Optimal linear control systems with input derivative constraints

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Synopsis

A very significant result in modern control theory is that, for a linear, finite-dimensional, dynamical system, the state feedback law derived from a quadratic-loss-function minimisation problem is linear. The paper applies the results of this optimal control theory to a class of problems in which the feedback law is realised by a linear dynamical system. The quadratic loss function of interest in this case consists of terms involving time derivatives of the input vector as well as the usual terms involving the input and state vectors. Optimal control problems of this type may arise, for example, when the input force or energy is to be included in the cost terms of the performance index. An advantage of the controllers resulting from the optimisation procedure is that they are dynamic, and thus possess finite bandwidth; accordingly they can be used when there is a limitation to the bandwidth of a communication channel over which feedback signals are transmitted.

1 Introduction

The problem considered in the paper is the derivation of a state feedback law for a linear, finite-dimensional system to minimise a quadratic loss function consisting of terms involving time derivatives of the input vector as well as the usual terms involving the input and state vectors.

A number of practical situations can be envisaged where such a control law could be required. As a light-hearted example, we could conceive of some mechanical object such as a slot car, together with its controls, as a system whose input is a potentiometer setting, representing the velocity of the car. It could be postulated that step changes in the velocity of the car are almost possible because of its very low weight; however the motor of the slot car will burn out if the electric power which it is converting into mechanical power exceeds a certain value. Thus it would be desired to limit the value of $uu$, representing in a rough sense the mechanical power of the driving motor, even though the dynamical equations describing the car and its controls do not involve $u$ because of the car's extremely low weight.

To achieve this limitation on $uu$, a loss function could be assumed which heavily weighted the term, in addition to involving quadratic terms in $u$ and the state variable $x$. (Presumably position here.)

It will actually be seen in the paper that to yield a well defined problem it is necessary to include in the loss function a term involving $u^t$ if one involving $uu$ is included. This can however have a very small weight relative to that of $uu$ to achieve the sort of performance desired.

For linear, finite-dimensional, completely controllable systems, Kalman has shown that in the case when the quadratic loss function consists of the input- and state-vector terms only, and, provided that certain conditions are specified to make the problem well defined, the minimisation problem results in a linear feedback law. In order to apply this theory to the problem of interest, the plant is first augmented with an integrator in cascade with each input line. The performance index for the original system may then be rewritten, so that it consists of terms involving only the input and state vectors of the augmented system. (The inputs to the original system will actually be some of the states of the augmented system.) Applying the optimal control theory to the augmented system in order to minimise the specified performance index of the original system results in a constant linear feedback law for the augmented system, which may be interpreted as a linear dynamical-system feedback controller for the original system.

Section 2 reviews the appropriate optimal control theory, and Section 3 applies it to the augmented system in order to obtain an optimal feedback law for the original system. The feedback controller determined for the original system itself possesses states, and the initial values of these states for which the performance index is minimised are also calculated in Section 3. The concluding Section considers the extension of the results to the case where the system states are not directly measurable, but rather only linear combinations of them corrupted by noise are available.

2 Review of optimal control problem

The plants under consideration are linear, finite-dimensional, time-invariant, dynamical systems. It is assumed that their behaviour may be represented by the differential system

\[ x_1 = F_1 x_1 + G_1 u_1 \]

where $x_1$ is an $n$-vector, the state, and for the present discussion is considered also as the output vector. The input $u_1$ is an $m$-vector, while the matrices $F_1$ and $G_1$ are constant and of dimension $n \times n$ and $n \times m$, respectively. It will also be assumed that

(a) the pair $(F_1, G_1)$ is completely controllable, or equivalently,

\[ \text{rank} (G_1, F_1G_1, F_1G_1^2, \ldots, F_1^{n-1}G_1) = n \]

(b) the matrix $R_1$ is positive-definite-symmetric.

(c) the matrix $Q_1 - S_1 R_1^{-1} S_1$ is nonnegative-definite-symmetric.

Also, in order to guarantee stability of the closed-loop system obtained when minimising eqn. 3, the following is needed:

(d) the pair $(F_1, H_1)$, where $H_1$ is any solution of $H_1^* H_1 = Q_1 - S_1 R_1^{-1} S_1$, is completely observable, or equivalently

\[ \text{rank} (H_1^*, F_1 H_1, \ldots, (F_1^*)^{n-1} H_1^*) = n \]

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The control law used in the solution of the optimal regulator problem is now stated.

Control law 1: Consider the plant of eqn. 1, and assume that (a)-(d) hold. Then there exists an optimal control \( v^* \) which minimises eqn. 3, and is given by

\[
u^* = -R^{-1}(S_t + G_t P_t)x_t.
\]

where \( P_t \) is defined later. Further, the value of the associated performance index is

\[
V^*(x(0)) = x(0)^T P_t x(0).
\]

The matrix \( P_t \) is positive-definite and is defined as

\[
P_t = \lim_{t \to \infty} \Pi_t(t)
\]

where \( \Pi_t(t) \) is the solution of the Riccati differential equation

\[
\Pi_t = \Pi_t F_t + F_t^T \Pi_t - \Pi_t G_t R_t^{-1} G_t^T \Pi_t + (Q_t - S_t R_t^{-1} S_t)
\]

with the initial condition

\[
\Pi_t(0) = 0
\]

With the assumptions listed, both \( \Pi_t \) and \( P_t \) are always well defined.

3 Optimal control with input derivative constraints

The plants under consideration are as in the previous Section. Introducing a notation change, the state-space equation is

\[
\dot{x} = F x + G u
\]

The controllability condition is therefore written as

(i) The pair \((F, G)\) is completely controllable, or equivalently,

\[
\text{rank} \begin{bmatrix} G & G^2 & \cdots & G^{n-1} \end{bmatrix} = n
\]

We define the optimal regulator problem with input derivative constraints as the selection of a control \( u \) which returns a nonzero initial state \( x(0) \) to the zero state, and which minimises the performance index

\[
V(x(0), u) = \int_0^T \left[ x^T Q x + 2u^T S x + u^T R u + 2\dot{u}^T T x + 2\ddot{u}^T Z \dot{x} \right] dt
\]

This performance index consists of quadratic terms involving not only the state vector \( x \) and the input vector \( u \), but also the derivative of the input vector \( u \). Restrictions on \( Q, S \) etc. will be explained subsequently.

In order to consider the performance index of eqn. 11 using the optimal control theory reviewed in the previous Section, the original system of eqn. 9 is augmented with an integrator in cascade with each input line, as in Fig. 1. The interpretations as conditions on the plant of eqn. 9 with performance index of eqn. 11 of assumptions (a)-(d) applied to the augmented system are as follows:

(ii) The matrix \( R_2 \) (i.e. \( Z \)) is positive-definite-symmetric.

(iii) The matrix \( [Q_2 - S_2 R_2^{-1} S_2]^T \), or equivalently,

\[
[Q \cdot S^T \cdot S \cdot R^T] Z^{-1} W T
\]

is nonnegative-definite-symmetric.

(iv) The pair \((F_2, H)\) is completely observable, for any \( H \), such that

\[
H^* H = Q_2 - S_2 R_2^{-1} S_2
\]

or equivalently,

\[
H^* H = [Q \cdot S^T \cdot R^T] Z^{-1} W T
\]

Note: The observability condition may be written as

\[
\text{rank} \begin{bmatrix} H' & F_2 H' & \cdots & F_2^{n-1} H' \end{bmatrix} = n + m
\]

or equivalently

\[
\text{rank} \begin{bmatrix} H' & F_2 H' & \cdots & F_2^{n-1} H' \end{bmatrix} = n + m
\]

Certainly, if from eqns. 20 and 21 \( H^* H \) is positive-definite-symmetric, the rank of \( H' \) (\( n + m \)), and (iv) is satisfied.
The optimum choice for $u^*$ and the corresponding performance index is

$$V^*(x(0)) = x(0)P_{21}x(0)$$

where $P_{21}$ is the limit as $t 	o -\infty$ of the solution $P(t)$ of the Riccati differential equation (eqn. 26). The control law is given by

$$u^* = -R^{-1}(S_2 + G_2^TP)x_2$$

where $P$ is the limit as $t \to -\infty$ of the solution $P(t)$. These results may now be interpreted in terms of the original system of eqn. 9. Eqn. 25, the linear control law for the augmented system, may be written as

$$u^* = -Z^{-1}(T + P_{22})u^* - Z^{-1}(W + P_{21})x$$

where $P_{21}, P_{22}$ are $m \times n$ and $n \times n$ matrices obtained by partitioning $P$ as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

We observe that, if $u^*(0)$ is specified, eqn. 28 may be interpreted as a linear dynamical feedback law for the original system to minimise the performance index (eqns. 15 or 11). The minimum performance index associated with the optimum control $u^*$ is given from eqn. 28 as

$$V^*(x(0), u^*(0)) = x(0)P_{11}x(0) + 2[u^*(0)]P_{21}x(0) + [u^*(0)]P_{22}u^*(0)$$

It may or may not be possible to set $u^*(0)$ in practice; in some cases the value might be totally unknown, in others an initial value might automatically be taken as $u^*(0) = 0$. But in case $u^*(0)$ can be arbitrarily chosen, the choice which minimises the performance index (eqn. 30) with respect to all possible $u^*(0)$ is obtained from a rewriting of eqn. 30

$$V^*(x(t), u^*(t)) = x(t)P_{11}x(t) + 2[u^*(t)]P_{21}x(t) + [u^*(t)]P_{22}u^*(t)$$

The optimum choice for $u^*$ is clearly

$$u^*(t) = -P_{22}^{-1}P_{21}x(t)$$

and the corresponding performance index is

$$V^*(x(t), u^*(t)) = x(t)P_{11}x(t)$$

which is certainly less than that obtained for $u^*(t) = 0$:

$$V^*(x(t), u(t) = x(t)P_{11}x(t)$$

If 2 shows the optimal augmented system with its linear feedback law. By including initial conditions on the integrators, this schematic may be viewed as the original system having a dynamical controller minimising the performance index of eqn. 11 (see also Fig. 3). We conclude that the controller may be realised by a linear time-invariant, dynamical system, whose input is the output of the system given from eqn. 9, i.e., having state-space equations given in eqn. 28, and thus having a transfer function

$$C(s) = \left\{ sI + Z^{-1}(T + P_{22})^{-1}Z^{-1}(W + P_{21}) \right\}$$

\[ \text{Fig. 2} \quad \text{Optimal closed-loop augmented system} \]

These results may be summarised as Control law 2: For the plant of eqn. 9, there exists an optimal control $u^*$ which minimises the performance index (eqn. 11) containing input derivative constraints, provided conditions (i)-(iv) are satisfied. The control $u^*$ is given from the state equation

$$u^* = -Z^{-1}(T + P_{22})u^* - Z^{-1}(W + P_{21})x$$

with initial conditions considered later. The minimum performance index associated with the optimal control $u^*$ is

$$V^*(x(0), u^*(0)) = x(0)P_{11}x(0) + 2[u^*(0)]P_{21}x(0) + [u^*(0)]P_{22}u^*(0)$$

A value of $u^*(0)$ may be selected which minimises the index (eqn. 25) (see eqns. 31, 32 and 33). That is, for

$$u^*(0) = -P_{22}^{-1}P_{21}x(t)$$

we have

$$V^*(x(t), u^*(0)) = x(t)P_{11}x(t)$$

whereas that obtained for $u^*(0) = 0$ is

$$V^*(x(t), 0) = x(t)P_{11}x(t)$$

For these equations, $P_{11}, P_{21}$ and $P_{22}$ are $n \times n$ and $n \times n$ matrices obtained by partitioning $P$ as in eqn. 29, where $P$ is the limit as $t \to -\infty$ of the solution $P(t)$ of the Riccati differential equation (eqn. 36) with initial condition

$$P(0) = \begin{bmatrix} F & 0 \\ G & 0 \end{bmatrix}$$

$$0 = \begin{bmatrix} Q & 0 \\ S & R \end{bmatrix} \begin{bmatrix} W & 0 \\ T & 0 \end{bmatrix}$$

\[ \text{Fig. 3} \quad \text{Optimal closed-loop system (input derivative constraints)} \]

4 Concluding remarks

Many of the extensions of optimal control theory are also applicable to the problem dealt with in the paper. One such extension worthy of note is that when the plant outputs, being linear transformations of the states, are only available with additive Gaussian noise, and there is also Gaussian noise at the inputs, the same feedback law given

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in control law I may be used, together with a state estimator, to minimise the expected value of the performance index. For the augmented system, the best estimate of the states (minimum-variance estimate) is available from a state estimator constructed for the original system together with the inputs to the original system; so to minimise the expected value of the performance index (eqn. 11) for the system of eqn. 9 is a straightforward application of available theory together with the results of the paper.

As is pointed out in Reference 1, the optimal control law resulting for the augmented system will be stable; since the plant with its dynamic controller constitutes a rearrangement of the augmented system with its constant controller, stability of the closed-loop system is assured (irrespective of the stability of the plant itself).

It is of interest to note than an implication of the paper is that feedback controllers with dynamics may be optimal for some performance index. Not all dynamical feedback controllers will be optimal, of course, and it would be interesting to characterise the optimal ones. The optimal controllers in essence possess finite bandwidth, and accordingly may be used appropriately when signalling of the optimal control must take place over a channel of restricted bandwidth.

5 References

1 Kalman, R. E.: "When is a linear control system optimal?", Trans., ASME, 1964, [D], 86, pp. 1-10