. 5. The signal z_K in the figure can be calculated as

$$z_K = R_K(\bar{w}) = R_K(w + e_{12}) = \partial R_{K(w)}(e_{12}) + \bar{z}_K.$$

This equation shows that the mapping $(e_1, e_2, z_K) \mapsto \bar{z}_K$ is bounded because of the global Lipschitz continuity of R_K. The null internal stability of the feedback system as on the right-hand side in the figure proves that the mapping $(e_1, e_2, z_K, z_G) \mapsto (u, y)$ is bounded and the strong internal stability of the system on the left-hand side in the figure follows directly. □

The equivalence between well-posedness and null well-posedness does not hold in general. However, if both R_G and R_K have special constructions as in (11), then all three well-posedness notions coincide as follows:

Lemma 18. *Consider a well-posed feedback system $\{G, K\}$ with a kernel representation $R_K : L_{2e}^{m+p} \to L_{2e}^m$ of K which has a construction as in* (11). *Then there exists a kernel representation $R_G : L_{2e}^{m+p} \to L_{2e}^p$ of G such that the system $\{G, K\}$ with $R_{\{G,K\}}$ is null well posed.*

Proof. Use the notations R_K^{smth} and R_K^{cnst} in (13) and let R_G be the trivial kernel representation $R_G(u, y) := y - G(u)$. From Lemma 15 one can observe that $R_K^\#$ has the construction as in (11). Since $K(y) = R_K^\#(y, 0)$, K has the same construction, namely K has the form of

$$K(y) = K^{\text{smth}}(y) + K^{\text{cnst}} \cdot y$$

with a smoothing operator K^{smth} and a constant matrix K^{cnst}. Here R_K^{cnst} is a (well defined) kernel representation of K^{cnst}. Therefore, it is similar to the trivial kernel representation of K^{cnst}, i.e. $R_K^{\text{cnst}}(y, u) = M(K^{\text{cnst}} y - u)$ holds with a nonsingular constant matrix M. Hence, the kernel representation of the feedback system $z_{GK} = R_{\{G,K\}}(w)$ can be described by

$$\begin{pmatrix} z_K \\ z_G \end{pmatrix} = \underbrace{\begin{pmatrix} R_K^{\text{smth}}(w) \\ 0 \end{pmatrix}}_{\text{smoothing}} + \underbrace{\begin{pmatrix} M & 0 \\ 0 & \text{Id} \end{pmatrix} \begin{pmatrix} -\text{Id} & K^{\text{cnst}} \\ -G & \text{Id} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}}_{\text{non-smoothing}}.$$

It now follows from Lemma 15 and (7) that

$$\text{null well-posedness} \Leftrightarrow \begin{pmatrix} -\mathrm{Id} & K^{\mathrm{cnst}} \\ -G & \mathrm{Id} \end{pmatrix}^{-1} : \begin{matrix}\text{weakly}\\ \text{Lipschitz}\end{matrix} \Leftrightarrow \text{well-posedness.}$$

This relation proves the lemma. □

Lemma 18 only proves the null well-posedness of the feedback system with the trivial kernel representation R_G of G. The class of all kernel representations of G such that the feedback system is null well posed is given by $R_0 R_{\{G,K\}}$ with any well-defined kernel representation R_0 of a zero operator (Paice & van der Schaft, 1996) (where $R_{\{G,K\}}$ is invertible). Lemmas 16 and 18 imply that if either of the kernel representations R_G or R_K has a construction as in (11) then the three well-posedness notions are equivalent in some sense.

Furthermore, we investigate the property of differential coprimeness of the kernel representation $R_{\{G,K\}}$.

Lemma 19. *Consider an invertible globally Lipschitz kernel representation $R_\Sigma : L_{2e}^m \to L_{2e}^m$. Then R_Σ^{-1} is globally Lipschitz if and only if R_Σ is uniformly differentially coprime.*

Proof. First, we show the relation

$$\partial(R_\Sigma^{-1})_{(z_\Sigma)} = (\partial R_{\Sigma(R_\Sigma^{-1}(z_\Sigma))})^{-1}.$$

This equation is proved from the following calculation:

$$\begin{aligned} \tilde{z}_\Sigma &:= \partial R_{\Sigma(w)}(\tilde{w}) = R_\Sigma(w+\tilde{w}) - R_\Sigma(w) \\ &\Leftrightarrow z_\Sigma + \tilde{z}_\Sigma = R_\Sigma(w+\tilde{w}) \\ &\Leftrightarrow w + \tilde{w} = R_\Sigma^{-1}(z_\Sigma + \tilde{z}_\Sigma) \\ &\Leftrightarrow \tilde{w} = R_\Sigma^{-1}(z_\Sigma + \tilde{z}_\Sigma) - R_\Sigma^{-1}(z_\Sigma) = \partial(R_\Sigma^{-1})_{(z_\Sigma)}(\tilde{z}_\Sigma), \end{aligned}$$

where $z_\Sigma = R_\Sigma(w)$. Then we obtain the Bezout identity

$$\partial R_{\Sigma(w)} X_{(w)} = \mathrm{Id}, \quad X_{(w)} := (\partial R_{\Sigma(w)})^{-1} = \partial(R_\Sigma^{-1})_{(R_\Sigma(w))}.$$

Hence, the operator R_Σ^{-1} is globally Lipschitz if and only if $X_{(w)}$ has a finite gain uniformly over $w \in L_{2e}$, i.e. R_Σ is uniformly differentially coprime. □

Lemma 19 is a generalized version of Remark 10 allowing global Lipschitz continuity, and it also implies the double Bezout identity in the linear case. Using Lemma 19, we can establish the property of a differential coprime kernel representation $R_{\{G,K\}}$.

Theorem 20. *Consider a feedback system $\{G,K\}$ with kernel representations R_G and R_K. Suppose $R_{\{G,K\}}$ is uniformly differentially coprime, and either R_G or R_K has a construction as in (11). Then the feedback system $\{G,K\}$ with the kernel representation $R_{\{G,K\}}$ is strongly internally stable.*

Proof. The proof follows straightforwardly from Lemma 16, Lemma 19 and Theorem 17. □

Uniform differential coprimeness of $R_{\{G,K\}}$ implies strong internal stability of the feedback system $\{G,K\}$ provided either R_G or R_K has a construction as in (11) (see Theorem 20). This property will be used in the parametrization of *all* internally stabilizing plant and controller pairs in the next subsection.

3.2. Parametrization of all internally stabilizing plant and controller pairs

This subsection discusses the parametrization of all internally stabilizing controllers, in contrast to Theorem 13 (Theorem 14) which provides the parametrization of all *strongly* internally stabilizing controllers (pairs). Before stating the results, we give an important remark on strong detectability. While a strongly detectable kernel representation R_Σ is globally Lipschitz by definition, the converse will also hold if it is completely controllable.

Lemma 21. *Consider a kernel representation $R_\Sigma^{x^0} : L_{2e}^{m+p} \to L_{2e}^p$. Suppose it is completely controllable. Then the following properties are equivalent.*

(i) *$R_\Sigma^{x^0}$ is strongly detectable.*
(ii) *$R_\Sigma^{x^0}$ is globally Lipschitz.*
(iii) *The following inequality holds for $\forall u, v \in L_2^{m+p}$.*

$$\|R_\Sigma^0(u+v) - R_\Sigma^0(u)\| \leqslant \gamma \|v\|. \tag{17}$$

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are obvious. Hence, we only show (iii) ⇒ (i). From Remark 3, for $\forall x^1, x^2 \in \mathscr{X}^0$, there exists $T_1, T_2 > 0$ and $u_1, u_2 \in L_2^m$ such that

$$\delta_{T_1} R_\Sigma^{x^1}(v_1)(t) = R_\Sigma^0(\delta_{T_1}(v_1) + \tau_{T_1}(u_1))(t) \quad \forall t \geqslant T_1, \tag{18}$$

$$\delta_{T_2} R_\Sigma^{x^2}(v_2)(t) = R_\Sigma^0(\delta_{T_2}(v_2) + \tau_{T_2}(u_2))(t) \quad \forall t \geqslant T_2 \tag{19}$$

holds for $\forall v_1, v_2 \in L_{2e}^m$. One can observe the relation $\delta_T R_\Sigma^0(v) = R_\Sigma^0(\delta_T v)$ holds for $\forall T \geqslant 0$ and $\forall v \in L_{2e}^m$ because any operator is supposed to have its state-space realization in the form (1) and the origin is its equilibrium. Suppose now $T_1 \geqslant T_2$ without loss of generality. Then, applying $\delta_{(T_1 - T_2)}$ to (19) reduces to

$$\begin{aligned} \delta_{T_1} R_\Sigma^{x^2}(v_2)(t) &= \delta_{(T_1-T_2)} \delta_{T_2} R_\Sigma^{x^2}(v_2)(t) \\ &= \delta_{(T_1-T_2)} R_\Sigma^0(\delta_{T_2}(v_2) + \tau_{T_2}(u_2))(t) \\ &= R_\Sigma^0(\delta_{(T_1-T_2)}\delta_{T_2}(v_2) + \delta_{(T_1-T_2)}\tau_{T_2}(u_2))(t) \\ &= R_\Sigma^0(\delta_{T_1}(v_2) + \tau_{T_1}\delta_{(T_1-T_2)}(u_2))(t) \quad \forall t \geqslant T_1. \end{aligned} \tag{20}$$

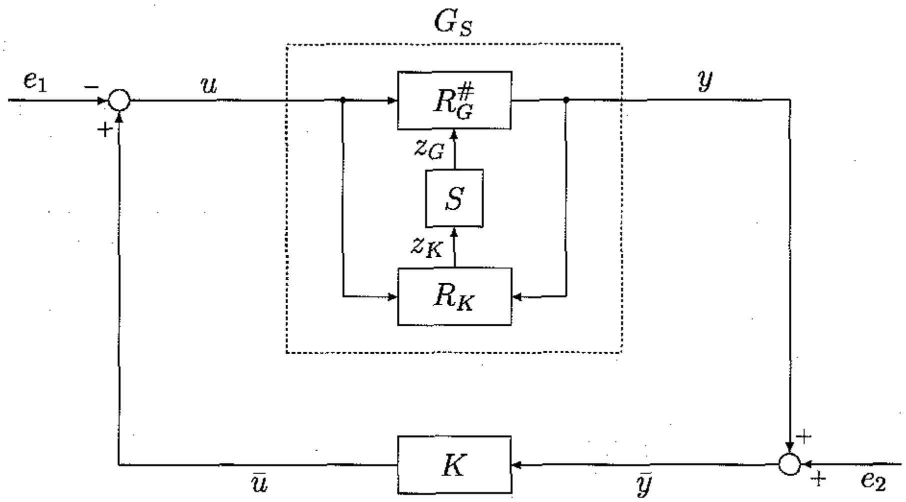

Fig. 6. Configuration of the parametrization $\{G_S, K\}$.

Eqs. (17), (18) and (20) prove

$$\begin{aligned}&\|R_\Sigma^{x^1}(v_1) - R_\Sigma^{x^2}(v_2)\|\\ &\quad = \|\delta_{T_1} R_\Sigma^{x^1}(v_1) - \delta_{T_1} R_\Sigma^{x^2}(v_2)\|\\ &\quad \leqslant \|R_\Sigma^0(\delta_{T_1}(v_1) + \tau_{T_1}(u_1))(t)\\ &\quad\quad - R_\Sigma^0(\delta_{T_1}(v_2) + \tau_{T_1}\delta_{(T_1-T_2)}(u_2))(t)\|_{[T_1,\infty)}\\ &\quad \leqslant \gamma\|\delta_{T_1}(v_1 - v_2) + \tau_{T_1}(u_1 - \delta_{(T_1-T_2)}u_2)\|\\ &\quad = \gamma\|v_1 - v_2\| + \gamma\|\tau_{T_1}(u_1 - \delta_{(T_1-T_2)}u_2)\|\\ &\quad =: \gamma\|v_1 - v_2\| + \phi(x^1, x^2).\end{aligned}$$

In this equation $\phi(x^1, x^1) = 0$ holds for $\forall x^1 \in \mathscr{X}^0$ because if $x^1 = x^2$ then we can always choose $T_1 = T_2$ and $u_1 = u_2$. This completes the proof. □

Lemma 21 suggests that the global Lipschitz continuity of a kernel representation is an important property for the parametrization. The parametrization of *all* internally stabilizing controllers is stated now.

Theorem 22. *Consider a null internally stable feedback system $\{G, K\}$ with kernel representations $R_G : L_{2e}^{m+p} \to L_{2e}^p$ and $R_K : L_{2e}^{m+p} \to L_{2e}^m$. Suppose R_K is uniformly differentially coprime with a weakly Lipschitz operator $X_{(w)}$ in* (5), *completely controllable and has a construction as in* (11). *Then the parametrization of all weakly Lipschitz plants which are internally stabilized by K is given by G_S in* (10) *with any weakly Lipschitz bounded operator S.*

Differential coprimeness plays an important role to connect the different definitions of internal stability. The parametrization stated in Theorem 22 is a generalized version of that based on left factorizations given in Linard et al. (1999), but the assumptions made here are much weaker than those in the former result. The configuration of the parametrization is shown in Fig. 6. K is a stabilizing controller for G and S is a bounded free parameter.

Proof of Theorem 22. From Theorem 17, the null internal stability implies the strong internal stability, and the parametrization gives all plants and their kernel representations which are *strongly* internally stabilized by K with R_K from Theorem 13. Hence, what we have to show is the fact that any plant which is internally stabilized by K is contained in this parametrization.

Suppose a weakly Lipschitz plant G_1 such that the feedback system $\{G_1, K\}$ is internally stable is given. Then Lemma 18 implies that there exists a kernel representation R_{G_1} of G_1 such that the feedback system $\{G_1, K\}$ with the kernel representation $R_{\{G_1,K\}}$ is null well-posed. Further one can observe from Lemma 16 that the construction of R_K as in (11) implies that the system $\{G_1, K\}$ with $R_{\{G_1,K\}}$ is strongly well-posed. Hence, the operator G_1 can be represented[1] by G_S in Fig. 6 with a weakly Lipschitz operator S. We will show S is bounded. The operator $E_{\{G_S,K\}}$ is weakly Lipschitz and bounded because of the internal stability of $\{G_S, K\}$. The uniform differential coprimeness of R_K implies the existence of a weakly Lipschitz operator $X_{(w)}$ which has a finite gain uniformly over w such that

$$\partial R_{K(w)} X_{(w)} = \mathrm{Id} \quad (21)$$

[1] In the proof of Theorem 13, it was shown that any plant G_1 with R_{G_1} so that the feedback system $\{G_1, K\}$ with a given R_K is strongly well posed is parametrized by (10) under the assumption that the nominal feedback system $\{G, K\}$ with $R_{\{G,K\}}$ is strongly well posed. (See Fujimoto & Sugie, 2000; Paice & van der Schaft, 1996; Fujimoto, 2000 for more detail.)

holds. Let the external input be chosen as $(e_1, e_2) = X_{(w)}(v)$ for some v. We now prove that the loop signal (u, y) is uniquely determined for every $v \in L_{2e}$, and that the mapping of $v \mapsto (u, y)$ is weakly Lipschitz. Using the notations R_K^{smth} and R_K^{cnst} in (13), equation (21) reduces to

$$R_K^{\text{cnst}} X_{(w)} = \text{Id} - \partial R_{K(w)}^{\text{smth}} X_{(w)}. \tag{22}$$

Further, from Lemma 18, we can obtain a kernel representation

$$z_{G_S K} = R_{E_{\{G_S,K\}}}(e_{12}, w) = \begin{pmatrix} R_K(w + e_{12}) \\ R_{G_S}(w) \end{pmatrix}$$

such that $\{G_S, K\}$ is null well-posed. Substituting $e_{12} = X_{(w)}(v)$ and (22),

$$\begin{aligned} z_{G_S K} &= \begin{pmatrix} R_K^{\text{smth}}(w + X_{(w)}(v)) + R_K^{\text{cnst}} w + R_K^{\text{cnst}} X_{(w)}(v) \\ R_{G_S}(w) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} R_K^{\text{smth}}(w + X_{(w)}(v)) - \partial R_{K(w)}^{\text{smth}} X_{(w)}(v) \\ 0 \end{pmatrix}}_{\text{smoothing}} \\ &\quad + \underbrace{\begin{pmatrix} R_K^{\text{cnst}} \\ R_{G_S} \end{pmatrix}(w) + \begin{pmatrix} v \\ 0 \end{pmatrix}}_{\text{non-smoothing}}. \end{aligned}$$

Thus, it follows from Lemma 15 and the null well-posedness of $\{G_S, K\}$ that the mapping of $v \mapsto (u, y)$ under $z_{G_S K} = 0$ exists and is weakly Lipschitz.

On the other hand, the mapping $e_{12} \mapsto z_K$ can be described by

$$z_K = R_K(w) = -(R_K(\bar{w}) - R_K(w)) = -\partial R_{K(w)}(e_{12}).$$

because $R_K(\bar{w}) = 0$. Hence, from (21) the mapping $v \mapsto z_K$ is obtained by

$$z_K = -\partial R_{K(w)} X_{(w)}(v) = -v. \tag{23}$$

This equation holds for all $v, z_K \in L_{2e}$ from the fact that the mapping $v \mapsto w$ is weakly Lipschitz. Connecting operators (23): $z_K \mapsto v$, $X_{(w)}: v \mapsto e_{12}$, $E_{\{G_S,K\}}: e_{12} \mapsto w$ and $R_G: w \mapsto z_G$, $S: z_K \mapsto z_G$ is described by

$$z_G = S(z_K) = R_G E_{\{G_S,K\}} X_{(w)}(-z_K).$$

It now follows from the (uniform) boundedness of R_G, $E_{\{G_S,K\}}$ and $X_{(w)}$ that the operator $S: z_K \mapsto z_G$ is bounded. This completes the proof. □

An alternative sufficient condition for the complete controllability assumption in Theorem 22 is given in the following lemma.

Lemma 23. *Consider a causal operator $\Sigma^{x^0}: L_{2e}^m \to L_{2e}^p$ which has a state-space realization as in* (1) *with a kernel representation $R_\Sigma^{x^0}: L_{2e}^m \times L_{2e}^p \to L_{2e}^p$. Suppose $(R_\Sigma^{\#})^{x^0}(u, 0)$ has the same state-space realization as that of Σ^{x^0}. Then R_Σ is completely controllable if Σ is completely controllable.*

Remark 7, Lemmas 15 and 23 suggest that if the controller K is globally Lipschitz and completely controllable then the trivial kernel representation

$$R_K(y, u) = u - K(y) \tag{24}$$

is a good candidate kernel representation which satisfies the assumptions in Theorem 22, because it is uniformly differentially coprime, completely controllable and has a construction as in (11).

Next we give the parametrization of *all* internally stabilizing plant and controller pairs which allows both plant and controller to vary.

Theorem 24. *Consider a feedback system $\{G, K\}$ with kernel representations $R_G: L_{2e}^{m+p} \to L_{2e}^p$ and $R_K: L_{2e}^{m+p} \to L_{2e}^m$. Suppose $R_{\{G,K\}}$ is uniformly differentially coprime and completely controllable, and it has a construction as in* (11). *Then the parametrization of all internally stabilizing plant and controller pairs G_S and K_Q is given by* (12) *with all internally stable feedback systems $\{S, Q\}$ with the kernel representation $R_{\{S,Q\}}$.*

This theorem can be proved in a manner similar to the proof of Theorem 22, and the proof is omitted for the reason of space, see Fujimoto (2000) for a complete proof. In addition to the assumptions in Theorem 14, the complete controllability of the state-space realizations is assumed in Theorem 24. (Note that the differential coprimeness of $R_{\{G,K\}}$ is equivalent to the global Lipschitz continuity of both $R_{\{G,K\}}$ and $R_{\{G,K\}}^{-1}$ from Lemma 19.) This additional assumption derives the parametrization of *all* internally stabilizing plant and controller pairs, whereas Theorem 14 gives all strongly/null internally stabilizing pairs. Theorem 22 will be applied to closed-loop identification in the following section by designing a controller K such that the differential coprimeness assumption is satisfied.

4. Application to closed-loop identification

This section demonstrates how the parametrization given in the previous section is utilized for closed-loop identification. Consider the configuration as shown in Fig. 7. G denotes the plant to be identified. G_0 is a nominal model obtained from a prior knowledge. K is a stabilizing controller for both G and G_0 with R_{G_0} and R_K their kernel representations, respectively. We use the (shorthand) notations $r_{12} := (r_1, r_2)$ and r_3 for the known reference signals, $d_{12} := (d_1, d_2)$ for the unknown disturbance signals and $w := (u, y)$ for the measurable loop signals. Conventionally, r_3 would be taken as zero, and $R_K^{\#}$ would be replaced by K. It suits us to consider the

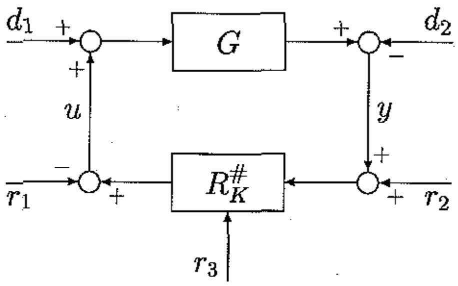

Fig. 7. Configuration of closed-loop identification.

more general arrangement. We make some assumptions pertinent to the application of Theorem 22.

Assumptions

(A1) R_K is completely controllable.
(A2) R_K is differentially coprime.
(A3) R_K has a construction as in (11).
(A4) The system $\{G_0, K\}$ with R_{G_0} and R_K is null internally stable.
(A5) The system $\{G, K\}$ is internally stable.

Assumptions (A1)–(A3) are necessary for the equivalence between internal stabilities investigated in Section 3.1 and we have to design a controller K satisfying those assumptions. Assumptions (A4) and (A5) are the internal stability which is necessary for the parametrization. We assume different internal stabilities of the feedback system with the nominal plant in Assumption (A4) and the feedback system with the actual plant to be identified in Assumption (A5). The *null* internal stability in Assumption (A4) is necessary to utilize the parametrization in Theorem 22 and the internal stability in Assumption (A5) is required by closed-loop identification.

From Remark 7, Lemmas 15, 21 and 23, we can use another set of assumptions instead of Assumptions (A1)–(A4).

(A1′) K is completely controllable.
(A2′) K is globally Lipschitz.
(A3′) K has a construction as in (11).
(A4′) The system $\{G_0, K\}$ with R_{G_0} and R_K in (24) is null internally stable.

Suppose Assumptions (A1)–(A5) (or (A1′)–(A4′), (A5)) are satisfied. Then G can be depicted as Fig. 8 with a bounded weakly Lipschitz operator S by Theorem 22, where R_K and $R_K^{\#}$ have the same initial condition. The objective of this section is to identify the operator S using the reference input signal r_1, r_2 and r_3 and the measurable loop signal w. It should be noticed that from Theorem 22 all internally stable $\{G, K\}$ are always *strongly* internally stable with R_K and some kernel representation R_G of G. Hence, the mapping $(r_1, r_2, r_3) \mapsto (u, y)$ is bounded and we can use all signals r_1, r_2 and r_3 as reference signals for identification.

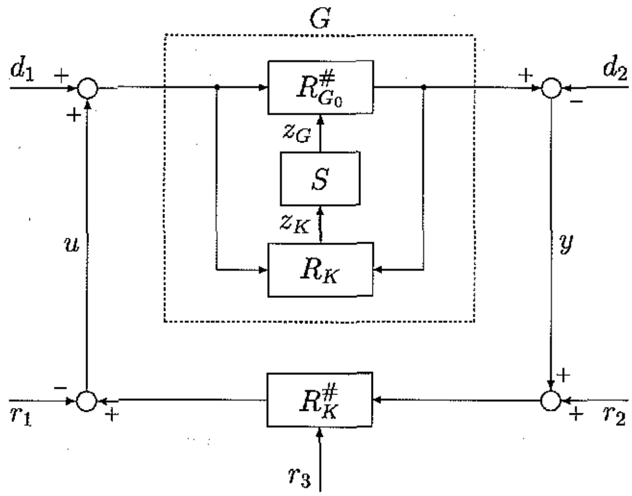

Fig. 8. Identification of the Youla parameter S.

It can be readily observed that the input and output signals z_K and z_G of the operator S are described by

$$z_G = R_{G_0}(w + d_{12}) = R_{G_0}(w) + \partial R_{G_0(w)}(d_{12}),$$

$$z_K = R_K(w + d_{12}) = R_K(w) + \partial R_{K(w)}(d_{12}).$$

The signals z_K and z_G are input and output signals of the operator S but they are not measurable because of the unknown noise d_{12}. Define the known signals $\bar{z}_K := R_K(w)$ and $\bar{z}_G := R_{G_0}(w)$, we can rewrite the configuration of identification in Fig. 8 into an equivalent configuration depicted in Fig. 9. Note that the open-loop input $\bar{z}_K$ is described by

$$\bar{z}_K = r_3 - \partial R_{K(w)}(r_{12}). \tag{25}$$

Therefore, the closed-loop identification problem can be described by an open-loop identification one depicted in Fig. 10.

First, consider the case that we can choose r_3 arbitrarily. Suppose we choose $r_{12} = 0$. Then we can choose an arbitrary $\bar{z}_K \in L_{2e}$ by letting r_3 equal to the desired open-loop input (the desired value of $\bar{z}_K$).

Next, consider the case that we cannot choose r_3 arbitrarily, e.g. a fixed controller is implemented and we cannot inject any nonzero external signal r_3 into the operator K. This situation is the typical setting used in the preliminary results (e.g. Dasgupta & Anderson, 1996; Linard et al., 1999). Here we need another theorem to ensure the fact that we can still choose the open-loop input signal $\bar{z}_K$ arbitrarily.

Theorem 25. *Consider the closed-loop identification as in Fig.* 8, *and suppose Assumptions* (A1)–(A5) (*or

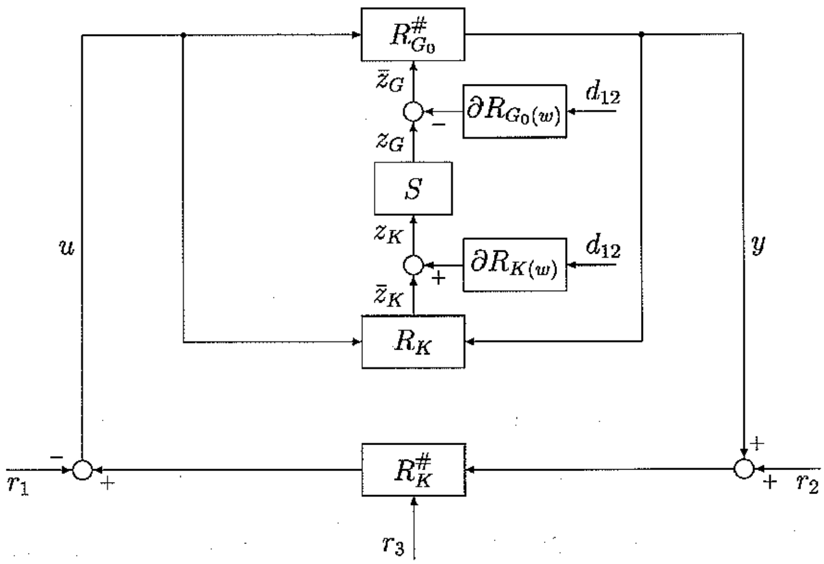

Fig. 9. Conversion to nonstandard open-loop identification problem.

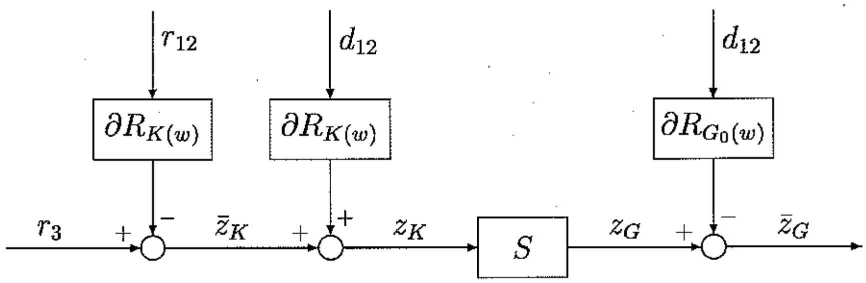

Fig. 10. Non-standard open-loop identification problem.

(A1′)–(A4′), (A5)) *hold. Then,*

(i) *we can choose an arbitrary open-loop input* $\bar{z}_K \in L_{2e}$ *by choosing* $r_1 \in L_{2e}$:

$$r_1 = (\partial R_{K(w)}(r_{12}))^{\#}(r_2, r_3 - \bar{z}_K) \tag{26}$$

when $r_2, r_3, d_1, d_2 \in L_{2e}$ *are prescribed,*

(ii) *we can choose an arbitrary open-loop input* $\bar{z}_K \in L_2$ *by choosing* $r_{12} \in L_2$:

$$r_{12} = X_{(w)}(r_3 - \bar{z}_K) \tag{27}$$

when $r_3, d_1, d_2 \in L_2$ *are prescribed. Here* $X_{(w)}$ *is a uniformly bounded operator satisfying* (21).

Proof. (i) and (ii) follow straightforwardly from the input signals (26) and (27), respectively, if the feedback systems with inputs (26) and (27), which are feedbacks indeed, are strongly well-posed. Because (25) reduces to

$$r_3 - \bar{z}_K = \partial R_{K(w)}(r_{12})$$

and both (26) and (27) assign $\bar{z}_K$ the desired open-loop input. The strong well-posedness of the system is proved based on Lemma 15 and the techniques in the proof of Theorem 22. The complete proof is omitted for the reason of space (Fujimoto (2000) contains a complete proof). □

Theorem 25 implies that even if the disturbances d_1 and d_2 are not zero, we can choose the open-loop input signal $\bar{z}_K$ arbitrarily. That is, we obtain the open-loop identification problem shown in Fig. 10. However, this configuration is not a standard open-loop identification problem because both the input and output signals are contaminated by noise and the unknown additive signals have correlation with the reference signal $\bar{z}_K$. To overcome such problems, we assume a high SNR (signal-to-noise ratio) as in Dasgupta and Anderson (1996) and Linard et al. (1999). More precisely, we make the following assumptions.

Assumptions

(A6) $\|d_{12}\| \ll \|w\|$.

(A7) $\|d_{12}\| \ll \dfrac{1}{\|R_K\|_L}\|\bar{z}_K\|$.

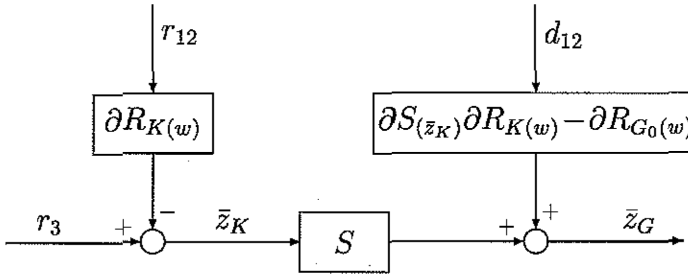

Fig. 11. Conversion to standard open-loop identification problem.

Assumption (A6) allows us to regard the operators $\partial R_{G_0(w)}(\cdot)$ and $\partial R_{K(w)}(\cdot)$ as being modellable by time-varying linear systems when they are acting on d_{12}. These time-varying systems result from the linearization around the trajectory produced by the input w. Furthermore, employing the differential operator of S we obtain

$$\bar{z}_G = S(\bar{z}_K) + (\partial S_{(\bar{z}_K)}\ \partial R_{K(w)} - \partial R_{G_0(w)})(d_{12}).$$

The closed-loop identification problem has been transformed into another open-loop identification problem as shown in Fig. 11. Assumption (A7) also allows us to regard the operator $\partial S_{(\bar{z}_K)}$ as a time-varying linear system around the trajectory produced by the input $\bar{z}_K$. Therefore, we now obtain a standard open-loop identification problem. Here the operator $(\partial S_{(\bar{z}_K)} \partial R_{K(w)} - \partial R_{G_0(w)})$ is a time-varying linear system and the unknown additive signal has no correlation with the reference signal $\bar{z}_K$. Details about the nonlinear identification procedure are given in Dasgupta and Anderson (1996).

If the reference signal r_3 is available, then the easiest way for closed-loop identification is to set $r_{12} = 0$ and to set r_3 as the desired open-loop input $\bar{z}_K$, provided Assumption (A6) holds. If Assumption (A6) fails, then we should enlarge the loop signal w by choosing appropriate $r_{12} \neq 0$.

Finally, one can deduce the open-loop identification problem by assuming conditions (A1)–(A7). In general, however, it is not so easy to design a controller K satisfying these assumptions. The simplest way to fulfill them is to employ a linear (or globally Lipschitz) controller K for a given nonlinear plant G, because Assumptions (A1)–(A3) are automatically satisfied if K is linear (or globally Lipschitz).

5. Conclusion

This paper has discussed the parametrization of all nonlinear plants which are internally stabilized by a given controller and its application to closed-loop identification. Differentially coprime kernel representations have been introduced and used to clarify the equivalence between three different notions of well-posedness and internal stability definitions available in the literature. The authors believe that the results in the paper provide new insights to the stability of nonlinear feedback systems and extend the applicability of closed-loop identification to a wider class of nonlinear systems.

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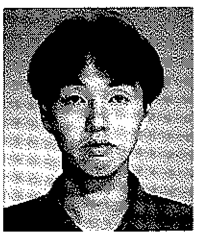

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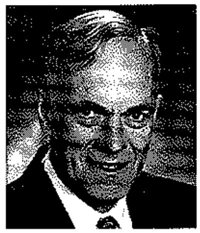

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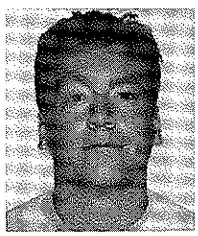

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