Optimal linear control systems with input derivative constraints

J. B. Moore, Ph.D., and Prof. B. D. O. Anderson, Ph.D.

Synopsis

A very significant result in modern control theory is that, for a linear, finite-dimensional, dynamical system, the state feedback law derived from a quadratic-loss-function minimisation problem is linear. The paper applies the results of this optimal control theory to a class of problems in which the feedback law is realised by a linear dynamical system. The quadratic loss function of interest in this case consists of terms involving time derivatives of the input vector as well as the usual terms involving the input and state vectors. Optimal control problems of this type may arise, for example, when the input force or energy is to be included in the cost terms of the performance index. An advantage of the controllers resulting from the optimisation procedure is that they are dynamic, and thus possess finite bandwidth; accordingly they can be used when there is a limitation to the bandwidth of a communication channel over which feedback signals are transmitted.

1 Introduction

The problem considered in the paper is the derivation of a state feedback law for a linear, finite-dimensional system to minimise a quadratic loss function consisting of terms involving time derivatives of the input vector as well as the usual terms involving the input and state vectors.

A number of practical situations can be envisaged when such a control law could be required. As a light-hearted example, we could conceive of some mechanical object such as a slot car,* together with its controls, as a system whose input u is a potentiometer setting, representing the velocity of the car. It could be postulated that step changes in the velocity u of the car are almost possible because of its very low weight; however the motor of the slot car will burn out if the electric power which it is converting into mechanical power exceeds a certain value. Thus it would be desired to limit the value of $u\dot{u}$, representing in a rough sense the mechanical power of the driving motor, even though the dynamical equations describing the car and its controls do not involve $\dot{u}$ because of the car's extremely low weight.

To achieve this limitation on $u\dot{u}$, a loss function could be assumed which heavily weighted the term, in addition to involving quadratic terms in u and the state variable x (presumably position here).

It will actually be seen in the paper that to yield a well defined problem it is necessary to include in the loss function a term involving $\dot{u}^2$ if one involving $u\dot{u}$ is included. This can however have a very small weight relative to that of $u\dot{u}$ to achieve the sort of performance desired.

For linear, finite-dimensional, completely controllable systems, Kalman[1] has shown that in the case when the quadratic loss function consists of the input- and state-vector terms only, and, provided that certain conditions are specified to make the problem well defined, the minimisation problem results in a linear feedback law. In order to apply this theory to the problem of interest, the plant is first augmented with an integrator in cascade with each input line. The performance index for the original system may then be rewritten, so that it consists of terms involving only the input and state vectors of the augmented system. (The inputs to the original system will actually be some of the states of the augmented system.) Applying the optimal control theory to the augmented system in order to minimise the specified performance index of the original system results in a constant linear feedback law for the augmented system, which may be interpreted as a linear dynamical-system feedback controller for the original system.

* A slot car is a miniature electrically driven and controlled car used in competitive racing. Control of the car is essentially achieved by setting the speed of the car with a rheostat. Excessive speed settings will lead to the car skidding, or leaving the track on a curve.

Paper 5430 C, first received 9th May and in revised form 21st August 1967

Dr. Moore and Prof. Anderson are with the Department of Electrical Engineering, University of Newcastle, Newcastle, NSW, Australia

Section 2 reviews the appropriate optimal control theory,[1] and Section 3 applies it to the augmented system in order to obtain an optimal feedback law for the original system. The feedback controller determined for the original system itself possesses states, and the initial values of these states for which the performance index is minimised are also calculated in Section 3. The concluding Section considers the extension of the results to the case where the system states are not directly measurable, but rather only linear combinations of them corrupted by noise are available.

2 Review of optimal control problem

The plants under consideration are linear, finite-dimensional, time-invariant, dynamical systems. It is assumed that their behaviour may be represented by the differential system

$$\dot{x}_1 = F_1 x_1 + G_1 u_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x_1 is an $n$vector, the state, and for the present discussion is considered also as the output vector. The input u_1 is an $m$vector, while the matrices F_1 and G_1 are constant and of dimension $n \times n$ and $n \times m$, respectively. It will also be assumed that

(a) the pair (F_1, G_1) is completely controllable,[2] or equivalently,

$$\text{rank}\,(G_1, F_1G_1, F_1^2G_1, \ldots, F_1^{n-1}G_1) = n \quad . \quad . \quad (2)$$

In the optimal regulator problem, the object is to return a nonzero initial state $x_1(0)$ to the zero state by selecting a control u_1 which minimises the performance index

$$V\{x_1(0), u_1\} = \int_0^\infty (x_1'Q_1x_1 + 2u_1'S_1x_1 + u_1'R_1u_1)dt \quad (3)$$

(The prime denotes matrix transpositions.) This performance index is a function of the initial state $x_1(0)$ and the control u_1 employed over the interval $(0, \infty)$.

In order that a minimisation problem be well defined, it is required that

(b) the matrix R_1 is positive-definite-symmetric.

It is normally assumed, though it is not always necessary for the existence of a minimum, that

(c) the matrix $Q_1 - S_1'R_1^{-1}S_1$ is nonnegative-definite-symmetric.

Also, in order to guarantee stability of the closed-loop system obtained when minimising eqn. 3, the following is needed:

(d) the pair (F_1, H_1), where H_1 is any solution of $H_1'H_1 = Q_1 - S_1'R_1^{-1}S_1$, is completely observable,[2] or equivalently

$$\text{rank}\,(H_1', F_1'H_1', \ldots (F_1')^{n-1}H_1') = n \quad . \quad . \quad . \quad . \quad (4)$$

The control law used in the solution of the optimal regulator problem is now stated.[1]

Control law 1: Consider the plant of eqn. 1, and assume that (*a*)–(*d*) hold. Then there exists an optimal control u_1^* which minimises eqn. 3, and it is given by

$$u_1^* = -R_1^{-1}(S_1 + G_1'P_1)x_1 \quad . \; . \; . \; . \; . \; . \quad (5)$$

where P_1 is defined later. Further, the value of the associated performance index is

$$V^*\{x(0)\} = x_1'(0)P_1x_1(0) \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

The matrix P_1 is positive-definite and is defined as

$$P_1 = \lim_{t \to -\infty} \Pi_1(t) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

where $\Pi_1(t)$ is the solution of the Riccati differential equation

$$-\dot{\Pi}_1 = \Pi_1F_1 + F_1'\Pi_1 - \Pi_1G_1R^{-1}G_1'\Pi_1 + (Q_1 - S_1'R_1^{-1}S_1) \quad . \; . \; . \; . \quad (8)$$

with the initial condition

$$\Pi_1(0) = 0$$

With the assumptions listed, both Π_1 and P_1 are always well defined.

3 Optimal control with input derivative constraints

The plants under consideration are as in the previous Section. Introducing a notation change, the state–space equation is

$$\dot{x} = Fx + Gu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

The controllability condition is therefore written as

(i) The pair (F, G) is completely controllable, or equivalently,

$$\text{rank}\,(G, FG, F^2G, \dots, F^{n-1}G) = n \quad . \; . \; . \quad (10)$$

We define the optimal regulator problem with input derivative constraints as the selection of a control u which returns a nonzero initial state $x(0)$ to the zero state, and which minimises the performance index

$$V\{x(0), u\} = \int_0^\infty (x'Qx + 2u'Sx + u'Ru + 2\dot{u}'Tu + 2\dot{u}'Z\dot{u} + 2\dot{u}'Wx)dt \quad . \; . \; . \; . \quad (11)$$

This performance index consists of quadratic terms involving not only the state vector x and the input vector u, but also the derivative of the input vector $\dot{u}$. Restrictions on Q, S etc. will be explained subsequently.

In order to consider the performance index of eqn. 11 using the optimal control theory reviewed in the previous Section, the original system of eqn. 9 is augmented with an integrator in cascade with each input line, as in Fig. 1. The

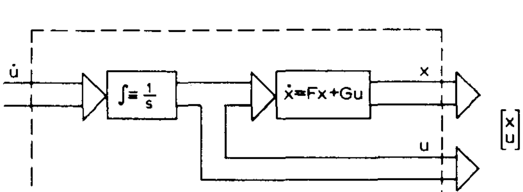

Fig. 1
Augmented system

input vector u_2 of the augmented system is then the derivative of the input to the original plant, i.e. $\dot{u}$; the state vector x_2 of the augmented system is therefore $[x' \,\vdots\, u']'$ and the state–space equations for the augmented system are given by

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \dot{u} \quad . \; . \; . \; . \quad (12)$$

or in shorthand notation,

$$\dot{x}_2 = F_2x_2 + G_2u_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

The performance index of eqn. 11 may be rearranged to incorporate the variables of the augmented system as

$$V\{x_1(0), u_2\} = \int_0^\infty (x_2'Q_2x_2 + 2u_2'S_2x_2 + u_2'R_2u_2)dt \quad (14)$$

or equivalently,

$$V\{x(0), u(0), u\} = \int_0^\infty \{[x' \mid u'] \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + 2\dot{u}'[W \,\vdots\, T] \begin{bmatrix} x \\ u \end{bmatrix} + \dot{u}'Z\dot{u}\}dt \quad . \quad (15)$$

The identification of Q_2, S_2 and R_2 from Q, S etc. is clear from eqns. 14 and 15.

We point out later that a constant linear feedback law for the augmented system of eqn. 13 determined from the minimisation of the performance index (eqn. 14) using control law 1 may be interpreted as a linear dynamical law which minimises the performance index of eqn. 11 for the unaugmented original system of eqn. 9. Before calculating this feedback law, conditions (*a*)–(*d*) in Section 2 will be viewed as conditions on the augmented system of eqn. 13 and its performance index (eqn. 14) and interpreted in terms of conditions on the original system of eqn. 9 and its performance index (eqn. 11).

The necessary and sufficient condition that the augmented system be completely controllable is

$$\text{rank}\,(G_2, F_2G_2, F_2^2G_2, \dots, F_2^{n+m-1}G_2) = n + m \quad . \quad (16)$$

or equivalently,

$$\text{rank}\left(\begin{bmatrix} 0 \\ I_m \end{bmatrix}, \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ I_m \end{bmatrix}, \dots, \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix}^{n+m-1}\begin{bmatrix} 0 \\ I_m \end{bmatrix}\right) = n + m \quad . \quad (17)$$

or

$$\text{rank}\left(\begin{bmatrix} 0 \\ I_m \end{bmatrix}, \begin{bmatrix} G \\ 0 \end{bmatrix}, \begin{bmatrix} FG \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} F^{n+m-2}G \\ 0 \end{bmatrix}\right) = n + m \quad . \; . \; . \; . \quad (18)$$

or

$$\text{rank}\,(G, FG, F^2G, \dots, F^{n+m-2}G) = n \quad . \; . \; . \quad (19)$$

Noting that m is positive, a straightforward application of the Cayley–Hamilton theorem[3] allows replacement of eqn. 19 by the equivalent statement (i). Thus the augmented system is completely controllable if, and only if, the original system is completely controllable.

The interpretations as conditions on the plant of eqn. 9 with performance index of eqn. 11 of assumptions (*b*), (*c*) and (*d*) applied to the augmented system are as follows:

(ii) The matrix R_2 (i.e. Z) is positive-definite-symmetric.

(iii) The matrix $[Q_2 - S_2'R_2^{-1}S^2]$, or equivalently,

$$\begin{bmatrix} Q & S' \\ S & R \end{bmatrix} - \begin{bmatrix} W' \\ T' \end{bmatrix} Z^{-1}[W \quad T]$$

is nonnegative-definite-symmetric.

(iv) The pair (F_2, H) is completely observable, for any H, such that

$$H'H = Q_2 - S_2'R_2^{-1}S_2 \quad . \; . \; . \; . \; . \; . \; . \quad (20)$$

or equivalently,

$$H'H = \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} - \begin{bmatrix} W' \\ T' \end{bmatrix} Z^{-1}[W \,\vdots\, T] \quad . \; . \; . \quad (21)$$

Note: The observability condition may be written as

$$\text{rank}\,\{H', F_2'H', \dots, (F_2')^{n+m-1}H'\} = n + m \quad . \quad (22)$$

or equivalently

$$\text{rank}\left(H', \begin{bmatrix} F' & 0 \\ G' & 0 \end{bmatrix} H', \dots, \begin{bmatrix} F' & 0 \\ G' & 0 \end{bmatrix}^{n+m-1} H'\right) = n + m \quad . \; . \; . \; . \quad (23)$$

Certainly, if from eqns. 20 and 21 $H'H$ is positive-definite-symmetric, the rank of H' is $(n + m)$, and (iv) is satisfied.

Again, in the special case when $S = W = T = 0$, eqn. 23 may be written as

$$\text{rank}\left(\begin{bmatrix} H_1' & 0 \\ 0 & H_2' \end{bmatrix}, \begin{bmatrix} F' & H_1' \\ G' & H_2' \end{bmatrix}, \ldots \begin{bmatrix} (F')^{n+m-1}H_1' \\ (G')^{n+m-1}H_2' \end{bmatrix}\right) = n + m \qquad (24)$$

where $H_1'H_1 = Q$ and $H_2'H_2 = R$.

It is evident that if R is positive-definite as distinct from nonnegative-definite, and, if (F, H_1) is completely observable, eqn. 16 will hold. Thus, as an alternative to (iv):

(v) With $S = W = T = 0$, the matrix R is positive-definite-symmetric, and the pair (F, H_1) is completely observable where H_1 is any solution of $H_1'H_1 = Q$.

Of course, in broad terms (v) relates complete observability of the augmented system to complete observability of the original plant.

For the augmented system of eqn. 13, if the conditions (i)–(iv) are satisfied, control law 1 may be applied to determine an optimal control u_2^* which minimises the performance index eqn. 14. The control law is given by

$$u_2^* = -R_2^{-1}(S_2 + G_2'P)x_2 \qquad (25)$$

where P is the limit as $t \to -\infty$ of the solution $\Pi(t)$ of the Riccati differential equation (eqn. 26) with initial condition $\Pi(0) = 0$:

$$-\dot{\Pi} = \Pi F_2 + F_2'\Pi - \Pi G_2 R_2^{-1} G_2' \Pi + (Q_2 - S_2' R_2^{-1} S_2) \qquad (26)$$

The minimum performance index associated with the optimum control u_2^* is

$$V^*\{x_2(0)\} = x_2'(0)Px_2(0) \qquad (27)$$

These results may now be interpreted in terms of the original system of eqn. 9. Eqn. 25, the linear control law for the augmented system, may be written as

$$\dot{u}^* = -Z^{-1}[W \mid T] - [0 \mid I_m]P\begin{bmatrix} x \\ u^* \end{bmatrix}$$

or $$\dot{u}^* = -Z^{-1}(T + P_{22})u^* - Z^{-1}(W + P_{21})x \qquad (28)$$

where P_{21}, P_{22} are $m \times n$ and $m \times m$ matrices obtained by partitioning P as

$$P = \begin{bmatrix} P_{11} & P_{21}' \\ P_{21} & P_{22} \end{bmatrix} \qquad (29)$$

We observe that, if $u^*(0)$ is specified, eqn. 28 may be interpreted as a linear dynamical feedback law for the original system to minimise the performance index (eqn. 15 or 11). The minimum performance index associated with the optimum control u^* is given from eqn. 28 as

$$V^*\{x(0), u^*(0)\} = x'(0)P_{11}x(0) + 2\{u^*(0)\}'P_{21}x(0) + \{u^*(0)\}'P_{22}u^*(0) \qquad (30)$$

It may or may not be possible to set $u^*(0)$ in practice; in some cases the value might be totally unknown, in others an initial value might automatically be taken as $u^*(0) = 0$. But in case $u^*(0)$ can be arbitrarily chosen, the choice which minimises the performance index (eqn. 30) with respect to all possible $u^*(0)$ is obtained from a rewriting of eqn. 30

$$V^*\{x(0), u^*(0)\} = x'(0)(P_{11} - P_{21}'P_{22}^{-1}P_{21})x(0) + \{u^*(0) + P_{22}^{-1}P_{21}x(0)\}'P_{22}\{u^*(0) + P_{22}^{-1}P_{21}x(0)\} \qquad (31)$$

The optimum choice for $u^*(0)$ is clearly

$$u^{**}(0) = -P_{22}^{-1}P_{21}x(0) \qquad (32)$$

and the corresponding performance index is

$$V^{**}\{x(0)\} = x'(0)(P_{11} - P_{21}'P_{22}^{-1}P_{21})x(0) \qquad (33)$$

which is certainly less than that obtained for $u^*(0) = 0$:

$$V^*\{x(0), 0\} = x'(0)P_{11}x(0) \qquad (34)$$

Fig. 2 shows the optimal augmented system with its linear feedback law. By including initial conditions on the integrators, this schematic may be viewed as the original system having a dynamical controller minimising the performance index of eqn. 11 (see also Fig. 3). We conclude that the controller may be realised by a linear time-invariant, dynamical system, whose input is the output of the system given from eqn. 9, i.e. x, having state-space equations given in eqn. 28, and thus having a transfer function

$$C(s) = -\{sI + Z^{-1}(T + P_{22})\}^{-1}Z^{-1}(W + P_{21}) \qquad (35)$$

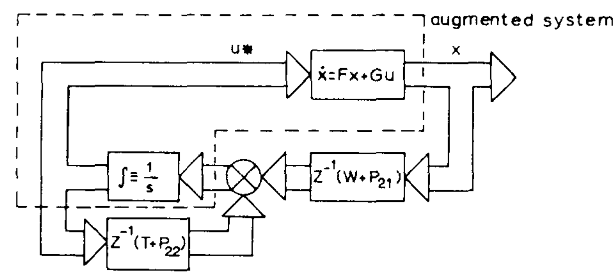

Fig. 2
Optimal closed-loop augmented system

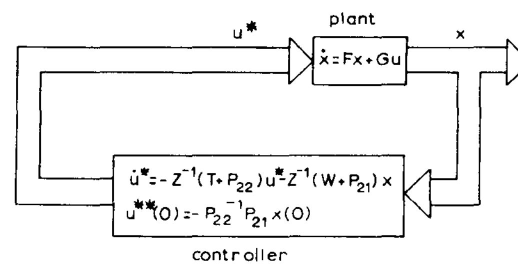

Fig. 3
Optimal closed-loop system (input derivative constraints)

These results may be summarised as

Control law 2: For the plant of eqn. 9, there exists an optimal control u^* which minimises the performance index (eqn. 11) containing input derivative constraints, provided conditions (i)–(iv) are satisfied. The control u^* is given from the state equation

$$\dot{u}^* = -Z^{-1}(T + P_{22})u^* - Z^{-1}(W + P_{21})x \qquad (28)$$

with initial conditions considered later. The minimum performance index associated with the optimal control u^* is

$$V^*\{x(0), u^*(0)\} = x'(0)P_{11}x(0) + 2\{u^*(0)\}'P_{21}x(0) + \{u^*(0)\}'P_{22}u^*(0) \qquad (30)$$

A value of $u^*(0)$ may be selected which minimises the index (eqn. 25) (see eqns. 31, 32 and 33). That is, for

$$u^{**}(0) = -P_{22}^{-1}P_{21}x(0) \qquad (32)$$

we have

$$V^{**}\{x(0)\} = x'(0)(P_{11} - P_{21}'P_{22}^{-1}P_{21})x(0) \qquad (33)$$

whereas that obtained for $u^*(0) = 0$ is

$$V^*\{x(0), 0\} = x'(0)P_{11}x(0) \qquad (34)$$

For these equations, P_{11}, P_{21} and P_{22} are $n \times n$, $m \times n$ and $m \times m$ matrices obtained by partitioning P as in eqn. 29, where P is the limit as $t \to -\infty$ of the solution $\Pi(t)$ of the Riccati differential equation (eqn, 36) with initial condition $\Pi(0) = 0$:

$$-\dot{\Pi} = \Pi\begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F' & 0 \\ G' & 0 \end{bmatrix}\Pi - \Pi\begin{bmatrix} 0 \\ I_m \end{bmatrix}Z^{-1}[0 \mid I_m]\Pi + \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} - \begin{bmatrix} W' \\ T' \end{bmatrix}Z^{-1}[W \mid T] \qquad (35)$$

4 Concluding remarks

Many of the extensions of optimal control theory are also applicable to the problem dealt with in the paper.

One such extension worthy of note is that when the plant outputs, being linear transformations of the states, are only available with additive Gaussian noise, and there is also Gaussian noise at the inputs, the same feedback law given in control law 1 may be used, together with a state estimator, to minimise the expected value of the performance index. For the augmented system, the best estimate of the states (minimum-variance estimate) is available from a state estimator constructed for the original system together with the inputs to the original system; so to minimise the expected value of the performance index (eqn. 11) for the system of eqn. 9 is a straightforward application of available theory together with the results of the paper.

As is pointed out in Reference 1, the optimal control law resulting for the augmented system will be stable; since the plant with its dynamic controller constitutes a rearrangement of the augmented system with its constant controller, stability of the closed-loop system is assured (irrespective of the stability of the plant itself).

It is of interest to note than an implication of the paper is that feedback controllers with dynamics may be optimal for some performance index. Not all dynamical feedback controllers will be optimal, of course, and it would be interesting to characterise the optimal ones. The optimal controllers in essence possess finite bandwidth, and accordingly may be used appropriately when signalling of the optimal control must take place over a channel of restricted bandwidth.

5 References

1 KALMAN, R. E.: 'When is a linear control system optimal?', *Trans., ASME*, 1964, [D], **86**, pp. 1-10

2 KALMAN, R. E.: 'Mathematical description of linear dynamical systems', *SIAM J. Control*, 1963, **1**, (2), pp. 152-192

3 GANTMACHER, F. R.: 'The theory of matrices' (Chelsea, New York, 1949)