# On State-Estimation of a Two-State Hidden Markov Model with Quantization

Louis Shue, Subhrakanti Dey, *Member, IEEE*, Brian D. O. Anderson, *Fellow, IEEE*, and Franky De Bruyne

***Abstract*—We consider quantization from the perspective of minimizing filtering error when quantized instead of continuous measurements are used as inputs to a nonlinear filter, specializing to discrete-time two-state hidden Markov models (HMMs) with continuous-range output. An explicit expression for the filtering error when continuous measurements are used is presented. We also propose a quantization scheme based on maximizing the mutual information between quantized observations and the hidden states of the HMM.**

***Index Terms*—Hidden Markov model, nonlinear filter, quantization.**

## I. Introduction

IN COMMUNICATION systems, a fundamental problem is the question of how much degradation in performance a receiver suffers when incoming data are quantized. Quantization may arise as a direct consequence of digitizing real-world (and hence continuous-valued) signals for further processing or in situations where bandwidth of a digital communication channel is limited, and it is necessary to minimize the number of bits used in representing a given signal. In this work, we will consider the effects of quantization on the performance of a nonlinear filter, with the signal source assuming the form $Y_k = h(X_k) + v_k$, where $X_k$ is a two-state Markov chain, with $h(X_k) \in \{-1, 1\}$, and $v_k$ is independent and identically distributed (i.i.d.) noise. The function of the filter is to estimate $X_k$ given a sequence of the quantized versions of the continuous measurements $Y_k$.

Much of the previous research in quantization aims to reduce the distortion between the original and quantized signals under some minimum mean-squared-error criteria [5], [6], [13], where the principal measure of performance is the error (or function of) between the input and output of the quantizer. Other authors have considered quantization where the quantized data is used to form a test of hypotheses for the purposes of signal detection [1], [3], [8]. Our approach differs from the previous research in a number of ways. First, we concentrate on continuous-valued observations obtained from a hidden Markov model (HMM) instead of i.i.d. sources. Furthermore, the primary focus here is to minimize the probability of error of a nonlinear filter with quantized observations as inputs.

Some of the questions we address are as follows. 1) Is there an optimal strategy for choosing the quantization levels that minimizes the filtering error? 2) How does filtering error vary with noise for different numbers of quantization levels $M$? In particular, is there a rule of thumb relating signal-to-noise ratio (SNR), loss of performance through quantization, and the number of quantization levels?

At this stage, we have only partial answers to both questions. For the first question, we propose two quantization schemes, based on the maximization of the mutual information and Kullback–Leibler (KL) divergence measures, respectively. A comparison of these schemes will be made by simulations, where we will show that the filtering error when a small number of quantization levels (obtained by maximizing the mutual information) are used can approach the filtering error when using continuous measurements. We also provide an analysis for computing the maximum *a posteriori* (MAP) error as a function of SNR for a two-state discrete-time HMM with continuous-range output (this is a discrete-time version of a continuous-time result obtained by Wonham [12] with very different methods). In principle, a similar analysis can be carried out for cases where quantization has been implemented. However, these extensions have not been developed in the present paper.

## II. Signal Model

Consider a probability space $(\Omega, \mathcal{F}, P)$. In this space, consider a first-order, homogeneous discrete-time two-state Markov chain $X_k \in \{1, 2\}$, $k$ denoting time; the actual state levels are $h(X_k) \in \{-1, 1\}$ such that $h(1) = -1$, $h(2) = 1$. The Markov chain evolves according to the state transition probability matrix $A = (a_{ij})$, where $a_{ij} = P(X_{k+1} = i | X_k = j)$. For this paper, we will, for ease of computation, assume that $A$ is symmetric (i.e., doubly stochastic) and has the form $A = \left[\begin{smallmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{smallmatrix}\right]$, where $0 < \alpha < 1$. The observation process $Y_k \in \mathbb{R}$ is defined as $Y_k = h(X_k) + v_k$, where $v_k$ is i.i.d. and $\mathcal{N}(0, \sigma^2)$.[1] Consequently, the observations $Y_k$ are conditionally distributed as

$$\begin{aligned} P(Y_k \in dy | X_k = i) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_k - h(i))^2}{2\sigma^2}\right] dy \\ &= b_i(y_k)\, dy \end{aligned} \tag{1}$$

[1]The calculations for the filtering error (see Section III) in fact hold true for other types of noise than Gaussian, provided the noise distribution is such that $P(Y_k \in dy | X_k = i)$, $\forall\, i$ can be specified and is time invariant.

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where $b_i(y_k) = (1/\sqrt{2\pi\sigma^2}) \exp[-((y_k - h(i))^2/2\sigma^2)]$. Throughout this paper, we will adopt the convention that lowercase variables correspond to an actual value taken by the respective stochastic process, i.e., $x_k$ corresponds to $X_k$, etc.

The objective of quantization is to partition the observations $Y_k$ into $M$ distinct intervals when an $M$-level quantizer is used. The output of the quantizer is $Y_k^q = m$ when $\ell_{m-1} \le Y_k < \ell_m$, $m \in \{1, 2, \cdots, M\}$, where $\ell_{m-1}$ and $\ell_m$ are the limits of the $m$th quantization level, $\ell_0 = -\infty$, and $\ell_M = \infty$. This leads to the construction of a measurement matrix $C = (c_{mi})$, where

$$\begin{aligned} c_{mi} &= P(Y_k^q = m | X_k = i) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\ell_{m-1}}^{\ell_m} \exp\left[-\frac{(y - h(i))^2}{2\sigma^2}\right] dy. \end{aligned} \tag{2}$$

Denote the filtered probability density vector as $\Pi_{k|k}$, with the $i$th entry $i \in \{1, 2\}$ being $P(X_k = i | Y_0 = y_0, Y_1 = y_1, \cdots, Y_k = y_k)$. Denote the corresponding filtered probability vector when quantized measurements are used as $\Pi_{k|k}^q$. The filtered probability vectors are updated by the following recursive relations [9]:

$$\Pi_{k+1|k+1} = \frac{1}{1_N' B_{y_{k+1}} A \Pi_{k|k}} B_{y_{k+1}} A \Pi_{k|k} \tag{3a}$$

$$\Pi_{k+1|k+1}^q = \frac{1}{1_N' C_{y_{k+1}^q} A \Pi_{k|k}^q} C_{y_{k+1}^q} A \Pi_{k|k}^q \tag{3b}$$

where $1_N'$ denotes an $(1 \times N)$ vector of ones, $B_{y_{k+1}} = \mathrm{diag}[b_1(y_{k+1})\ b_2(y_{k+1})]$, and $C_{y_{k+1}^q} = \mathrm{diag}[c_{j1} \quad c_{j2}]$ when $y_{k+1}^q = j$.

We will also make use of the following concepts from information theory [2].

*Definition II.1 (Entropy, Conditional Entropy):* The entropy of a discrete random variable $X$ and the conditional entropy of $X$, given another discrete random variable $Y$, are, respectively, $H(X) = -\sum_x P(x) \log P(x)$ and $H(X|Y) = -\sum_{x,y} P(x, y) \log P(x|y)$.

*Definition II.2 (Mutual Information):* The mutual information between a discrete random variable $X$ and another discrete random variable $Y$ is given by

$$\begin{aligned} I(X; Y) &= \sum_{x, y} P(x, y) \log \frac{P(x, y)}{P(x)P(y)} \\ &= H(X) - H(X|Y). \end{aligned} \tag{4}$$

*Remark II.1:* Note here that $P(x) = P(X = x)$, $P(y) = P(Y = y)$, $P(x, y) = P(X = x, Y = y)$, $P(x|y) = P(X = x|Y = y)$.

*Definition II.3 (KL Divergence):*

$$K_{ij} = K(C_{\bullet i}, C_{\bullet j}) = \sum_m c_{mi} \log \frac{c_{mi}}{c_{mj}} \triangleq E_{C_{\bullet i}} \log \frac{C_{\bullet i}}{C_{\bullet j}} \tag{5}$$

where $C_{\bullet i} = P(Y_k^q | X_k = i)$, $i = 1, 2$ denotes the conditional probability distributions of the quantized observations.

We will further assume that for any $i \neq j \in \{1, 2\}$, the KL divergence exists, and $\min_{i,j} K_{ij} > 0$.

## III. Filter Performance

In this section, we will derive the filtering error probability for a discrete-time and two-state HMM with continuous-range observations as a function of SNR in Section III-A. In Section III-B, we will consider filtering when quantized measurements are used; the different schemes will be compared via a series of simulations in Section IV.

### A. Continuous Measurements

Let $q_k = \Pi_{k|k}(1) - \Pi_{k|k}(2)$. By application of (3a), we have the following recursion:

$$q_k = \frac{C_k^1 - C_k^2 + (1 - 2\alpha)(C_k^1 + C_k^2) q_{k-1}}{C_k^1 + C_k^2 + (1 - 2\alpha)(C_k^1 - C_k^2) q_{k-1}} \tag{6}$$

where $C_k^1 = e^{[-(Y_k+1)^2/2\sigma^2]}$, $C_k^2 = e^{[-(Y_k-1)^2/2\sigma^2]}$, and $Y_k$ denotes the continuous range observation.

We will evaluate the probability of error for an HMM filter by considering the joint process $\{X_k, q_k\}$. Denote the stationary densities (for the existence and uniqueness of solution see Remark III.1; for further details, see the Appendix) as $\pi^-(q)\, dq = P(X_k = 1, q_k \in (q, q + dq))$, $\pi^+(q)\, dq = P(X_k = 2, q_k \in (q, q + dq))$, where $-1 \le q \le 1$. Then, we have

$$\begin{aligned} &P(\text{Filter Error}) \\ &\quad = P(X_k = 2, \hat{X}_k = 1) + P(X_k = 1, \hat{X}_k = 2) \\ &\quad = P(X_k = 2, \Pi_{k|k}^1 > \Pi_{k|k}^2) + P(X_k = 1, \Pi_{k|k}^1 \le \Pi_{k|k}^2) \\ &\quad = P(X_k = 2, q_k > 0) + P(X_k = 1, q_k \le 0) \\ &\quad = \int_0^1 \pi^+(q)\, dq + \int_{-1}^0 \pi^-(q)\, dq \\ &\quad = 2 \int_0^1 \pi^+(q)\, dq \end{aligned} \tag{7}$$

where the last step follows from the symmetry condition $\pi^-(q) = \pi^+(-q)$ (see [12] for more details).

It is well known that the joint process $\{X_k, q_k\}$ is a Markov process with a mixed transition probability kernel $P(X_{k+1} = i, q_{k+1} \in (q, q + dq) | X_k = j, q_k = \tilde{q}) = a_{ij} S_i(q, \tilde{q})\, dq$, where, by a change of variables [7, ch. 5]

$$\begin{aligned} &S_i(q, \tilde{q})\, dq \\ &\quad = P(q_{k+1} \in (q, q + dq) | X_{k+1} = i, q_k = \tilde{q}) \\ &\quad = P\left\{ \frac{C_{k+1}^1 - C_{k+1}^2 + (1 - 2\alpha)(C_{k+1}^1 + C_{k+1}^2) q_k}{C_{k+1}^1 + C_{k+1}^2 + (1 - 2\alpha)(C_{k+1}^1 - C_{k+1}^2) q_k} \in (q, q + dq) \,\middle|\, X_{k+1} = i, q_k = \tilde{q} \right\} \\ &\quad = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(g(q, \tilde{q}) - h(i))^2}{2\sigma^2}\right] \frac{\sigma^2}{1 - q^2}\, dq \end{aligned} \tag{8}$$

and for a given $\{q_{k+1}, q_k\}$ pair, the unique value of $y_{k+1}$ consistent with (6) is given by

$$\begin{aligned} y_{k+1} &= g(q_{k+1}, q_k) \\ &= \frac{-\sigma^2}{2} \ln\left[\frac{(1 + q_{k+1})(1 - q_k + 2\alpha q_k)}{(1 - q_{k+1})(1 + q_k - 2\alpha q_k)}\right]. \end{aligned} \tag{9}$$

Then, it can be shown that the densities $\pi^{\pm}(q)$ are the steady-state solutions to the following integral equation, where $\pi^{-}(q)$ corresponds to $X_k = 1$ and $\pi^{+}(q)$ corresponds to $X_k = 2$:

$$f_{k+1}^{i}(q)\,dq = P(X_{k+1} = i, q_{k+1} \in (q,\, q + dq)) = \sum_{j=1}^{2} a_{ij} \left[ \int_{-1}^{1} S_i(q,\, \tilde{q}) f_k^j(\tilde{q})\, d\tilde{q} \right] dq. \quad (10)$$

See the Appendix for a proof of (10).

In this paper, we will solve (10) numerically by discretizing the integration in (10)

$$\begin{bmatrix} F^{k+1}(1,\, q) \\ F^{k+1}(2,\, q) \end{bmatrix} = T \begin{bmatrix} F^{k}(1,\, \tilde{q}) \\ F^{k}(2,\, \tilde{q}) \end{bmatrix} \quad (11)$$

where for $j = 1, 2$ and $i = 1, 2 \cdots, n$, $F^k(j, q) = [f_k^j(q_1) \quad f_k^j(q_2) \quad \cdots \quad f_k^j(q_n)]'$, and (in block-matrix form) $T = \begin{bmatrix} a_{11} D_1 \Delta\tilde{q} & a_{12} D_1 \Delta\tilde{q} \\ a_{21} D_2 \Delta\tilde{q} & a_{22} D_2 \Delta\tilde{q} \end{bmatrix}$, and $D_\ell = (S_\ell(q_i,\, \tilde{q}_j)),\ i, j = 1, 2, \cdots, n$. Finally, by observing the normalization condition $\sum_i F^k(i, q)\Delta q = 1$ since $[F^k(1, q) \quad F^k(2, q)]'$ is a joint density vector, at each step of the recursion, the steady-state solution can be found.

*Remark III.1:* The discretization and normalizing at each step of the recursion, are in fact equivalent to updating the vectors $[F^k(1, \tilde{q}) \quad F^k(2, \tilde{q})]'$ by a normalized $T$, provided the initial vector $[F^0(1, \tilde{q}) \quad F^0(2, \tilde{q})]'$ is also normalized. Let $T_{\rm norm} = T\Lambda^{-1}$, where $\Lambda$ is a diagonal matrix such that $\Lambda_{ii} = \sum_i T_{ij},\ \forall i, j \in \{1, 2, \cdots, 2n\}$, so that $T_{\rm norm}$ is a stochastic matrix. The stochastic nature of $T_{\rm norm}$ in conjunction with the knowledge of its positivity (by discretizing $S_i(q, \tilde{q})$ such that $g(q_k,\, q_k + 1)$ from (9) remains bounded and avoiding the case $|q| = 1$) and the Perron–Frobenius theorem [10] ensure that (11) in fact has a unique positive steady-state solution. Finally, since these conclusions hold for arbitrary $n$, then by continuity arguments, a unique stationary solution must also exist for (10).

## B. Filtering Using Quantized Observations

In this section, we propose schemes for choosing the quantization levels from an information-theoretic perspective, keeping in mind that the ultimate goal is to derive quantization levels that will reduce the filtering error. In the subsequent discussions, uniform quantization will be used as the basis of comparisons for the schemes discussed in this paper (apart from the unquantized case).

In our first quantization method, we will determine the quantization intervals $\ell_i$ by maximizing the mutual information between the hidden Markov state $X_k$ and the quantized observation sequence $\mathcal{Y} = \{Y_0^q, Y_1^q, \cdots, Y_k^q\}$. However, we stress that this may not be an optimal means of determining the quantization levels. The justification of this approach may be regarded as a consequence of the fact that intuitively, filtering is a form of information processing (using the observation sequence), and hence, the accuracy of filtered estimates can be improved if the information content inherent in the filtered probability density (quantified in terms of the mutual information) is maximized in some sense.

From (4), the mutual information is maximized by minimizing $H(X_k|\mathcal{Y})$ as $H(X_k)$ is determined by the structure of the Markov chain. Noting that conditioning reduces the entropy [2, p. 27] so that

$$H(X_k|\mathcal{Y}) = H(X_k|Y_0^q,\, Y_1^q,\, \cdots,\, Y_k^q) \le H(X_k|Y_k^q) \quad (12)$$

it is seen that the computational task of minimizing $H(X_k|\mathcal{Y})$ can be simplified (albeit with approximations) by minimizing the approximate term $H(X_k|Y_k^q)$ instead. Assuming stationarity, the conditional entropy $H(X_k|\mathcal{Y})$ is then overbounded by

$$\begin{aligned} H(X_k|Y_k^q) &= -\sum_{i=1,2} \sum_m P(X_k = i,\, Y_k^q = m) \cdot \log P(X_k = i|Y_k^q = m) \\ &= -\sum_{i=1,2} \sum_m c_{mi} P(X_k = i) \log \frac{c_{mi} P(X_k = i)}{\sum_{j=1,2} c_{mj} P(X_k = j)} \\ &= -\frac{1}{2} \sum_m c_{m1} \log \frac{c_{m1}}{c_{m1} + c_{m2}} - \frac{1}{2} \sum_m c_{m2} \cdot \log \frac{c_{m2}}{c_{m1} + c_{m2}} \end{aligned} \quad (13)$$

where $P(X_k = 1) = P(X_k = 2) = (1/2)$ due to the symmetric nature of $A$ when stationarity is assumed.

*Remark III.2:* The minimization of $H(X_k|Y_k^q)$ means, in essence, that a lower bound to $I(X_k; \mathcal{Y})$ is determined. However, there is no guarantee that $H(X_k|Y_k^q)$ can be made arbitrarily close to $H(X_k|\mathcal{Y})$ in the process. The validity of our claims that $H(X_k|Y_k^q)$ is a reasonable approximation to $H(X_k|\mathcal{Y})$ and that minimizing the conditional entropy can lead to improved filter performance is only finally seen from simulation studies. It should also be noted that even though $H(X_k|Y_k^q)$ is concave in the distributions, $H(X_k|Y_k^q)$ may not be a convex function of the quantization parameters $\ell_i$, $i \in \{1, 2, \cdots, M\}$. To overcome possible problems from this, the minimization was initialized with the levels obtained from the uniform quantizer in our simulations; the minimization was performed using a simplex search routine (optimization toolbox from Matlab). We also noticed that provided the initial conditions for the minimization were within the interval $[-1 - 2\sigma,\, 1 + 2\sigma]$, the same minima resulted.

Since (13) is an approximation to the true conditional entropy $H(X_k|\mathcal{Y})$, it is natural to include more conditional terms in the minimization in order to improve the approximation. For example, one could tackle the same quantization problem by minimizing the following conditional entropy:

$$\begin{aligned} H(X_k|Y_k^q,\, Y_{k-1}^q) &= -\sum_{i=1,2} \sum_m \sum_n P(X_k = i,\, Y_k^q = m,\, Y_{k-1}^q = n) \\ &\quad \cdot \log P(X_k = i|Y_k^q = m,\, Yk - 1^q = n). \end{aligned} \quad (14)$$

In evaluating (14), the additional computations involved increases[2] by $O(M^2)$. In general, the additional computation cost in calculating $H(X_k|Y_k^q,\, Y_{k-1}^q,\, \cdots,\, Y_{k-i}^q)$ is $O(N^{2i} M^{2i})$ for

[2]We have assumed multiplications and additions to be equivalent operations.

an $N$-state HMM with $M$-level quantizer. This is naturally prohibitive if $M$ is large. However, since HMM filters exhibit an exponential forgetting property [11], in principle, there exists a $\Delta$ such that $H(X_k|Y_k^q, Y_{k-1}^q, \cdots, Y_{k-\Delta}^q)$ can approximate well the true conditional entropy $H(X_k|\mathcal{Y})$; hence, in situations where $\Delta$ is small, it may be possible to execute the optimization with as much accuracy as possible.

In this paper, we also propose a second rather ad hoc approach to quantization based on the KL information measure. Since each column of $C$ specifies a conditional (conditioned on $x_k$) distribution, the accuracy of filtered estimates may be improved if $C_{\bullet 1} = P(Y_k^q|X_k = 1)$ and $C_{\bullet 2} = P(Y_k^q|X_k = 2)$ are well "separated" so that each quantized observation $Y_k^q$ yields maximal information concerning $X_k$. Since KL divergence is one measure of the "distance" between distributions [2], this is then mathematically equivalent to maximizing the KL divergence between $C_{\bullet 1}$ and $C_{\bullet 2}$. We can point out a "theoretical" justification for this approach by recalling an asymptotic (asymptotic in the sense of $\alpha \to 0$) filtering result [4]. Adapting the notation to our present signal model, this result states that for small $\alpha$

$$\begin{aligned}\lim_{k\to\infty} & P(\text{Filter Error}) \\ &= \left(\sum_{i=1}^{2} P(X=i) \sum_{j\neq i} \frac{\alpha}{K_{ji}}\right) \log\left(\frac{\lambda}{\alpha}\right)(1+o(1)) \\ &= \frac{\alpha}{K_{21}} \log\left(\frac{\lambda}{\alpha}\right)(1+o(1)) \qquad (15)\end{aligned}$$

for some constant $\lambda > 0$, and we have used the fact that $P(X = 1) = P(X = 2) = 0.5$ and $K_{12} = K_{21}$ due to the symmetric nature of our problem. Clearly, this means that with a slow-moving Markov chain, the probability of filtering error can be reduced if the KL divergence is maximized.

## IV. Simulations

In this section, we present some simulations of HMM filters using the quantization schemes introduced in Section III-B. All simulations were obtained using data sets of 100 000 points, averaged over 10 sets; we used 1000 discretizing steps in the evaluation of the theoretical error curve. The transition probability matrix of the Markov chain is $A = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$. The filtering error was computed as the average number of incorrect estimates in each simulation and then averaged over the number of simulations.

### A. Continuous Measurements

As shown in Fig. 1, there is high fit between the theoretical error obtained from (7) and filtering error from simulation (using continuous-range measurements) for $\sigma \geq 0.7$. The discrepancy at small $\sigma$ is largely due to the method of discretization used because it was observed that for small $\sigma$ values, the error computed for the theoretical curve is moderately sensitive to the number of discretizing steps. Furthermore, it is clear that as $\sigma$ becomes small, the MAP filtered estimates will tend to the situation of either $\Pi_{k|k}(1) \gg \Pi_{k|k}(2)$ or $\Pi_{k|k}(2) \gg \Pi_{k|k}(1)$, i.e., $|q_k| \approx 1$, and there may arise some numerical difficulties in the approximations we have taken in the evaluation of (8).

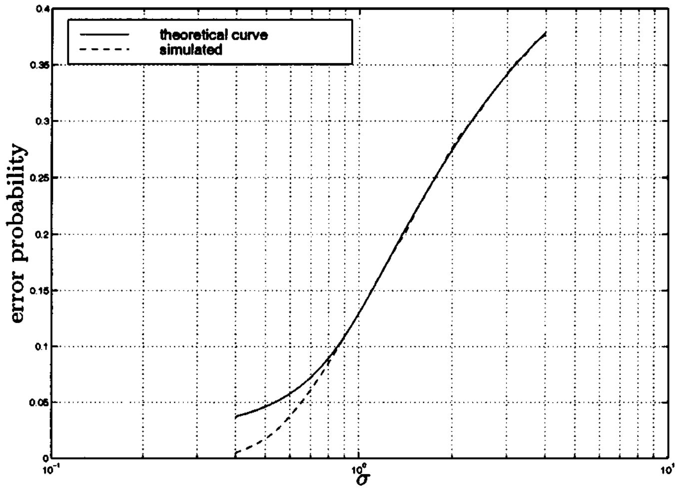


Fig. 1. Comparison of theoretical and simulated filtered error (using continuous measurements).

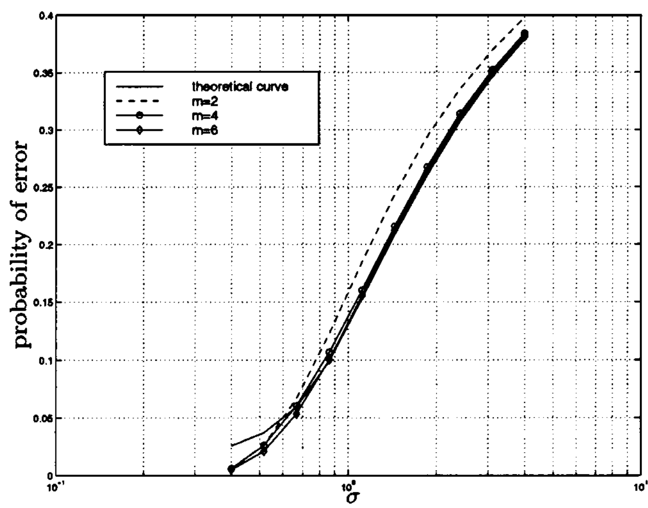


Fig. 2. Filtering error using (approximate) minimization of conditional entropy for different number of quantization levels.

### B. Comparison of Quantization Schemes

Representative error curves for different $\sigma$ values comparing the performance of filters obtained by quantization schemes are shown in Figs. 3 and 4. The errors have been normalized to the filtering error obtained when continuous measurements are used. From these results, one can conclude that quantization levels obtained by the approximate minimization of $H(X_k|\mathcal{Y})$ leads to significant improvement over the uniform case, particularly when $\sigma \leq 1$.

However, it was surprising to find that KL maximization did not lead to better filter performance. While KL maximization led to reduction of quantization noise (consistent with traditional nonuniform quantization schemes [5], [6]), the resulting filtering error remains comparable with errors obtained from filters constructed using uniform quantization. This may be understood by examining the resultant quantization levels (see Figs. 5–7). It can be seen that for small $\sigma$, minimizing conditional entropy places levels very close to $Y_k = 0$, relative to the other two schemes. This compacting of levels is likely the cause of improved filtering performance because as a result

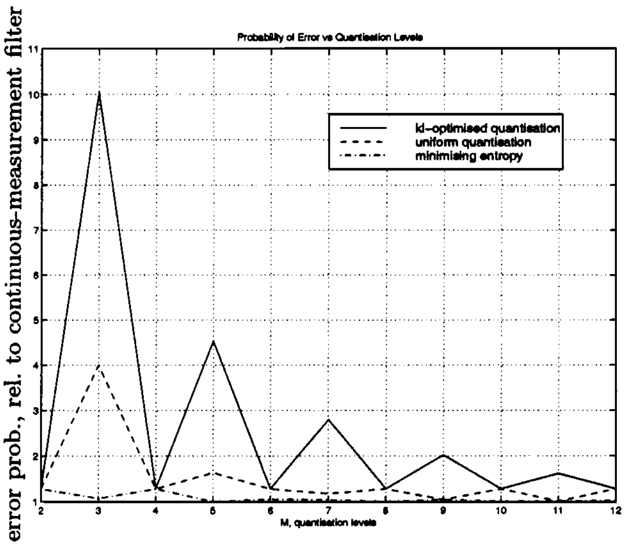


Fig. 3. Comparison of filtering error for different quantization schemes $\sigma = 0.4$.

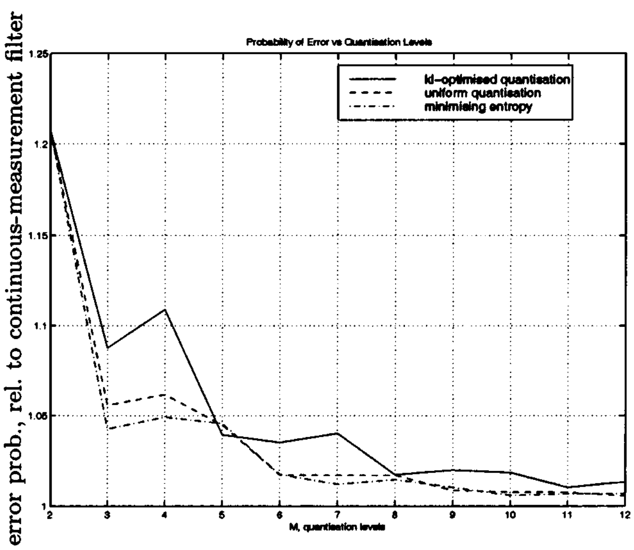


Fig. 4. Comparison of filtering error for different quantization schemes $\sigma = 1.113$.

of the symmetric placement of $X_k$, the region of ambiguity is when $Y_k$ is close to 0. However, for high $\sigma$ (e.g., Figs. 5–7), this effect is not so apparent, and there is similar performance for all three schemes. This also seems to indicate that for the purposes of estimation in a nonlinear framework, reduction of quantization noise may not be the right criterion to minimize.

*Remark IV.1:* For the HMM under consideration, no appreciable difference in the filtering error was observed when $H(X_k|Y_k^q, Y_{k-1}^q)$ as well as $H(X_k|Y_k^q, Y_{k-1}^q, Y_{k-2}^q)$ were minimized in order to derive the quantization levels. This may be due to the simplicity of the original signal model.

### C. Filtering Error at Different Quantization Levels

The error curves at differing number of quantization levels obtained by maximization of the mutual information is presented in Fig. 2. It was observed that when $M = 4$, the resulting filtering error is comparable with the theoretical curve,

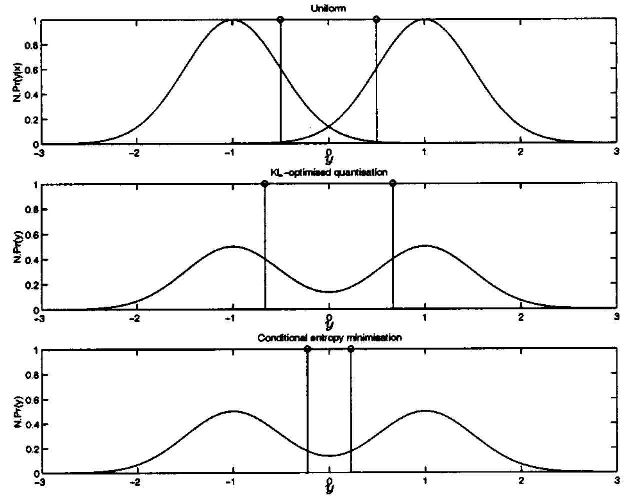


Fig. 5. Comparison of quantization intervals for different quantization schemes $M = 3, \sigma = 0.5$.

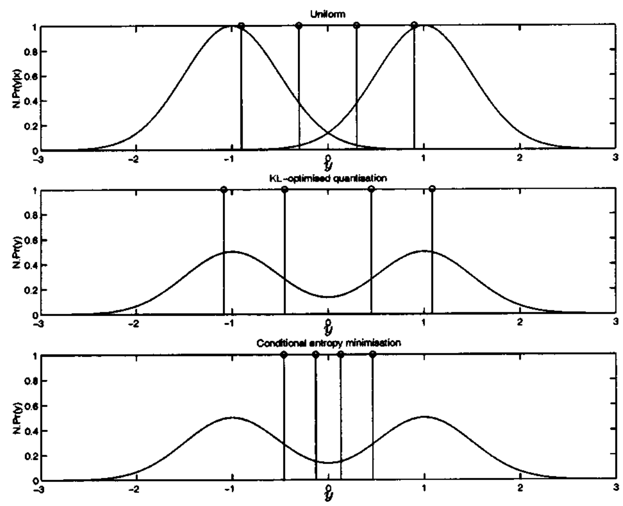


Fig. 6. Comparison of quantization intervals for different quantization schemes $M = 5, \sigma = 0.5$.

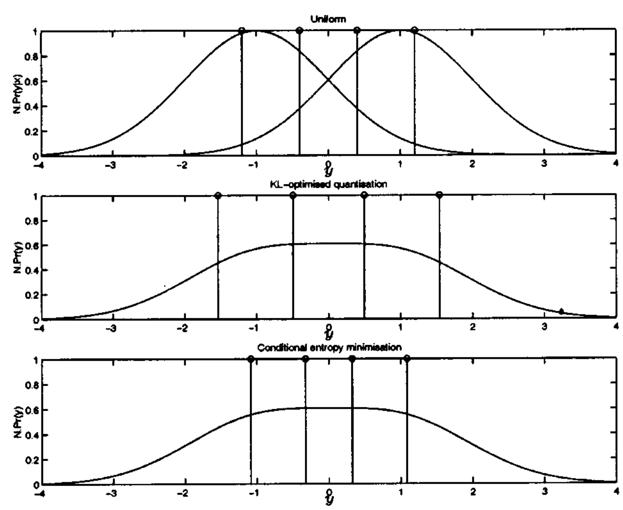


Fig. 7. Comparison of quantization intervals for different quantization schemes $M = 5, \sigma = 1$.

irrespective of SNR. However, more studies are required before any general rule for determining a "sufficient" number of quantization levels can be concluded.

## V. Conclusions

In this paper, explicit equations for evaluating the filtering error for a two-state HMM with continuous output have been presented. The error curve thus obtained indicates the expected filtering error when a filter using continuous measurements operates. Quantization levels have been derived using information-theoretic arguments. By maximizing the mutual information between the Markov state and the observation sequence, significant improvement in filtering performance at low noise compared with a filter obtained using uniform quantization can be obtained. The poor performance from quantization levels obtained by maximizing the KL divergence seems to indicate that a reduction in quantization noise (as a result of the KL-maximization scheme) may not necessarily lead to improved filter performance. It is speculated that this may be due to the nonlinear nature of HMM filters.

## Appendix
## Derivation of Joint Density Equations

In this section, we will derive the joint density equations. Let $q_k = \Pi_{k|k}(1) - \Pi_{k|k}(2)$. By (3a) and using the fact that $A = (a_{ij})$ is symmetric and $a_{21} = \alpha$, we have the recursive equation

$$q_{k+1} = \frac{C^1_{k+1} - C^2_{k+1} + (1-2\alpha)(C^1_{k+1} + C^2_{k+1})q_k}{C^1_{k+1} + C^2_{k+1} + (1-2\alpha)(C^1_{k+1} - C^2_{k+1})q_k} \quad (16)$$

where $-1 \le q_{k+1} \le 1$ for all $k$, $C^1_{k+1} = e^{[-(y_{k+1}+1)^2/2\sigma^2]}$, and $C^2_{k+1} = e^{[-(y_{k+1}-1)^2/2\sigma^2]}$. For a given $\{q_{k+1}, q_k\}$, the unique value of $y_{k+1}$ consistent with (16) is given by

$$\begin{aligned} y_{k+1} &= g(q_{k+1}, q_k) \\ &= \frac{-\sigma^2}{2} \ln\left[\frac{(1+q_{k+1})(1-q_k+2\alpha q_k)}{(1-q_{k+1})(1+q_k-2\alpha q_k)}\right]. \end{aligned} \quad (17)$$

To derive the density $f^i_k(q)$, $i = 1, 2$ for the joint process $\{X_k, q_k\}$, let us observe that

$$\begin{aligned} P(X_{k+1} &= i, q_{k+1} \in (q, q+dq), X_k = j, q_k \in (\tilde{q} + d\tilde{q})) \\ &= P(q_{k+1} \in (q, q+dq) | X_{k+1} = i, X_k = j, q_k = \tilde{q}) \\ &\quad \cdot P(X_{k+1} = i | X_k = j, q_k = \tilde{q}) f^j_k(\tilde{q})\, d\tilde{q} \\ &= a_{ij} S_i(q, \tilde{q}) f^j_k(\tilde{q})\, dq\, d\tilde{q} \end{aligned} \quad (18)$$

where, in the last line, the Markov property has been used, and

$$\begin{aligned} S_i&(q, \tilde{q})\, dq \\ &= P(q_{k+1} \in (q, q+dq) | X_{k+1} = i, X_k = j, q_k = \tilde{q}) \\ &= P(q_{k+1} \in (q, q+dq) | X_{k+1} = i, q_k = \tilde{q}). \end{aligned} \quad (19)$$

Now, by using (16) and a change of variables [7, ch. 5], for a given $\{q_{k+1}, q_k\}$ pair, we have

$$\begin{aligned} S_i&(q, \tilde{q})\, dq \\ &= P(q_{k+1} \in (q, q+dq) | X_{k+1} = i, q_k = \tilde{q}) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(g(q,\tilde{q}) - h(i))^2}{2\sigma^2}\right] \\ &\quad \cdot \left|\frac{dy_{k+1}}{dq_{k+1}}\right|_{q_{k+1}=q,\, q_k=\tilde{q}} dq \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(g(q,\tilde{q}) - h(i))^2}{2\sigma^2}\right] \frac{\sigma^2}{1-q^2}\, dq \end{aligned} \quad (20)$$

where $g(q, \tilde{q})$ is evaluated using (17) at $q_{k+1} = q$, and $q_k = \tilde{q}$. Consequently, for $i \in \{1, 2\}$

$$\begin{aligned} P(X_{k+1} &= i, q_{k+1} \in (q, q+dq)) \\ &= f^i_{k+1}(q)\, dq = \sum_j a_{ij} \left[\int_{-1}^{1} S_i(q,\tilde{q}) f^j_k(\tilde{q})\, d\tilde{q}\right] dq. \end{aligned} \quad (21)$$

We remark that $S_i(q, \tilde{q})$ can be verified to be continuous in the closed interval $[-1, 1]$.

We will solve (21) numerically by discretizing the integration in (21); each $q$, $\tilde{q}$ is replaced by $q_i = -1 + \Delta q/2 + (i-1)\Delta q$, $i \in \{1, 2, \cdots, n\}$, where $\Delta q = 2/n$, and $n$ is the number of points used in discretization. Next, by defining a discretized version of $f^j_k(\tilde{q})$, $j \in \{1, 2\}$ as

$$F^k(j, q) = [f^j_k(q_1) \quad f^j_k(q_2) \cdots \quad f^j_k(q_n)]' \quad (22)$$

(21) can be rewritten as a matrix product

$$\begin{bmatrix} F^{k+1}(1, q) \\ F^{k+1}(2, q) \end{bmatrix} = T \begin{bmatrix} F^k(1, \tilde{q}) \\ F^k(2, \tilde{q}) \end{bmatrix} \quad (23)$$

where

$$T = \begin{bmatrix} a_{11}S_1(q_1, \tilde{q}_1)\Delta\tilde{q} & \cdots & a_{12}S_1(q_1, \tilde{q}_n)\Delta\tilde{q} \\ \vdots & \ddots & \vdots \\ a_{11}S_1(q_n, \tilde{q}_1)\Delta\tilde{q} & \cdots & a_{12}S_1(q_n, \tilde{q}_n)\Delta\tilde{q} \\ a_{21}S_2(q_1, \tilde{q}_1)\Delta\tilde{q} & \cdots & a_{21}S_2(q_1, \tilde{q}_n)\Delta\tilde{q} \\ \vdots & \ddots & \vdots \\ a_{21}S_2(q_n, \tilde{q}_1)\Delta\tilde{q} & \cdots & a_{22}S_2(q_n, \tilde{q}_n)\Delta\tilde{q} \end{bmatrix}. \quad (24)$$

Since $[F^k(1, q) \quad F^k(2, q)]'$ is a joint density vector, we will introduce the normalizing condition $\sum_{i,j} f^j_k(q_i)\Delta q = 1$ at each step of the recursion. That is, at each step, having computed the left side of (23), the updated vector is divided by the sum of its entries, and the new vector is used at the next step of recursion. The steady-state solution can then be found iteratively.

The last two operations, i.e., using (23) and normalizing at each step of the recursion, are in fact equivalent to updating the vectors $[F^k(1, \tilde{q}) \quad F^k(2, \tilde{q})]'$ by a normalized $T$, provided the initial vector $[F^0(1, \tilde{q}) \quad F^0(2, \tilde{q})]'$ is also normalized. Let $T_{\text{norm}} = T\Lambda^{-1}$, where $\Lambda$ is a diagonal matrix where $\Lambda_{ii} =$

$\sum_i T_{ij}, \forall i, j \in \{1, 2, \cdots, 2n\}$ so that $T_{\rm norm}$ is a stochastic matrix. The stochastic nature of $T_{\rm norm}$ in conjunction with the knowledge of its positivity [by discretizing $S_i(q, \tilde{q})$ such that $g(q_k, q_k + 1)$ from (17) remains bounded and avoiding the case $|q| = 1]$ and the Perron–Frobenius theorem [10] ensure that (23), in fact, has a unique positive steady-state solution. Finally, since these conclusions hold for arbitrary $n$, then by continuity arguments, a unique stationary solution must also exist for (21).

*Remark 1:* $T$ is itself an approximation to a stochastic matrix since from (20) and with $\Delta q$ sufficiently small, $\sum_j S_i(q_j, \tilde{q}_\ell)\Delta\tilde{q} = \sum_j S_i(q_j, \tilde{q}_\ell)\Delta q \approx \int S_i(q, \tilde{q}_\ell)\, dq = 1$ for all $i \in \{1, 2\}$, and $\ell \in \{1, 2, \cdots, n\}$. Consequently, for $j \leq n$, $\sum_{i=1}^{2n} T_{ij} = \sum_{\ell=1}^{n} a_{11} S_1(q_\ell, q_j)\Delta\tilde{q} + \sum_{\ell=1}^{n} a_{21} S_2(q_\ell, q_j)\Delta\tilde{q} \approx a_{11} + a_{21} = 1$. This naturally relies on the fact that $q, \tilde{q}$ are discretized in the same manner.

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