Minimal Positive Realizations of Transfer Functions with Positive Real Poles

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***Abstract*—A standard result of linear-system theory states that a SISO rational nth-order transfer function always has an nth-order realization. In some applications, one is interested in having a realization with nonnegative entries (i.e., a positive system) and it is known that a positive system may not be minimal in the usual sense. In this paper, we give an explicit necessary and sufficient condition for a third-order transfer function with distinct real positive poles to have a third-order positive realization. The proof is constructive so that it is straightforward to obtain a minimal positive realization.**

***Index Terms*—Minimality, positive realizations, positive systems.**

I. Introduction and Motivation

The research presented in this paper was motivated by a research interest of the authors in the area of *positive linear systems* (see, for general overviews, [17], [16], and [9]). Positive systems are, by definition, systems whose variables take only nonnegative values. From a general point of view, they should be considered as very particular. From a practical point of view, however, such systems are anything but particular since positive systems are often encountered in applications.

In fact, positive-system examples include, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems (memories, warehouses, ...), promotional systems, compartmental systems (frequently used when modeling transport and accumulation phenomena of substances in human body), water and atmospheric pollution models, stochastic models where state variables must be nonnegative since they represent probabilities, and many other models commonly used in economy and sociology. Just to cite the most popular, consider the Leontief model used by economists for predicting productions and prices [14], the Leslie model used to study age-structured population dynamics [15], the hidden Markov models [23], [2] mainly adopted for speech recognition, the compartmental models [13] commonly encountered in pharmacokinetics and radio-nuclide tracer dynamics, and the birth and death processes, relevant to the analysis of queueing systems [24]. More recently, one has developed a MOS-based technology for discrete-time filtering, the so-called *charge routing networks*, in which the state variables are positive since they represent quantity of electrical charge [12], [5]. Interestingly enough, the motivation for the realization of filters using combinations of positive systems (see [6]) can also be found in the fields of optical fibers (as explained in the tutorial paper [19]), where positivity plays an important role since the signals are modulated as intensity variations on optical carriers whose coherence time is less than the shortest relevant delay in the system in such a way that they add on an intensity basis.

In recent times, a number of issues regarding positive systems have been addressed (such as positive orthant reachability [26]) but, for a number of reasons, the so-called positive realization problem has been the most studied (see, for example, the references cited in [3], [20], and [11]). In the following, we will restrict ourselves to consider only the SISO discrete-time case. The formulation of this problem is as follows.

A. The Positive-Realization Problem [1], [21]

Given a strictly proper rational transfer function $H(z)$, the triple $\{A, b, c^T\}$ is said to be a *positive realization* if $H(z) = \sum_{k\geq 1} c^T A^{k-1} b z^{-k}$ with A, b, c^T nonnegative (i.e., with nonnegative entries). The positive realization problem consists of providing answers to the questions:

- (*The existence problem*) Is there a positive realization $\{A, b, c^T\}$ of some finite dimension N and how may it be found?
- (*The minimality problem*) What is a minimal value for N?
- (*The generation problem*) How can we generate all possible positive realizations?

In [3] and [11], the existence problem has been completely solved and a means of constructing such realizations is also given there. In this paper, we shall not consider the interesting question of characterizing the relationship between equivalent realizations, and we shall concentrate on what we have termed the *minimality problem* (see, for example [22], [25], and [7]).

Such a problem is a key feature in many applications. Obviously, when designing a filter, one wishes to make use of a number of units that is "small as possible," in order to reduce space occupation and power consumption of the circuits. But this is not the only situation in which minimality is of paramount importance: one may think of the identification problem where one wishes to obtain information of the system structure from data measurements. This consideration alone justifies an effort in finding realization procedures which guarantees minimality. As shown in [8], the problem is quite intriguing, since the positivity constraint required on the system matrices, may "force" a given transfer function to have a minimal positive realization of order much greater than its degree, and, as also shown in [8],

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this possibility is not confined to some "pathological" systems, but seems to be a typical feature of most systems.

We shall give a partial answer to the minimality problem. In fact, we present here a step toward the solution of this problem by giving explicit necessary and sufficient conditions for a given third-order transfer function with distinct positive real poles to be realizable as a positive system of the same order. Such conditions are easily testable and the proof also provides a tool for constructing a positive realization when it exists.

II. Some Preliminaries

Next, we give a quick listing of basic results which will be needed in the sequel.

Lemma 1: [3] Let $H(z)$ be a rational transfer function with nonnegative impulse response. Then $H(z)$ has a positive realization of order N if and only if $c_1 H(c_2 z)$ has a positive realization of order N for any positive constants c_1, c_2.

Proof: If $H(z) = c^T(zI - A)^{-1}b$ where $A \in \mathbb{R}_+^{N\times N}$, $b \in \mathbb{R}_+^N$, $c \in \mathbb{R}_+^N$, then $c_1 H(c_2 z) = c_1 c^T(zI - c_2^{-1}A)^{-1}c_2^{-1}b$ and $\{c_2^{-1}A, c_2^{-1}b, c_1 c\}$ defines a positive realization for $c_1 H(c_2 z)$. ■

A set $\mathcal{K}$ is said to be a *cone* provided that $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$, if $\mathcal{K}$ contains an open ball of $\mathbb{R}^n$ then $\mathcal{K}$ is said to be solid, if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ then $\mathcal{K}$ is said to be *pointed*. A cone which is closed, convex, solid and pointed will be called a *proper cone*. A cone $\mathcal{K}$ is said to be polyhedral if it is expressible as the intersection of a finite family of closed half-spaces. The notation $\mathrm{cone}(v_1, \ldots, v_M)$ indicates the polyhedral closed convex cone consisting of all finite nonnegative linear combinations of vectors $v_1, \ldots, v_M$, and the vectors v_i will be called the generators of the cone.

Theorem 2: [21] Let $H(z)$ be a rational transfer function and let $\{F, g, h^T\}$ be a minimal realization of $H(z)$. Then, $H(z)$ has a positive realization if and only if there exists a polyhedral proper cone $\mathcal{K}$ such that

1) $F\mathcal{K} \subset \mathcal{K}$, i.e., $\mathcal{K}$ is F-invariant;
2) $\mathcal{K} \subset \mathcal{O}$
3) $g \in \mathcal{K}$

where $\mathcal{O} = \{x \mid h^T F^k x \geq 0, k = 0, 1, \ldots\}$ is called the *observability cone*. Moreover a positive realization $\{A, b, c^T\}$ is obtained by solving

$$FK = KA, \quad g = Kb, \quad c^T = h^T K$$

where K is such that $\mathcal{K} = \mathrm{cone}(K)$.

The above theorem provides a geometrical interpretation of the positive realization problem: given any minimal realization of a transfer function, then to any positive realization of order N corresponds an invariant cone $\mathcal{K}$ with N edges satisfying conditions (1)–(3) and vice versa.

III. Main Result

Since the case of first- or second-order transfer functions is trivial (see [3] and [21]), we shall only consider the case of third-order transfer functions in the sequel. We shall also restrict attention to transfer functions with three distinct positive real poles. From Lemma 1 we can assume, without loss of generality, the dominant pole of $H(z)$ to be $\lambda_1 = 1$.

Theorem 3: Let

$$H(z) = \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \frac{r_3}{z-\lambda_3}$$

be a third-order transfer function (i.e., $r_1, r_2, r_3 \neq 0$) with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $H(z)$ has a third-order positive realization if and only if the following conditions hold:

1) $r_1 > 0$
2) $r_1 + r_2 + r_3 \geq 0$
3) $(1-\overline{\eta})r_1 + (\lambda_2 - \overline{\eta})r_2 + (\lambda_3 - \overline{\eta})r_3 \geq 0$
4) $(1-\eta)^2 r_1 + (\lambda_2 - \eta)^2 r_2 + (\lambda_3 - \eta)^2 r_3 \geq 0$ for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$ where[1]

$$\overline{\eta} = \max\left\{\frac{1+\lambda_2+\lambda_3 - 2\sqrt{(\lambda_2-\lambda_3)^2 + (1-\lambda_2)(1-\lambda_3)}}{3}, 0\right\}.$$

Proof: (*Sufficiency*) Lemma 1 and condition (1) allow us to assume, without loss of generality, $r_1 = 1$. Assume conditions (1)–(4) are satisfied. We will prove that one can construct the following positive realization of $H(z)$ by appropriate choice for the real parameters ξ_1 and ξ_2:

$$A = \begin{pmatrix} a_{11}(\xi_1,\xi_2) & 0 & a_{13}(\xi_1) \\ 1 & \xi_2 & 0 \\ 0 & 1 & \xi_1 \end{pmatrix} \tag{1}$$

$$b = \begin{pmatrix} b_1(\xi_1,\xi_2) \\ b_2(\xi_1) \\ 1 + r_2 + r_3 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\begin{aligned} a_{11}(\xi_1,\xi_2) &= 1 + \lambda_2 + \lambda_3 - \xi_1 - \xi_2 \\ a_{13}(\xi_1) &= (1-\xi_1)(\lambda_2 - \xi_1)(\lambda_3 - \xi_1) \\ b_1(\xi_1,\xi_2) &= (1-\xi_1)(1-\xi_2) + (\lambda_2-\xi_1)(\lambda_2-\xi_2)r_2 \\ &\quad + (\lambda_3-\xi_1)(\lambda_3-\xi_2)r_3 \\ b_2(\xi_1) &= 1 - \xi_1 + (\lambda_2-\xi_1)r_2 + (\lambda_3-\xi_1)r_3 \end{aligned}$$

and

$$\xi_1 = \begin{cases} \lambda_3, & \text{if } 1 - \lambda_3 + (\lambda_2 - \lambda_3)r_2 \geq 0 \\ (1 + \lambda_2 r_2 + \lambda_3 r_3)/(1 + r_2 + r_3), & \text{otherwise} \end{cases} \tag{2}$$

and ξ_2 is such that

$$\begin{aligned} f(\xi_1,\xi_2) = -\xi_1^2 - \xi_2^2 - \xi_1\xi_2 + (1+\lambda_2+\lambda_3)(\xi_1+\xi_2) \\ - \lambda_2 - \lambda_3 - \lambda_2\lambda_3 = 0. \end{aligned}$$

It is easily verified that ξ_1 is continuous at $1 - \lambda_3 + (\lambda_2 - \lambda_3)r_2 = 0$.

According to (2), in the following we shall consider two cases.

[1] It is worth noting that indeed $0 \leq \overline{\eta} < \lambda_3$. In fact, the right-hand inequality reduces to $0 < (1-\lambda_2)^2$ with simple calculations.

(Case 1): Assume $1-\lambda_3+(\lambda_2-\lambda_3)r_2 \geq 0$, i.e., $\xi_1=\lambda_3$. Then, $f(\lambda_3,\xi_2) = -(1-\xi_2)(\lambda_2-\xi_2)$ so that by choosing $\xi_2=\lambda_2$, to force $f(\lambda_3,\xi_2)$ to zero, (1) reduces to

$$A=\begin{pmatrix}1&0&0\\1&\lambda_2&0\\0&1&\lambda_3\end{pmatrix},\quad b=\begin{pmatrix}(1-\lambda_3)(1-\lambda_2)\\1-\lambda_3+(\lambda_2-\lambda_3)r_2\\1+r_2+r_3\end{pmatrix}$$
$$c_J^T=(0\quad 0\quad 1).$$

By condition (2), with $r_1=1$, this is a positive realization of $H(z)$.

(Case 2): Assume $1-\lambda_3+(\lambda_2-\lambda_3)r_2 < 0$, i.e., $\xi_1 = (1+\lambda_2 r_2+\lambda_3 r_3)/(1+r_2+r_3)$ [note that, from conditions (2) and (3), $1+r_2+r_3>0$ since we assumed $r_1=1$]. Then, ξ_1 is such that

$$\overline{\eta}\leq\xi_1<\lambda_3 \tag{3}$$

holds. In fact, the left hand inequality readily follows from conditions (2) and (3), and the right-hand inequality from $1-\lambda_3+(\lambda_2-\lambda_3)r_2<0$. With this choice, we obtain with simple manipulations

$$b=\begin{pmatrix}\dfrac{(\lambda_2-\lambda_3)^2r_2r_3+(1-\lambda_2)^2r_2+(1-\lambda_3)^2r_3}{1+r_2+r_3}\\0\\1+r_2+r_3\end{pmatrix}.$$

One can establish nonnegativity of b in the following way. By evaluating the left side of condition (4) (with $r_1=1$) for $\eta=\xi_1$ [see (3)], one obtains a quantity which factorizes as follows:

$$\begin{aligned}&(1-\eta)^2+(\lambda_2-\eta)^2r_2+(\lambda_3-\eta)^2r_3|_{\eta=\xi_1}\\&\quad=\frac{(\lambda_2-\lambda_3)^2r_2r_3+(1-\lambda_2)^2r_2+(1-\lambda_3)^2r_3}{1+r_2+r_3}.\end{aligned}$$

By invoking condition (4) and condition (2) with $r_1=1$, the nonnegativity of b is then immediate. To prove nonnegativity of A in (1), it suffices to note that when $\overline{\eta}\leq\xi_1<\lambda_3$ [as required by (3)], the values of ξ_2 which ensure $f(\xi_1,\xi_2)=0$ are such that $\lambda_2<\xi_2<1$, as shown in Appendix A.

(Necessity): In the sequel we will make use of the Jordan canonical realization $\{A_J,b_J,c_J\}$ of $H(z)$

$$A_J=\begin{pmatrix}1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix},\quad b_J=\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix},\quad c_J=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

and of the corresponding observability cone $\mathcal{O}$ defined as

$$x_1,x_2,x_3\,|\,x_1+\lambda_2^kx_2+\lambda_3^kx_3\geq 0,\qquad k\geq 0. \tag{4}$$

Since $H(z)$ is positively realizable, then its impulse response $h_k=r_1+\lambda_2^{k-1}r_2+\lambda_3^{k-1}r_3$ is nonnegative so that conditions (1) and (2) immediately follows by considering $k\to\infty$ and $k=1$. Lemma 1 and condition (1) again now allow us to assume, without loss of generality, $r_1=1$.

It remains to prove the necessity of conditions (3) and (4).

Throughout the proof we shall assume that $\mathcal{K}=\mathrm{cone}(K)$ is a proper cone with three generators satisfying the conditions of Theorem 2, i.e.,

1) $A_J\mathcal{K}\subset\mathcal{K}$, i.e., $\mathcal{K}$ is A_J-invariant
2) $\mathcal{K}\subset\mathcal{O}$
3) $b_J\in\mathcal{K}$

and from the same theorem we know that such a cone $\mathcal{K}$ exists. Now, what we have to prove is that, given any cone $\mathcal{K}$ satisfying the requirements of Theorem 2, then conditions (3) and (4) follow.

The proof is basically organized as follows. In the *first step* we will consider a special cone $\mathcal{K}^*$, with three generators, satisfying the first two conditions of Theorem 2, i.e.,

1) $A_J\mathcal{K}^*\subset\mathcal{K}^*$, i.e., $\mathcal{K}^*$ is A_J-invariant
2) $\mathcal{K}^*\subset\mathcal{O}$

and in the *second step* we shall make use of the cone $\mathcal{K}^*$ in order to split the proof in two parts: the case in which $\mathcal{K}\subset\mathcal{K}^*$ and the case in which $\mathcal{K}\not\subset\mathcal{K}^*$.

When $\mathcal{K}\subset\mathcal{K}^*$, since by hypothesis $b_J\in\mathcal{K}$, we also have $b_J\in\mathcal{K}^*$ so that, for the first case, we shall prove that $b_J\in\mathcal{K}^*$ implies conditions (3) and (4). On the other hand, for the second case, i.e., $\mathcal{K}\not\subset\mathcal{K}^*$, we shall prove that $b_J\notin\mathcal{K}^*$, but $b_J\in\mathcal{K}\subset\mathcal{O}$ implies conditions (3) and (4) thus finishing the necessity part of the proof.

(Step 1): Consider then the cone

$$\mathcal{K}^*=\mathrm{cone}(K^*)=\mathrm{cone}\begin{pmatrix}1&0&0\\-\dfrac{1-\lambda_3}{\lambda_2-\lambda_3}&1&0\\\dfrac{1-\lambda_2}{\lambda_2-\lambda_3}&-1&1\end{pmatrix}.$$

To show A_J-invariance note that

$$(K^*)^{-1}A_JK^*=\begin{pmatrix}1&0&0\\\dfrac{(1-\lambda_2)(1-\lambda_3)}{\lambda_2-\lambda_3}&\lambda_2&0\\0&\lambda_2-\lambda_3&\lambda_3\end{pmatrix}$$

is a nonnegative matrix so that each A_J-transformed generator of $\mathcal{K}^*$ is a nonnegative combination of the generators of $\mathcal{K}^*$.

To show that $\mathcal{K}^*\subset\mathcal{O}$ it suffices to note that each generator of $\mathcal{K}^*$ lies in $\mathcal{O}$. In fact, for the first generator, the inequalities (4) become

$$1-\lambda_2^k\frac{1-\lambda_3}{\lambda_2-\lambda_3}+\lambda_3^k\frac{1-\lambda_2}{\lambda_2-\lambda_3}=0,\qquad\text{for } k=0,1 \tag{5}$$

and, for $k\geq 2$, we have

$$\begin{aligned}&1-\lambda_2^k\frac{1-\lambda_3}{\lambda_2-\lambda_3}+\lambda_3^k\frac{1-\lambda_2}{\lambda_2-\lambda_3}\\&\quad=1-\lambda_2^m\frac{\lambda_2^2(1-\lambda_3)}{\lambda_2-\lambda_3}+\lambda_3^m\frac{\lambda_3^2(1-\lambda_2)}{\lambda_2-\lambda_3}\\&\quad:=y(m)\geq 0\end{aligned} \tag{6}$$

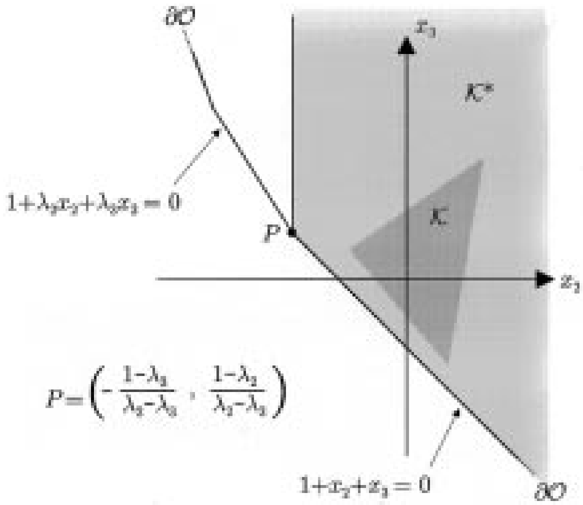

Fig. 1. A planar section ($x_1 = 1$) of the cones $\mathcal{K}$, $\mathcal{K}^*$, and $\mathcal{O}$.

where $m \geq 0$. Nonnegativity of $y(m)$ for $m \geq 0$ follows if one considers $y(m)$ as the impulse response of the following positive system:

$$A_+ = \begin{pmatrix} \lambda_2 & 0 & \lambda_2^2 \dfrac{(1-\lambda_3)(1-\lambda_2)}{\lambda_2-\lambda_3} \\ \lambda_2-\lambda_3 & \lambda_3 & (1-\lambda_2)(1-\lambda_3)(\lambda_2+\lambda_3) \\ 0 & 0 & 1 \end{pmatrix}$$

$$b_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 \\ 1 \\ (1-\lambda_2)(1-\lambda_3) \end{pmatrix}$$

i.e., $y(m) = c_+^T A_+^m b_+$, $m \geq 0$ which can be directly checked with some long but straightforward calculations. For the second and third generator of $\mathcal{K}^*$, inequalities (4) reduce to

$$\begin{aligned} \lambda_2^k - \lambda_3^k \geq 0, &\qquad k \geq 0 \\ \lambda_3^k \geq 0, &\qquad k \geq 0 \end{aligned}$$

which obviously hold.

(Step 2): As previously stated, we shall split the proof in two parts: $\mathcal{K} \subset \mathcal{K}^*$ (case *A*) and $\mathcal{K} \not\subset \mathcal{K}^*$ (case *B*).

(Case A): Assume $\mathcal{K} \subset \mathcal{K}^*$. The scenario is depicted in Fig. 1. Since by hypothesis $b_J \in \mathcal{K}$, then necessarily $b_J \in \mathcal{K}^*$ and we prove next that conditions (3) and (4) hold. To do this let us rewrite b_J (with $r_1 = 1$) as a nonnegative combination of the generators of $\mathcal{K}^*$, i.e.,

$$b_J = \begin{pmatrix} 1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{1-\lambda_3}{\lambda_2-\lambda_3} & 1 & 0 \\ \dfrac{1-\lambda_2}{\lambda_2-\lambda_3} & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$$

with $\alpha, \beta \geq 0$. Hence, conditions (3) and (4) (with $r_1 = 1$) reduce, respectively, to

$$\begin{aligned} &\alpha(\lambda_2-\lambda_3) + \beta(\lambda_3 - \overline{\eta}) \geq 0 \\ &\alpha(\lambda_2-\lambda_3)(\lambda_2+\lambda_3-2\eta) + \beta(\lambda_3-\eta)^2 \\ &\quad + (1-\lambda_2)(1-\lambda_3) \geq 0 \end{aligned}$$

which can be easily seen to hold for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$.

In sum, we have just proved that whenever a cone $\mathcal{K}$ with three generators satisfies the conditions of Theorem 2 and is contained in $\mathcal{K}^*$, then conditions (3) and (4) must hold. It remains to prove now that conditions (3) and (4) still hold when the cone $\mathcal{K}$ is *not* contained in $\mathcal{K}^*$, i.e., case *B* below.

(Case B): Assume $\mathcal{K} \not\subset \mathcal{K}^*$.

Then, $\mathcal{K}$ has *at least* one generator k_i which lies in $\mathcal{O}$, but not in $\mathcal{K}^*$. As shown in Appendix B, any of such generators $k_i = (k_{i1}, k_{i2}, k_{i3})^T$ can be always chosen to satisfy all of the following conditions

$$\begin{aligned} &1)\quad k_{i1} = 1 \\ &2)\quad k_{i2} < -\frac{1-\lambda_3}{\lambda_2-\lambda_3} \\ &3)\quad k_{i1} + \lambda_2^h k_{i2} + \lambda_3^h k_{i3} \geq 0, \qquad \text{for each } h \geq 1. \end{aligned} \tag{7}$$

Now, we will consider *three subcases*: the cone $\mathcal{K}$ has *exactly* one (case *B.1*), two (case *B.2*) or three (case *B.3*) generators satisfying conditions (7). The proof is rather lengthy so that only a sketch of the proof is given here. The interested reader can find a detailed proof in [4].

(Case B.1): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7). Now, *three subcases* can be considered: the cone $\mathcal{K}$ has *exactly* one (case *B.1.1*), two (case *B.1.2*), or three (case *B.1.3*) generators satisfying the following conditions:

$$\begin{aligned} &1)\quad k_{j2} < 0 \\ &2)\quad k_{j3} > 0. \end{aligned} \tag{8}$$

Note that the first condition of (8) and $\mathcal{K} \subset \mathcal{O}$ imply $k_{j1} > 0$. In fact, if $k_{j1} = 0$ inequalities (4) become $k_{j2}\lambda_2^h + k_{j3}\lambda_3^h \geq 0$ which cannot hold for $h \to \infty$, the left-hand term approaching the negative value $k_j \lambda_2^h$. Then, without loss of generality, we shall assume $k_{j1} = 1$.

We shall prove in the following that a cone $\mathcal{K}$ with one generator satisfying conditions (7) and *one* or *three* generators satisfying conditions (8) *cannot fulfill the conditions of Theorem 2*. The remaining case of a cone $\mathcal{K}$ with one generator satisfying conditions (7) and *two* generators satisfying conditions (8) will be split in two subcases. It will be shown that only one of such two subcases admits invariant cones satisfying the conditions of Theorem 2. Finally, it will be proved that in the latter case conditions (3) and (4) must hold. All the various subcases are graphically represented in Fig. 2.

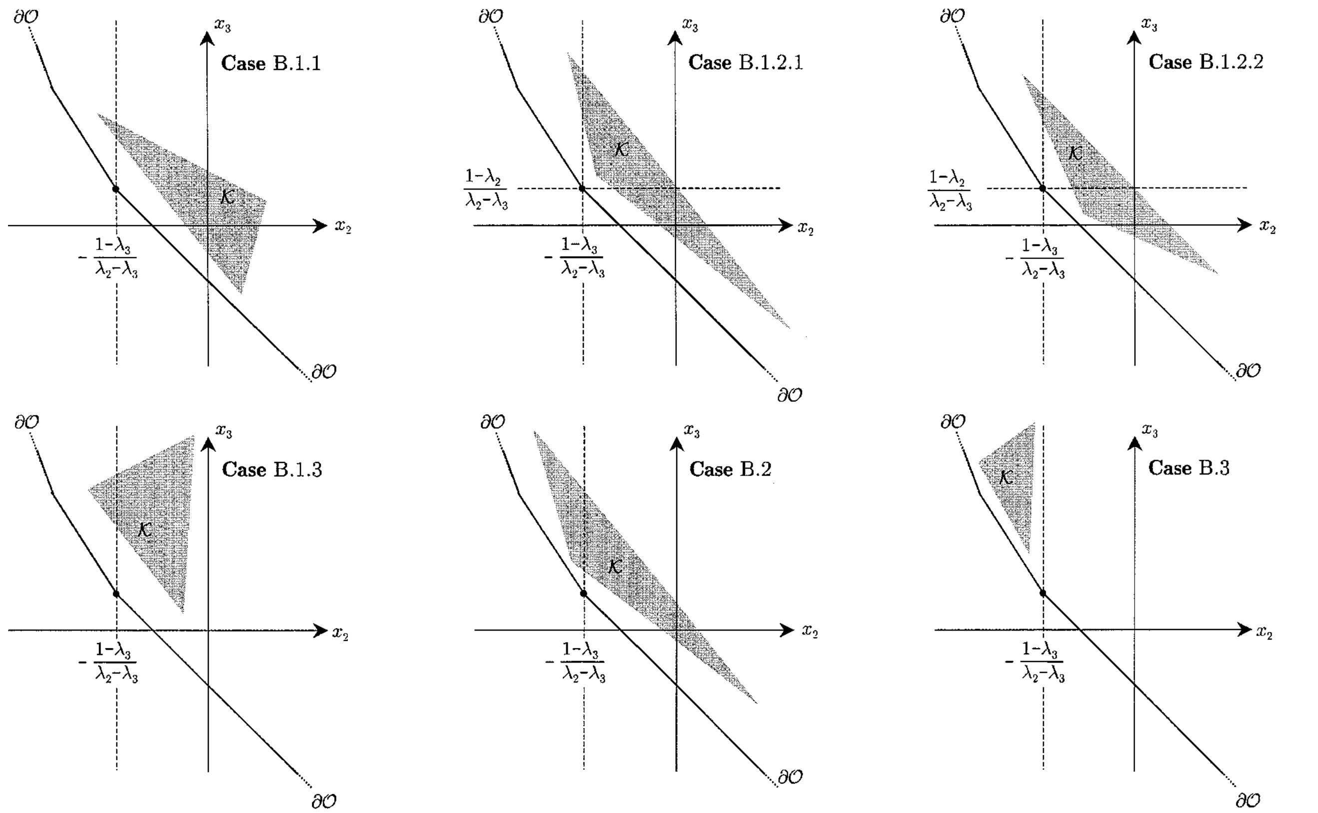

Fig. 2. Planar sections $(x_1 = 1)$ of the cones $\mathcal{K}$ and $\mathcal{O}$ for all the subcases considered analyzing Case B.

(Case B.1.1): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7) and exactly one generator satisfying conditions (8),[2] then such a cone cannot fulfill the conditions of Theorem 2.

The sketch of the proof is as follows: given the cone $\mathcal{K}$ fulfilling the conditions of Theorem 2 and with one generator satisfying conditions (7) and (8), one can prove that it is always possible to construct a larger cone $\mathcal{K}_c$, i.e., $\mathcal{K} \subset \mathcal{K}_c$, with the same properties of $\mathcal{K}$. Then one can prove that such a cone $\mathcal{K}_c$ cannot exist thus, as a consequence, ruling out the existence of $\mathcal{K}$.

(Case B.1.2): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7) and exactly two generators satisfying conditions (8), then such a cone may fulfill the conditions of Theorem 2 only in some particular cases.

To prove this, consider the following *two subcases*: the cone $\mathcal{K}$ has *exactly* one (case *B.1.2.2*) or two (case *B.1.2.1*) generators satisfying the following additional condition:

$$k_{i3} > \frac{1-\lambda_2}{\lambda_2-\lambda_3}. \tag{9}$$

(Case B.1.2.1): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7), exactly two generators satisfying conditions (8) and exactly two generators satisfying conditions (9), then such a cone cannot fulfill the conditions of Theorem 2.

The sketch of the proof is as follows: given the cone $\mathcal{K}$ fulfilling the conditions of Theorem 2 and with the generators satisfying the above conditions, one can prove that it is always possible to construct a larger cone $\mathcal{K}_c$, i.e., $\mathcal{K} \subset \mathcal{K}_c$, with the same properties of $\mathcal{K}$. Then one can prove that such a cone $\mathcal{K}_c$ cannot exist thus, as a consequence, ruling out the existence of $\mathcal{K}$.

(Case B.1.2.2): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7), exactly two generators satisfying conditions (8) and exactly one generator satisfying conditions (9),[3], then such a cone may fulfill the conditions of Theorem 2.

The sketch of the proof is as follows: given $\mathcal{K}$, one can prove that it is always possible to construct a cone $\mathcal{Q} := \mathrm{cone}(q_1, q_2, q_3)$ such that the following properties hold:

$$\begin{aligned} A_J\mathcal{Q} &\subset \mathcal{Q}, \qquad \text{i.e., } \mathcal{Q} \text{ is } A_J\text{-invariant}\\ \mathcal{Q} &\subset \mathcal{O}\\ b_J &\in \mathcal{Q}\\ A_J q_1 &= a_{11}q_1 + a_{21}q_2\\ A_J q_2 &= a_{22}q_2 + a_{32}q_3\\ A_J q_3 &= a_{13}q_1 + a_{33}q_3\\ q_1 &= \left(1, -\frac{1-\lambda_3}{\lambda_2-\lambda_3}, \frac{1-\lambda_2}{\lambda_2-\lambda_3}\right)^T \end{aligned}$$

for some nonnegative a_{ij}.

Then, one can prove that conditions (3) and (4) of the Theorem are satisfied.

[2]Obviously, those two generators necessarily coincide. In fact, from conditions (2) and (3) (for $h = 1$) of (7) it follows that $k_{i3} > (1-\lambda_2)/(\lambda_2-\lambda_3) > 0$ so that the generator k_i satisfying conditions (7) also satisfies both of conditions (8).

[3]Obviously, the generator satisfying (7), say v, coincides with the one satisfying (9) and one of the two satisfying (8), say v, u.

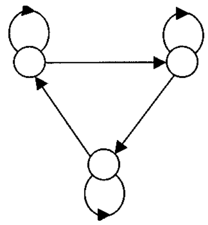

Fig. 3. The structure graph of A.

(Case B.1.3): Assume $\mathcal{K}$ has exactly one generator satisfying conditions (7) and exactly three generators satisfying conditions (8). This is not possible. In fact, from A_J-invariance, since without loss of generality $k_i = (1, k_{i2}, k_{i3})^T$, it follows that

$$\lim_{k\to\infty} A_J^k k_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \subset \mathcal{K} \tag{10}$$

and consequently the vector $(1, 0, 0)^T$ must be a nonnegative combination of the generators of $\mathcal{K}$. Since each generator of $\mathcal{K}$ satisfies conditions (8), then the vector $(1, 0, 0)^T$ must satisfy conditions (8) thus arriving at a contradiction as this is clearly not the case.

(Case B.2): Assume $\mathcal{K}$ has exactly two generators satisfying conditions (7), then such a cone cannot fulfill the conditions of Theorem 2.

The sketch of the proof is as follows: given the cone $\mathcal{K}$ fulfilling the conditions of Theorem 2 and with one generator satisfying conditions (7) and (8), one can prove that it is always possible to construct a larger cone $\mathcal{K}_c$, i.e., $\mathcal{K} \subset \mathcal{K}_c$, with the same properties of $\mathcal{K}$. Then one can prove that such a cone $\mathcal{K}_c$ cannot exist thus, as a consequence, ruling out the existence of $\mathcal{K}$.

(Case B.3): Assume $\mathcal{K}$ has exactly three generators satisfying conditions (7).

This is not possible. In fact, using the same argument of case *B.1.3*, one can show that the vector $(1, 0, 0)^T$ must satisfy conditions (7) thus arriving at a contradiction, condition (2) of (7) being not satisfied. ∎

It is worth noting that according to the sufficiency part of the proof the zero pattern of a positive realization can always be chosen as follows:

$$A = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{pmatrix}, \quad b = \begin{pmatrix} * \\ * \\ * \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}.$$

The structure graph of A is depicted in Fig. 3. In the first case considered in (2), $a_{13} = 0$ and in the second case, $b_2 = 0$. Thus in both cases, there are six zeros among the 15 entries of A, b, and c. When $1 - \lambda_3 + (\lambda_2 - \lambda_3) r_2 = 0$, both $a_{13} = 0$ and $b_2 = 0$.

It is worth rewriting the main result of this paper, i.e., Theorem 3, in terms of other input–output quantities. With obvious substitutions one can easily obtain the following.

Theorem 4: Let

$$H(z) = \sum_{k=1}^{\infty} h_k z^{-k}$$

be a third-order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $H(z)$ has a third-order positive realization, if and only if, the following conditions hold:

1) $h_3 - (\lambda_2 + \lambda_3) h_2 + \lambda_2 \lambda_3 h_1 > 0$
2) $h_1 \geq 0$
3) $h_2 - \overline{\eta} h_1 \geq 0$
4) $h_3 - 2h_2\eta + h_1\eta^2 \geq 0$ for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$ where

$$\overline{\eta} = \max\left\{ \frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3}, 0 \right\}.$$

Theorem 5: Let

$$H(z) = \frac{a_2 z^2 + a_1 z + a_0}{(z-1)(z-\lambda_2)(z-\lambda_3)}$$

be a third-order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $H(z)$ has a third-order positive realization, if and only if, the following conditions hold:

1) $a_0 + a_1 + a_2 > 0$
2) $a_2 \geq 0$
3) $a_1 + (1 + \lambda_2 + \lambda_3 - \overline{\eta}) a_2 \geq 0$
4) $(\eta^2 - 2\alpha_1\eta + \alpha_0) a_2 + (1 + \lambda_2 + \lambda_3 - 2\eta) a_1 + a_0 \geq 0$ for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$ where

$$\overline{\eta} = \max\left\{ \frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3}, 0 \right\}$$

and

$$\alpha_1 = 1 + \lambda_2 + \lambda_3$$
$$\alpha_0 = 1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2.$$

A. An Application: Size Identifiability of Compartmental Models

A *compartmental system* is a system which is made up of a finite number of macroscopic subsystems, called compartments, each of which is homogeneous and well-mixed, and the compartments interact by exchanging material (see, for example, [13], [27]). Since the state variables are the quantities of material contained in each compartment, then compartmental systems belong to the class of positive systems [16]. One of the most important problems in the analysis of compartmental systems is the determination of internal structures, in particular the number of compartments, from input–output observations [18]. In terms of system theory, this problem belongs to that of realization, specifically to that of positive realization.

Usually, the number of compartments is assumed on the basis of some *a priori* knowledge in the field from which the process

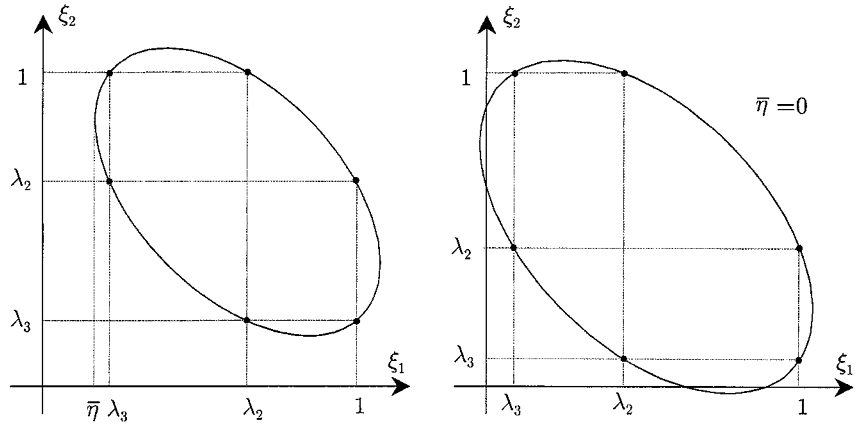

Fig. 4. The case of an ellipse entirely contained in the first quadrant or including points in the 2nd and 4th quadrants.

under study arises. Nevertheless, in many biomedical applications [27], it is unclear whether to include certain compartments in the model that has to be identified. The result presented in this paper allows to evaluate whether a three compartments assumption is consistent with the input–output data measurements.

Consider, for example, the case one has assumed a given process to be described by three compartments and the corresponding estimated impulse response h_k is the following:

$$h_k = 1 - 16 \cdot (0.4)^{k-1} + 75 \cdot (0.2)^{k-1}$$

for $k \geq 1$. Using Theorem 4, one can validate the three compartmental assumpion. The value of $\overline{\eta}$ is 0.0526 and conditions (1)–(3) are immediately verified to hold. By contrast, condition (4) is not fulfilled, for example, when $\eta = 0.15$. As a consequence the three compartments hypothesis results to be false.

IV. Concluding Remarks

In this paper, we have studied the problem of finding a positive realization of minimal order. This problem arises in the context of positive linear systems which are often encountered in applications. We have proved that a simple test on the residues (or impulse response or coefficients of the numerator and poles of the transfer function) can be performed to ascertain whether a given third-order transfer function with positive real pole is minimally positively realizable or not. The proof of the theorem is constructive so that a minimal positive realization can be readily obtained.

Appendix A

To prove nonnegativity of A in (1), we need to determine the locus of points in the ξ_1, ξ_2 plane satisfying $f(\xi_1, \xi_2) = 0$. The form of f makes evident that the locus is an ellipse which is symmetrical around the line $\xi_1 = \xi_2$. The following points are all easily verified to lie on the ellipse: $(\xi_1, \xi_2) = (1, \lambda_2), (1, \lambda_3), (\lambda_2, \lambda_3)$ and their reflections $(\xi_1, \xi_2) = (\lambda_2, 1), (\lambda_3, 1), (\lambda_3, \lambda_2)$. The ellipse always includes points in the first quadrant, and may include points in the 2nd and 4th quadrants, as shown in Fig. 4. In the case of an ellipse contained in the first quadrant, a straightforward calculation establishes that $\overline{\eta}$ is the abscissa of the left most point on the ellipse. The diagrams also make clear that if $\overline{\eta} \leq \xi_1 < \lambda_3$, as required by (3), then ξ_2 necessarily satisfies $\lambda_2 < \xi_2 < 1$.

Appendix B

We prove now that any generator $k_i = (k_{i1}, k_{i2}, k_{i3})^T$ which lies inside $\mathcal{O}$ and outside $\mathcal{K}^*$, can be always chosen to satisfy all of the following conditions

1) $k_{i1} = 1$
2) $k_{i2} < -\dfrac{1-\lambda_3}{\lambda_2 - \lambda_3}$
3) $k_{i1} + \lambda_2^h k_{i2} + \lambda_3^h k_{i3} \geq 0, \qquad$ for each $h \geq 1$.

Assume $k_{i1} = 0$, then $k_i \in \mathcal{K}^*$. In fact, we can write

$$\begin{pmatrix} 0 \\ k_{i2} \\ k_{i3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{1-\lambda_3}{\lambda_2-\lambda_3} & 1 & 0 \\ \dfrac{1-\lambda_2}{\lambda_2-\lambda_3} & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix}$$

with

$$\alpha = k_{i2} \geq 0$$

as one can readily see from inequality (4) when $k \to \infty$, and

$$\beta = k_{i2} + k_{i3} \geq 0$$

as one can readily see from inequality (4) when $k = 0$. Hence, we are allowed to assume without loss of generality $k_{i1} = 1$. Suppose now that

$$k_{i2} \geq -\frac{1-\lambda_3}{\lambda_2-\lambda_3}$$

then $k_i \in \mathcal{K}^*$. In fact, we can write

$$\begin{pmatrix} 1 \\ k_{i2} \\ k_{i3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{1-\lambda_3}{\lambda_2-\lambda_3} & 1 & 0 \\ \dfrac{1-\lambda_2}{\lambda_2-\lambda_3} & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$$

with

$$\alpha = k_{i2} + \frac{1-\lambda_3}{\lambda_2-\lambda_3} \geq 0$$

by assumption, and

$$\beta = k_{i3} + k_{i2} + \frac{1-\lambda_3}{\lambda_2-\lambda_3} - \frac{1-\lambda_2}{\lambda_2-\lambda_3} = 1 + k_{2i} + k_{3i} \geq 0$$

as one can readily see from inequality (4) when $k=0$. It follows that any generator $k_i = (k_{i1}, k_{i2}, k_{i3})^T$ which lies inside $\mathcal{O}$ and outside $\mathcal{K}^*$, can be always chosen to satisfy all of conditions (7).

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