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Brief Paper

Extended H_∞ control — H_∞ control with unstable weights[☆]

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Abstract

This paper presents a complete solution to an extended H_∞ control problem for a generalized plant comprising a physical system cascaded with weights containing poles in the closed right half-plane. Necessary and sufficient conditions for a solution to exist are obtained which guarantee closed-loop stability of the physical system and controller as well as achieving the H_∞ norm constraint on the closed loop of the transfer function matrix of the generalized plant and controller interconnection. The set of all feasible controllers is characterized. Conditions for solvability are a minimal variation on those appearing in the standard H_∞ control problem. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

For the last decade, H_∞ control has been intensively studied (Glover & Doyle, 1988; Doyle, Glover, Khargonekar & Francis, 1989). However, the nonstandard H_∞ problem in which the generalized plant has unstable weights has not been solved completely. Such a nonstandard problem arises naturally in treating the H_∞ servo problem where weights are chosen to have $j\omega$-axis poles (especially at the origin), and also in treating the H_∞ filtering and LTR problems for unstable plants, since the plants in these problems are regarded as weights in the control problem setting. Note that the LMI approach to H_∞ control (Gahinet & Apkarian, 1994; Iwasaki & Skelton, 1994) is not directly applicable to the above nonstandard problem owing to the fact that no internally stabilizing controller exists.

Hosoe, Zhang and Kono (1992) first considered H_∞ control problems with weights having $j\omega$-axis poles, within the scope of the mixed sensitivity problem. Meinsma (1995) treated unstable and nonproper weights also in the mixed sensitivity problem. Mita, Kuriyama and Liu (1993, 1996) dealt with more general generalized plants which have both input and output weights having purely $j\omega$-axis poles, and showed that the solvability condition as well as the controller parametrization of the problem are given by those for the standard problem, provided stabilizing solutions of the two algebraic Riccati equations (AREs) are replaced by the so-called quasi-stabilizing solutions. However, the derivation heavily depended on fulfillment of certain technical conditions and only $j\omega$-axis poles and not open right half-plane poles were taken into account. Kuriyama, Anderson and Mita (1995) solved the same problem without use of these extra assumptions and called it *extended H_∞ control*. However, the related proof was not given in that conference paper.

The purpose of this paper is to give a complete (and comparatively straightforward) proof of the extended H_∞ control result for generalized plants which have weights containing *poles in the closed right half-plane* (excluding $s = \infty$). The theory will be developed by making use of the notion of *comprehensive stability* (Liu, Zhang & Mita, 1995) which ensures internal stability of the closed-loop system except for the weights. The H_∞ servo-control and filtering problems can be easily solved in the framework of extended H_∞ control theory (see, Mita, Xin & Anderson, 1997), and are not discussed here because of space limits.

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This paper is organized as follows. In Section 2, the standard H_∞ control results are reviewed. In Section 3, we formulate the problem to be studied. In Section 4, first, the parametrization of all controllers which guarantee comprehensive stability is given, and next the solution to the extended H_∞ control problem (Theorem 3) is presented. In Section 5, the proof of Theorem 3 is given. Section 6 offers conclusions.

In what follows, we express the star product of M_1 and M_2 by $M = M_1 * M_2$ so that $F_l(M_1, F_l(M_2, K)) = F_l(M, K)$ holds, where $F_l(*, *)$ denotes the standard lower linear fractional map (Zhou, Doyle & Glover, 1995). Moreover, $\lambda_i(A)$ and $\{\lambda_i(A)\}$ denote the ith eigenvalue and the set of all the eigenvalues of a matrix A, respectively. $\mathscr{R}(A)$ denotes the range space of a matrix A. RH_∞ stands for proper and stable rational functions and BH_∞ represents the subset of RH_∞ whose elements have H_∞ norm less than 1.

2. Preliminaries on the standard H_∞ control problem

The generalized plant considered is described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]\begin{bmatrix} w \\ u \end{bmatrix}, \quad (1)$$

where w is the external input, u the control input, z the controlled output, and y the measurement output. It is well known that the *standard H_∞ control* problem for $G(s)$ in (1) is to find the control law $u = K(s)y$ to stabilize the closed-loop system (G, K) and to ensure

$$G_{zw}(s) = F_l(G, K) \in BH_\infty, \quad (2)$$

where G_{zw} is the closed-loop transfer function from w to z.

The following *standard assumptions* are normally used (Glover & Doyle, 1988).

(A1) (A, B_2) is stabilizable and (A, C_2) is detectable;
(A2) the matrices D_{12} and D_{21} are of full column rank and full row rank, respectively; and
(A3) the realizations of $G_{12}(s)$ and $G_{21}(s)$ induced from (1) have no $j\omega$-axis invariant zeros.

Using assumption (A2), we define $D_{12}^\dagger$, $D_{12}^\perp$, $D_{21}^\dagger$, and $D_{21}^\perp$ such that

$$\begin{bmatrix} D_{12}^\dagger \\ D_{12}^\perp \end{bmatrix}[D_{12} \quad (D_{12}^\perp)^T] = I,$$

$$\begin{bmatrix} D_{21} \\ (D_{21}^\perp)^T \end{bmatrix}[D_{21}^\dagger \quad D_{21}^\perp] = I. \quad (3)$$

The key conclusions are in the following result from Glover and Doyle (1988).

Theorem 1. *Suppose that the generalized plant $G(s)$ in* (1) *satisfies assumptions* (A1)–(A3). *Then the standard H_∞ control problem is solvable if and only if the following three conditions are satisfied*:

1. *The following ARE admits a unique stabilizing solution $X \geq 0$*:
$$X(A - B_2 D_{12}^\dagger C_1) + (A - B_2 D_{12}^\dagger C_1)^T X + X(B_1 B_1^T - B_2 E_{12}^{-1} B_2^T)X + C_1^T (D_{12}^\perp)^T D_{12}^\perp C_1 = 0, \quad (4)$$
where $E_{12} = D_{12}^T D_{12}$. By the stabilizing solution, we mean that
$$A_X = A - B_2 D_{12}^\dagger C_1 + (B_1 B_1^T - B_2 E_{12}^{-1} B_2^T)X \quad (5)$$
has all eigenvalues with negative real parts.
2. *The following ARE admits a unique stabilizing solution $Y \geq 0$*:
$$Y(A - B_1 D_{21}^\dagger C_2)^T + (A - B_1 D_{21}^\dagger C_2)Y + Y(C_1^T C_1 - C_2^T E_{21}^{-1} C_2)Y + B_1 D_{21}^\perp (D_{21}^\perp)^T B_1^T = 0, \quad (6)$$
where $E_{21} = D_{21} D_{21}^T$. By the stabilizing solution, we mean that
$$A_Y = A - B_1 D_{21}^\dagger C_2^T + Y(C_1^T C_1 - C_2^T E_{21}^{-1} C_2) \quad (7)$$
has all eigenvalues with negative real parts.
3. *$\rho(YX) < 1$, where $\rho(\cdot)$ denotes the maximum eigenvalue magnitude.*

Moreover, when these conditions hold, every H_∞ controller is expressed by

$$K_\infty(s) = F_l(M^\infty(s), N(s)), \quad (8)$$

where $M^\infty(s)$ and its parameters are given by

$$M^\infty(s) = \left[\begin{array}{c|cc} \hat{A} & -ZL_\infty & Z\hat{B}_2 E_{12}^{-1/2} \\ \hline F_\infty & 0 & E_{12}^{-1/2} \\ -E_{21}^{-1/2}\hat{C}_2 & E_{21}^{-1/2} & 0 \end{array}\right], \quad (9)$$

$$\hat{A} = A + B_1 B_1^T X + B_2 F_\infty + Z L_\infty \hat{C}_2,$$
$$F_\infty = -D_{12}^\dagger C_1 - E_{12}^{-1} B_2^T X,$$
$$L_\infty = -B_1 D_{21}^\dagger - Y C_2^T E_{21}^{-1},$$
$$\hat{B}_2 = B_2 + Y C_1^T D_{12},$$
$$\hat{C}_2 = C_2 + D_{21} B_1^T X, \; Z = (I - YX)^{-1}, \quad (10)$$

and $N(s) \in BH_\infty$ is a free parameter.

3. Problem formulation

Throughout this paper, we consider $G(s)$ in (1) with the structure as shown in Fig. 1, which includes the physical plant $P^0(s)$, input and output weights $W_w(s)$ and $W_z(s)$, with their minimal realizations given as

$$P^0(s) = \begin{bmatrix} P_{11}^0 & P_{12}^0 \\ P_{21}^0 & P_{22}^0 \end{bmatrix} = \left[\begin{array}{c|cc} A^0 & B_1^0 & B_2^0 \\ \hline C_1^0 & 0 & D_{12}^0 \\ C_2^0 & D_{21}^0 & 0 \end{array}\right], \tag{11}$$

$$W_w(s) = \left[\begin{array}{c|c} \hat{A}_w & \hat{B}_w \\ \hline \hat{C}_w & \hat{D}_w \end{array}\right], \quad W_z(s) = \left[\begin{array}{c|c} \hat{A}_z & \hat{B}_z \\ \hline \hat{C}_z & \hat{D}_z \end{array}\right]. \tag{12}$$

Assume that $W_w(s)$ and $W_z(s)$ possess unstable poles, expressed by $\lambda_i(A_w)$ and $\lambda_i(A_z)$, respectively, where A_w and A_z are square matrices with the maximal dimensions satisfying:

$$\hat{A}_w T_w = T_w A_w, \qquad T_z \hat{A}_z = A_z T_z \tag{13}$$

for a full column rank matrix T_w and a full row rank matrix T_z, where $\mathrm{Re}(\lambda_i(A_w)) \geq 0$ and $\mathrm{Re}(\lambda_i(A_z)) \geq 0$ hold.

From Fig. 1, no state of W_w is controllable from u, and no state of W_z is observable from y. Hence, if one of A_w and A_z as defined in (13) is not vacuous, it is impossible to find a controller stabilizing the closed-loop of (G, K). However, since the weighting functions are not part of the closed-loop system involving the physical plant $P^0(s)$ and controller $K(s)$ as shown in Fig. 1, the physical closed-loop system can be stabilized *even if* $W_w(s)$ and $W_z(s)$ have $j\omega$-axis unstable poles. Therefore, we recall the notions of *essential stability* (Mita et al., 1993) and *comprehensive stability* (Liu et al., 1995), respectively, as follows:

Definition 1. The closed-loop system (G, K) in Fig. 1 is essentially stable if the interconnection of the physical plant $P^0(s)$ and controller $K(s)$ is internally stable, or equivalently, if the only non-internally-stable modes of (G, K) are those associated with the input weighting $W_w(s)$ (via A_w) or the output weighting $W_z(s)$ (via A_z).

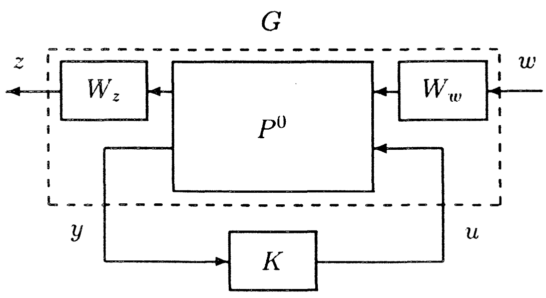

Fig. 1. Closed-loop structure.

Definition 2. The closed-loop system (G, K) in Fig. 1 is said to be comprehensively stable if it is essentially stable, *and* the closed-loop transfer function from w to z (after cancellations) is stable. When this is the case, K is called a comprehensively stabilizing controller.

Now we are ready to formulate the problem to be solved in this paper.

Definition 3. The *extended H_∞ control* problem for the generalized plant $G(s)$ depicted in Fig. 1 is to find a comprehensively stabilizing controller $K(s)$ such that (2) holds.

To solve the extended H_∞ control problem in this paper, according to Fig. 1, with quantities defined in (11)–(12), we assume:

(A0) The state-space matrices in the generalized plant $G(s)$ in (1) are given by

$$G(s) = \left[\begin{array}{ccc|cc} \hat{A}_w & 0 & 0 & \hat{B}_w & 0 \\ B_1^0\hat{C}_w & A^0 & 0 & B_1^0\hat{D}_w & B_2^0 \\ 0 & \hat{B}_z C_1^0 & \hat{A}_z & 0 & \hat{B}_z D_{12}^0 \\ \hline 0 & \hat{D}_z C_1^0 & \hat{C}_z & 0 & \hat{D}_z D_{12}^0 \\ D_{21}^0\hat{C}_w & C_2^0 & 0 & D_{21}^0\hat{D}_w & 0 \end{array}\right]. \tag{14}$$

(A1′)

$$\left(\begin{bmatrix} A^0 & 0 \\ \hat{B}_z C_1^0 & \hat{A}_z \end{bmatrix}, \begin{bmatrix} B_2^0 \\ \hat{B}_z D_{12}^0 \end{bmatrix} \right)$$

is stabilizable and

$$\left([D_{21}^0\hat{C}_w \quad C_2^0], \begin{bmatrix} \hat{A}_w & 0 \\ B_1^0\hat{C}_w & A^0 \end{bmatrix} \right)$$

is detectable. (Obviously, this is the minimum relaxation of (A1) consistent with Fig. 1.)

Also, we continue to impose assumption (A2). As to assumption (A3), its variation will be described in Section 4.2.

4. Solution to the extended H_∞ control problem

In this section, we will first present the necessary and sufficient condition for the existence of the comprehensively stabilizing controller $K(s)$, and provide its parameterization, which is a generalization of the Youla parameterization. Next, we will provide a complete solution to the extended H_∞ control, which is a generalization of the result for the standard H_∞ control.

4.1. Parametrization of comprehensively stabilizing controllers

Theorem 2. *Suppose that the generalized plant G(s) defined in* (1) *satisfies assumptions* (A0), (A1′) *and* (A2). *Let A_w and A_z are square matrices with the maximal dimensions satisfying* (13), *and let $n = \dim(A)$. Then*:

1. *A comprehensively stabilizing controller exists if and only if the following two conditions hold*:
   1-1. *There exists a full column rank matrix V satisfying*

$$(A - B_2 D_{12}^{\dagger} C_1)V = VA_w, \quad D_{12}^{\perp} C_1 V = 0, \tag{15}$$

$$\mathscr{R}(V) \cap \mathscr{C} = \emptyset, \tag{16}$$

   where $\mathscr{C}$ is the controllable space of (A, B_2) defined by $\mathscr{C} = \Sigma_{i=0}^{n-1} A^i \mathscr{R}(B_2)$;
   1-2. *There exists a full row rank matrix U satisfying*

$$U(A - B_1 D_{21}^{\dagger} C_2) = A_z U, \quad U B_1 D_{21}^{\perp} = 0, \tag{17}$$

$$\mathscr{R}(U^{\mathrm{T}}) \cap \mathscr{O} = \emptyset, \tag{18}$$

   where $\mathscr{O}$ is the observable space of (A, C_2) defined by $\mathscr{O} = \Sigma_{i=0}^{n-1} (A^{\mathrm{T}})^i \mathscr{R}(C_2^{\mathrm{T}})$.
2. *When the conditions of assertion* 1 *hold, every comprehensively stabilizing controller K is given by*

$$K(s) = F_l(M(s), Q(s)), \quad Q(s) \in RH_{\infty}, \tag{19}$$

   where

$$M = \left[\begin{array}{c|cc} A + B_2F + HC_2 & -H & B_2 \\ \hline F & 0 & I \\ C_2 & -I & 0 \end{array}\right]. \tag{20}$$

   Here, F is a matrix that satisfies

$$F = -D_{12}^{\dagger} C_1 + F_r, \quad F_r V = 0 \tag{21}$$

   for a parameter matrix F_r and makes all the eigenvalues of $A + B_2 F$ except $\{\lambda_i(A_w)\}$ stable; H is a matrix that satisfies

$$H = -B_1 D_{21}^{\dagger} + H_r, \quad UH_r = 0 \tag{22}$$

   for a parameter matrix H_r and makes all the eigenvalues of $A + HC_2$ except $\{\lambda_i(A_z)\}$ stable. Such matrices F_r and H_r always exist.

Proof. *Sufficiency*: From assumptions (A0) and (A1′), and (15) and (16), we can choose V_2 to construct a nonsingular matrix T as follows:

$$T = [V \quad V_2], \quad T^{-1} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{23}$$

which yields the following canonical form:

$$T^{-1}(A - B_2 D_{12}^{\dagger} C_1)T = \begin{bmatrix} A_w & * \\ 0 & A_2 \end{bmatrix},$$

$$T^{-1}B_2 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad D_{12}^{\perp} C_1 T = [0 \quad D_0], \tag{24}$$

where (A_2, B_{22}) is stabilizable, $D_0 = D_{12}^{\perp} C_1 V_2$. Hence, choosing F in (21) with $F_r = [0 \ F_2]T^{-1}$, where F_2 is any matrix so that $A_2 + B_{22}F_2$ is stable, we obtain

$$T^{-1}(A + B_2 F)T = \begin{bmatrix} A_w & * \\ 0 & A_2 + B_{22}F_2 \end{bmatrix}. \tag{25}$$

It follows that $A + B_2F$ is stable except for the $\{\lambda_i(A_w)\}$. Also, trivially, $F_r V = 0$.

Dually, we can choose U_1 to construct a nonsingular matrix S described as

$$S = \begin{bmatrix} U_1 \\ U \end{bmatrix}, \quad S^{-1} = [W_1 \quad W_2] \tag{26}$$

which yields the following canonical form:

$$S(A - B_1 D_{21}^{\dagger} C_2)S^{-1} = \begin{bmatrix} A_1 & * \\ 0 & A_z \end{bmatrix},$$

$$SB_1 D_{21}^{\perp} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \quad C_2 S^{-1} = [C_{21} \quad 0], \tag{27}$$

where (A_1, C_{21}) is detectable, and $D_1 = U_1 B_1 D_{21}^{\perp}$. By choosing H_1 such that $A_1 + H_1 C_{21}$ is stable, and choosing H in (22) with $H_r = S^{-1}[H_1^{\mathrm{T}} \ 0]^{\mathrm{T}}$, we obtain $UH_r = 0$ and

$$S(A + HC_2)S^{-1} = \begin{bmatrix} A_1 + H_1 C_{21} & * \\ 0 & A_z \end{bmatrix}. \tag{28}$$

Therefore, we can stabilize $A + HC_2$ except for the $\{\lambda_i(A_z)\}$.

Substituting K in (19) into (2) leads to

$$G_{zw} = F_l(G^Q, Q), \qquad G^Q = G * M, \tag{29}$$

where

$$G^Q = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & 0 & D_{12} \\ 0 & -C_2 & -D_{21} & 0 \end{array}\right]. \tag{30}$$

Applying the following similarity transformation:

$$\mathrm{diag}(T, S^{-1}) \tag{31}$$

to G^Q in (30), we obtain

$$G^Q = \left[\begin{array}{cc|cc} A_2 + B_{22}F_2 & -B_{22}FW_1 & B_{12} & B_{22} \\ 0 & A_1 + H_1C_{21} & U_1(B_1 + HD_{21}) & 0 \\ \hline (C_1 + D_{12}F)V_2 & -D_{12}FW_1 & 0 & D_{12} \\ 0 & -C_{21} & -D_{21} & 0 \end{array}\right]. \quad (32)$$

after removing the unobservable mode A_w and the uncontrollable mode A_z, where

$$B_{12} = L_2 B_1. \quad (33)$$

Since the (2,2) block of $G^Q(s)$ is zero, and both $A_2 + B_{22}F_2$ and $A_1 + H_1C_{21}$ are stable, any $Q(s) \in RH_\infty$ is a comprehensively stabilizing controller for $G^Q(s)$ in (30), so is $K(s)$ in (19) for $G(s)$ in Fig. 1.

Necessity: Conditions (15) and (17) were proven in Lemma 2 of Mita et al. (1996) for the case $\mathrm{Re}(\lambda_i(A_w)) = 0$ and $\mathrm{Re}(\lambda_i(A_z)) = 0$. They can be proven very similarly when $\mathrm{Re}(\lambda_i(A_w)) \geq 0$ and $\mathrm{Re}(\lambda_i(A_z)) \geq 0$. The detail is omitted for brevity.

As to condition (16), since $\{\lambda_i(A_w)\}$ define uncontrollable poles of $(A - B_2D_{12}^{\dagger}C_1, B_2)$ from assumption (A1′) and also unobservable poles of $(A - B_2D_{12}^{\dagger}C_1, D_{12}^{\perp}C_1)$ due to (15), then the well-known Kalman decomposition of $(A - B_2D_{12}^{\dagger}C_1, B_2, D_{12}^{\perp}C_1)$ leads to (16). Dually, we can prove (18). □

4.2. Main results

From (15) and (17), we know that the realizations of $G_{12}(s)$ and $G_{21}(s)$ induced from (1) and (14) have $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ as their invariant zeros, respectively. Therefore, when these zeros are on the $j\omega$-axis, the existence of the comprehensively stabilizing controller violates assumption (A3) required for the standard H_∞ control. To get around this difficulty, we introduce the following assumption instead of (A3).

(A3′) The realizations of $G_{12}(s)$ and $G_{21}(s)$ induced from (1) and (14) have no $j\omega$-axis invariant zeros, except for those inherited from the sets $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$, respectively.

Before proceeding further, we recall the notion of quasi-stabilizing solutions of the AREs (4) and (6) (Mita et al., 1993, 1996):

Definition 4. With the quantities defined in condition 1 of Theorem 2,

1. a real symmetrical solution X of (4) is called a quasi-stabilizing solution if it satisfies $XV = 0$ and all the eigenvalues of A_X in (5) have negative real parts except for those inherited from $\{\lambda_i(A_w)\}$;
2. a real symmetrical solution Y of (6) is called a quasi-stabilizing solution if it satisfies $UY = 0$ and all the eigenvalues of A_Y in (7) have negative real parts except for those inherited from $\{\lambda_i(A_z)\}$.

Then we obtain the following lemma:

Lemma 1. *Suppose that the generalized plant $G(s)$ in (1) satisfies assumptions (A0), (A1′), (A2) and (A3′). The quasi-stabilizing solution is unique if it exists.*

Proof. This lemma was proven in Mita et al. (1996) when $\mathrm{Re}(\lambda_i(A_w)) = 0$ and $\mathrm{Re}(\lambda_i(A_z)) = 0$. The same procedure will prove Lemma 1. The detail is omitted for brevity. □

The following result indicates how to solve for these quasi-stabilizing solutions.

Lemma 2. *Suppose that the quantities defined in condition 1 of Theorem 2 hold.*

(1) *The ARE (4) has a quasi-stabilizing solution X if and only if the ARE*

$$X_2A_2 + A_2^{\mathrm{T}}X_2 + X_2(B_{12}B_{12}^{\mathrm{T}} - B_{22}B_{22}^{\mathrm{T}})X_2 + D_0^{\mathrm{T}}D_0 = 0 \quad (34)$$

has a stabilizing solution X_2, where the coefficient matrices in (34) are defined as in (24) and (33). Moreover,

$$X = T^{-\mathrm{T}}\mathrm{diag}(0, X_2)T^{-1} = L_2^{\mathrm{T}}X_2L_2 \quad (35)$$

holds, where T and L_2 are defined as in (23).

(2) *The ARE (6) has a quasi-stabilizing solution Y if and only if the ARE*

$$Y_1A_1^{\mathrm{T}} + A_1Y_1 + Y_1(C_{11}^{\mathrm{T}}C_{11} - C_{21}^{\mathrm{T}}C_{21})Y_1 + D_1D_1^{\mathrm{T}} = 0 \quad (36)$$

has a stabilizing solution Y_1, where the coefficient matrices in (34) are defined as in (27) and $C_{11} = C_1W_1$. Moreover,

$$Y = S^{-1}\mathrm{diag}(Y_1, 0)S^{-\mathrm{T}} = W_1Y_1W_1^{\mathrm{T}} \quad (37)$$

holds, where S and W_1 are defined as in (26).

Proof. (1) This lemma can be proved directly by using Definition 4. The detail is omitted here. □

The solution to the extended H_∞ control problem is now given by the following main theorem with its proof presented in Section 5.

Theorem 3. *Suppose that the generalized plant $G(s)$ in* (1) *satisfies assumptions* (A0), (A1′), (A2) *and* (A3′). *With quantities defined in Theorem 2, then*

1. *the extended H_∞ control problem has a solution if and only if the AREs* (4) *and* (6) *admit quasi-stabilizing solutions $X \geq 0$ and $Y \geq 0$, respectively, such that $\rho(XY) < 1$;*
2. *every extended H_∞ controller is expressed by* (8)–(10) *with the quasi-stabilizing solutions X and Y.*

This theorem extends our previous results (Mita et al., 1993, 1996) in the sense that $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are allowed to have positive real parts and the restrictive conditions of these earlier works $C_2V = 0$ and $UB_2 = 0$ are not used.

5. Proof of Theorem 3

Necessity: Suppose that $K(s)$ is a controller for the extended H_∞ control problem. From Theorem 2, $K(s)$ can be expressed by (19) with F in (21) and H in (22), and $Q(s)$ is stable. Then $G_{zw} = F_l(G^Q, Q) \in BH_\infty$ holds, where G^Q has the realization (32) as exhibited in the proof of Theorem 2.

From (32), (24) and (27), we know that G^Q in (32) satisfying the standard assumptions (A1)–(A3). In what follows, we will complete the proof by applying Theorem 1 to G^Q in (32) for which the standard H_∞ control problem is solvable.

Let X^Q and Y^Q be the stabilizing solution of the ARE obtained by applying conditions 1 and 2 of Theorem 1 to the G^Q in (32), respectively. Using the structure of (32), we obtain that

$$X^Q = \mathrm{diag}(X_2, 0) \geq 0, \tag{38}$$

where X_2 is the stabilizing solution of the ARE (34). It follows from Lemma 2 that X in (35) is the quasi-stabilizing solution of (4).

Next, to calculate Y_Q, we apply the similarity transformation:

$$T^Q = \begin{bmatrix} I & L_2W_1 \\ 0 & I \end{bmatrix} \tag{39}$$

to (32) to obtain

$$G^Q = \left[\begin{array}{cc|cc} A_2 + B_{22}F_2 & -L_2HC_{21} & L_2HD_{21} & B_{22} \\ 0 & A_1 + H_1C_{21} & U_1(B_1 + HD_{21}) & 0 \\ \hline (C_1 + D_{12}F)V_2 & -C_{11} & 0 & D_{12} \\ 0 & -C_{21} & -D_{21} & 0 \end{array}\right] \tag{40}$$

which is convenient for solving the ARE corresponding to (6). Let $\tilde{Y}^Q$ be the stabilizing solution of the ARE obtained by applying condition 2 of Theorem 1 to the G^Q in (40). The dual version of X_Q in (38) implies that

$$\tilde{Y}^Q = \mathrm{diag}(0, Y_1) \geq 0, \tag{41}$$

where Y_1 is the stabilizing solution of the ARE (36). It is clear from Lemma 2 that $Y \geq 0$ in (37) is the quasi-stabilizing solution of (6).

Finally, from

$$Y^Q = T^Q\tilde{Y}^Q T^{Q'}, \tag{42}$$

then the third assertion of Theorem 1 leads to

$$\rho(Y^QX^Q) = \rho(L_2YL_2^{\mathrm{T}}X_2) = \rho(YX) < 1. \tag{43}$$

Sufficiency: From Lemma 2, X_2 as defined in (35) and Y_1 as defined in (37) are the non negatively stabilizing solutions of the AREs (34) and (36), respectively. Further, $A_2 - B_{22}B_{22}^{\mathrm{T}}X_2$ and $A_1 - YC_{21}^{\mathrm{T}}C_{21}$ are stable (Doyle et al., 1989). Then, it is easy to check that F_∞ and L_∞ satisfy (21) and (22). Therefore, in the following, we choose F and H in $M(s)$ in (20) as

$$F = F_\infty, \quad H = L_\infty \tag{44}$$

to obtain $G_{zw} = F_l(G^Q, Q)$ with G^Q in (32).

Now, using X_2 and Y_1, we see that $X^Q \geq 0$ in (38) and $Y^Q \geq 0$ in (42) are the stabilizing solutions for the two AREs corresponding to the generalized plant G^Q in (32), respectively, and satisfy $\rho(Y^QX^Q) < 1$ owing to (43). Therefore, the standard H_∞ problem for the G^Q in (32) is solvable and *every* H_∞ controller is given by the form:

$$Q_\infty(s) = F_l(R^\infty, N), \quad N \in BH_\infty, \tag{45}$$

where R^∞ is calculated from the formulas of (8)–(10). Moreover, since G^Q in (32) is stable and its (2,2) block is zero, then its standard H_∞ controller Q_∞ must be stable. Then, from (19) and (45), we obtain

$$K(s) = F_l(M, Q_\infty) = F_l(M * R^\infty, N), \tag{46}$$

which guarantees the comprehensive stability of (G, K) as well as $G_{zw} = F_l(G^Q, Q_\infty) \in BH_\infty$ for a free parameter $N(s) \in BH_\infty$. Therefore, we only have to verify that M^∞ in (9) and $M(s)$ in (20) with F and H chosen as in (44) satisfy $M^\infty = M * R^\infty$, or equivalently (Zhou et al., 1995):

$$R^\infty = JM^{-1}J * M^\infty, \qquad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{47}$$

The verification of (47) is straightforward calculation, where the similarity transformation (31) is used to delete unobservable mode A_w and the uncontrollable mode A_z in the state-space realization of the right-hand side of (47). This detail is omitted here. □

6. Conclusion

We have presented a complete ARE-based solution to the extended H_∞ control problem for generalized plants having unstable weights by using the notion of comprehensive stability. For such a problem, the LMI approach to H_∞ control (Gahinet & Apkarian, 1994; Iwasaki & Skelton, 1994) is not directly applicable owing to the fact that no internally stabilizing controller exists.

Remaining works include deriving extended H_∞ control solutions for the case where D_{12} and D_{21} no longer are of full column rank and row rank, respectively.

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