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Complete solution of the 4-block H^∞ control problem with infinite and finite jω-axis zeros

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SUMMARY

This paper discusses the 4-block H^∞ control problem with infinite and finite jω-axis invariant zeros in the state-space realizations of the transfer functions from the control input to the controlled output and from the disturbance input to the measurement output, where these realizations are induced from a stabilizable and detectable realization of the generalized plant. This paper extends the DGKF approach to the H^∞ control problem but permitting infinite and finite jω-axis invariant zeros by using the eigenstructures related to these zeros. Necessary and sufficient conditions are presented for checking solvability through checking the stabilizing solutions of two reduced-order Riccati equations and examining matrix norm conditions related to the jω-axis zeros. The parameterization of all suitable controllers is given in terms of a linear fractional transformation involving a certain fixed transfer function matrix and together with a stable transfer function matrix with gain less than 1 which is free apart from satisfying certain interpolation conditions. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: H^∞ control; infinite zeros; finite jω-axis zeros; Riccati equation; parameterization

1. INTRODUCTION

Consider a generalized plant with its stabilizable and detectable realization described as

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]\begin{bmatrix} w \\ u \end{bmatrix} \tag{1}$$

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where $z \in R^m$, $y \in R^q$, $w \in R^r$ and $u \in R^p$ are the controlled output, the measurement output, the disturbance input and the control input, respectively, and $A \in R^{n \times n}$. The H^∞ control problem is to find a *proper* control law $u(s) = K(s)y(s)$ which internally stabilizes the closed-loop system and satisfies $\|\Phi(s)\|_\infty < 1$, where $\Phi(s)$ is the closed-loop transfer function from w to z given by the following *lower* linear fractional transformation:

$$\Phi(s) = F_l(P; K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{2}$$

It is well known that the *standard* H^∞ control problem has been solved for some time [1–5], the word *standard* connoting that plant (1) satisfies the following assumptions:

A1. (A, B_2, C_2) is stabilizable and detectable.
A2. rank $D_{12} = p$, rank $D_{21} = q$.
A3. rank $\begin{bmatrix} -j\omega I + A & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + p$, rank $\begin{bmatrix} -j\omega I + A & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + q$, $\forall \omega$.

The assumption A1 is *necessary* for closed-loop stability. Assumptions A2 and A3 mean that $P_{12}(s)$ and $P_{21}(s)$, with their realizations induced from (1), do not have invariant zeros on the $j\omega$-axis including infinity (denoted by Ω_e in what follows). An H^∞ control problem is called *non-standard* or *singular* if A2 and/or A3 do/does not hold. The non-standard H^∞ control problem, which is often encountered in many practical cases, has attracted considerable research interest. Several methods have been proposed to treat the problem. The first one is to treat the Ω_e zeros by reducing the original problem to one without Ω_e zeros, e.g. through frequency domain loop shifting [6] or the cheap control technique [7]. However, if the modified problem does not have a solution with a given shifting perturbation ε, we do not know whether the modified problem has a solution or not for a smaller perturbation, so we cannot conclude whether or not the original problem has a solution. This disadvantage has been overcome by the analysis of the behaviour of the suboptimal solutions as ε tends to zero in Reference [8]. However, the stability of the closed loop is relaxed by allowing the existence of $j\omega$-axis poles and non-proper controllers in the cited paper. The second method proposed in Reference [9] originally just treats the infinite zeros, i.e. D_{12} is not injective and/or D_{21} is not surjective. The solvability condition is reduced to checking the conditions of two elegant quadratic matrix inequalities (QMIs). Recently, Stoorvogel [10] extended the QMI approach to treat both infinite and finite $j\omega$-axis zeros. Stoorvogel [9] points out that the existence of solutions of the QMI 'can be checked via a state-space transformation and investigating a reduced order Riccati equation'. Stoorvogel [10] points out that 'the quadratic matrix inequality reduces to a Riccati equation (of smaller dimension)' and comments that with zeros on the $j\omega$-axis, 'there do not exist stabilizing solutions of the algebraic Riccati equation'. A third method is to use the notion of J-lossless conjugation and descriptor-form representation of the system. For example, Hara *et al.* [11] gives the solvability condition for the 1-block H^∞ control problem with Ω_e-zeros; and without assumption A2, Xin and Kimura [12] solves the 4-block case based on the (J, J')-lossless factorization developed in Reference [13]. A fourth method is to give the solvability condition by using algebraic Riccati inequalities in References [14–16]. Scherer [15] shows how the QMI arising in a state feedback version of the H^∞ problem with finite $j\omega$-axis zeros can be replaced by an algebraic Riccati equation with stabilizing solution. Scherer [16] characterizes solvability of the H^∞ problem with finite or infinite zeros (see Theorem 13 and related discussion) in terms of the solutions of well-defined reduced-order Riccati equations (which have stabilizing solutions). On the other hand, Scherer [16] just proposes an algorithm to compute an H_{∞} controller without discussion of the parametrization of controllers. Finally, the fifth method uses the LMI approach, where the existence conditions for the general H^{∞} control problem are given in terms of 3 LMIs [17, 18].

In contrast to the various solvability problem approaches, there is little available on the parameterization of all controllers except LMI-based H^{∞} synthesis, where the H^{∞} controllers are parameterized by solutions of 3 LMIs [17, 18]. As to the extension of the Riccati-based parameterization of the controllers for the standard H^{∞} problem in Reference [2] to the singular problem, if one is content to use a perturbation method, it is possible to write down all controllers for the perturbed (standard) system, which controller will then be satisfactory. For a generalized plant without zeros on the imaginary axis, though no explicit characterization of all suitable controllers is given, an interesting characterization of all stable closed-loop systems with a suitable feedback is given in Reference [19], the close-loop systems are parameterized via a stable system with its H_{∞} norm less than 1 and satisfying two other relatively implicit conditions, and an approximate characterization is then given. Apart from these, plus an implicit characterization of the set of controllers for the 1-block problem in Reference [11], the authors are unaware of other contributions to the controller parameterization problem.

In this paper, instead of assumptions A2 and A3, we assume that

A4. $P_{12}(s)$ and $P_{21}(s)$ have full normal column and row ranks, respectively.

Such an assumption allows $P_{12}(s)$ and $P_{21}(s)$ with their realizations induced from (1) to have invariant zeros on Ω_e.

The first purpose of this paper is to extend the DGKF approach [1] to a 4-block H^{∞} control problem without constraints on the infinite or finite jω-axis zeros and to provide the necessary and sufficient conditions for its solvability in terms of *stabilizing* solutions of *Riccati equations* which take the same forms as those in DGKF's paper. At the very least, Theorem 1 in this paper fleshes out using a convenient coordinate based on the key ideas of Stoorvogel [9, 10] and Scherer [15, 16], ensuring that the Riccati equations encountered, because of the existence of stabilizing solutions, are computationally tractable. Moreover, the proof is short and uses a simple adjustment to standard DGKF ideas. However, in terms of the later parts of this paper, the key content of Theorem 1 is, as for the DGKF theory, a recasting of the controller parameterization problem as one for a 1-block plant. This is because this allows easy treatment of the Riccati-based controller parameterization problem, which is the second purpose of this paper. The parameterization of all suitable controllers is given in terms of a linear fractional transformation involving a certain fixed transfer function matrix and together with a stable transfer function matrix with gain less than 1, which is free apart from satisfying certain interpolation conditions related to infinite and finite imaginary zeros.

The paper is organized as follows: some preliminary results are introduced in Section 2, and solvability conditions are given in Section 3 with the proof given briefly in the appendices. The parameterization of all controllers is presented with the discussion of the connection to Reference [19] in Section 4. A numerical example is presented in Section 5. Section 6 concludes this paper.

Notations. The open left half complex plane, open right half complex plane and open complex plane are denoted by C_{-}, C_{+} and C, respectively. The jω − axis and jω − axis with infinity are denoted by Ω and Ω_e, respectively. The set of all $m \times r$ constant real matrices is denoted by $R^{m \times r}$. I_r denotes the identity matrix of size $r \times r$. $RH^{\infty}_{m \times r}$ denotes the set of all $m \times r$ rational stable proper matrices, and $BH^{\infty}_{m \times r}$ denotes the subset of $RH^{\infty}_{m \times r}$ with H^{∞}-norm less than 1. $\sigma(A)$ denotes the set of all eigenvalues of a matrix A. $\rho(X)$ is the maximum eigenvalue of X. $|A|$ denotes the determinant of a matrix A. Im A and Ker A denote the image space and null space of a matrix A, respectively. A^{T} denotes the transpose of a matrix A. We denote $G^{\sim}(s) := G^{\mathrm{T}}(-s)$ and express the star product of $M_1(s)$ and $M_2(s)$ by $M_1(s) * M_2(s)$ so that

$$F_l(M_1(s), F_l(M_2(s), K(s))) = F_l(M_1(s) * M_2(s), K(s)) \tag{3}$$

holds.

2. PRELIMINARIES

This section serves to define some key quantities used subsequently, through the introduction of several matrices and two special co-ordinate bases.

2.1. *Infinite eigenstructures*

Denote the system matrix pencils of $P_{12}(s)$ and $P_{21}^{\mathrm{T}}(s)$ with their realizations induced from (1) as $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$, respectively, where

$$P_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_A := \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \tag{4}$$

$$\tilde{P}_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{P}_A := \begin{bmatrix} A^{\mathrm{T}} & C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} & D_{21}^{\mathrm{T}} \end{bmatrix} \tag{5}$$

According to assumption A4, the above two pencils have full normal column rank.

Let $(v_1^1, \ldots, v_p^1)$ be a base of Ker P_E. Then the infinite eigenvectors of $-sP_E + P_A$ are defined in a manner virtually identical to the square pencil case [20, 21] by

$$P_E v_j^1 = 0, \qquad j = 1, \ldots, p \tag{6}$$

$$P_E v_j^{k+1} = P_A v_j^k, \qquad k = 1, \ldots, k_j - 1 \tag{7}$$

where $v_j^{k_j}$ is the last (highest) one of each infinite eigenvector chain, satisfying $P_A v_j^{k_j} \notin \mathrm{Im}\, P_E$. Now construct

$$V_\infty := [V_{\mathrm{r}} \quad V_{\mathrm{h}}] \tag{8}$$

where $V_{\mathrm{h}} \in R^{(n+p)\times p}$ contains all the *last* (*highest*) infinite eigenvectors and V_{r} contains all the remaining eigenvectors (in any order). From (7), we know that $P_A V_{\mathrm{r}} \in \mathrm{Im}\, P_E$ which leads to $[C_1 \quad D_{12}]\, V_{\mathrm{r}} = 0$; hence we can decompose $P_A V_\infty$ as

$$P_A V_\infty = \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} [V_{\mathrm{r}} \quad V_{\mathrm{h}}] =: \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix} \tag{9}$$

which yields

$$T := [A \quad B_2]\, V_{\mathrm{r}}, \qquad \hat{B}_2 := [A \quad B_2]\, V_{\mathrm{h}}, \qquad \hat{D}_{12} := [C_1 \quad D_{12}]\, V_{\mathrm{h}} \tag{10}$$

It follows from Lemma C.2 in Reference [22] that $\hat{D}_{12}$ has full column rank. Note that we can use Algorithm 4.1 in Reference [23] to present a numerically stable computation of the matrices in (10), see Reference [24].

Dually consider $P_{21}^{\mathrm{T}}(s)$. Now arrange all the infinite eigenvectors of $-s\tilde{P}_E + \tilde{P}_A$ as

$$\tilde{V}_\infty := [\tilde{V}_{\mathrm{r}} \quad \tilde{V}_{\mathrm{h}}] \tag{11}$$

where $\tilde{V}_{\mathrm{h}} \in R^{(n+q)\times q}$ contains all the *last* (*highest*) infinite eigenvectors and $\tilde{V}_{\mathrm{r}}$ contains all the remaining eigenvectors in any order. Dually to (10), we obtain

$$\tilde{T} := [A^{\mathrm{T}} \quad C_2^{\mathrm{T}}]\, \tilde{V}_{\mathrm{r}}, \qquad \hat{C}_2 := \tilde{V}_{\mathrm{h}}^{\mathrm{T}} \begin{bmatrix} A \\ C_2 \end{bmatrix}, \qquad \hat{D}_{21} := \tilde{V}_{\mathrm{h}}^{\mathrm{T}} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \tag{12}$$

It follows that $\hat{D}_{21}$ has full row rank.

2.2. *Finite* jω-*axis eigenstructures*

Let the jω-axis eigenspaces of $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$ be spanned by real matrices

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix}$$

respectively. It follows that there exist real matrices Λ_j and $\tilde{\Lambda}_j$ such that $\sigma(\Lambda_j) \subset \Omega$ and $\sigma(\tilde{\Lambda}_j) \subset \Omega$ hold, and

$$(-sP_E + P_A)\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}(-sI + \Lambda_j) \tag{13}$$

$$(-s\tilde{P}_E + \tilde{P}_A)\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 \\ 0 \end{bmatrix}(-sI + \tilde{\Lambda}_j) \tag{14}$$

Note that T_i and $\tilde{T}_i$ $(i = 1, 2)$ can be calculated by using numerically stable Algorithm 4.5 in Reference [23].

2.3. *Stable eigenstructures*

From the realization of $P_{12}(s)$ induced from (1), a realization of $P_{12}^{\mathrm{T}}(-s)\,P_{12}(s)$ is readily found, with its system matrix

$$W_{12}(s) := \begin{bmatrix} -sI + A & 0 & B_2 \\ -C_1^{\mathrm{T}}C_1 & -sI - A^{\mathrm{T}} & -C_1^{\mathrm{T}}D_{12} \\ D_{12}^{\mathrm{T}}C_1 & B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}}D_{12} \end{bmatrix} \tag{15}$$

Similarly, for $P_{21}(-s)\,P_{21}^{\mathrm{T}}(s)$, we obtain its system matrix

$$W_{21}(s) := \begin{bmatrix} -sI + A^{\mathrm{T}} & 0 & C_2^{\mathrm{T}} \\ -B_1B_1^{\mathrm{T}} & -sI - A & -B_1D_{21}^{\mathrm{T}} \\ D_{21}B_1^{\mathrm{T}} & C_2 & D_{21}D_{21}^{\mathrm{T}} \end{bmatrix} \tag{16}$$

Let real matrices

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix}$$

span the stable eigenspaces of $W_{12}(s)$ and $W_{21}(s)$, respectively (corresponding to eigenvalues in C_-). Then there exist real stable Λ_{12} and Λ_{21} such that

$$W_{12}(s)\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix}(-sI + \Lambda_{12}) \tag{17}$$

$$W_{21}(s)\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ 0 \end{bmatrix}(-sI + \Lambda_{21}) \tag{18}$$

Note that U_i and $\tilde{U}_i$ $(i = 1, 2, 3)$ can also be calculated by using numerically stable Algorithm 4.5 in Reference [23].

From Xin and Mita [25], we can obtain:

Lemma 1

Consider the generalized plant (1) under assumptions A1 and A4. Let the infinite, finite $j\omega$-axis and stable eigenstructures be defined as described above. Define

$$S := [U_1 \quad T_1 \quad T], \qquad \tilde{S} := [\tilde{U}_1 \quad \tilde{T}_1 \quad \tilde{T}] \tag{19}$$

Then S and $\tilde{S}$ are square and non-singular.

The matrices S and $\tilde{S}$ effectively define two special co-ordinate bases which are useful in this paper.

3. SOLVABILITY CONDITIONS

Now, we are ready to state the following result, which fleshes out in a conventional co-ordinate basis the key ideas of Stoorvogel [9, 10] and Scherer [15, 16].

Theorem 1

Under assumptions A1 and A4, the H^∞ control problem for the plant $P(s)$ in (1) is solvable if and only if the following four statements hold:

(i) The following Riccati equation has a stabilizing solution $X_r \geqslant 0$:

$$\begin{aligned} &X_r(A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_{r1}) + (A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_{r1})^{\mathrm{T}}X_r \\ &\quad + X_r(B_{r1}B_{r1}^{\mathrm{T}} - \hat{B}_{r2}E_{12}^{-1}\hat{B}_{r2}^{\mathrm{T}})X_r + C_{r1}^{\mathrm{T}}(I - \hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}})C_{r1} = 0 \end{aligned} \tag{20}$$

where $E_{12} := \hat{D}_{12}^{\mathrm{T}}\hat{D}_{12}$, and

$$A_r := L_1 A U_1, \qquad B_{r1} := L_1 B_1 \tag{21}$$

$$\hat{B}_{r2} := L_1 \hat{B}_2, \qquad C_{r1} := C_1 U_1 \tag{22}$$

$$[L_1^{\mathrm{T}} \quad L_2^{\mathrm{T}} \quad L_3^{\mathrm{T}}]^{\mathrm{T}} := S^{-1} = [U_1 \quad T_1 \quad T]^{-1} \tag{23}$$

where T, $\hat{B}_2$ and $\hat{D}_{12}$ are defined in (10), T_1 is defined in (13), and U_1 is defined in (17), respectively.

(ii) The following Riccati equation has a stabilizing solution $Y_r \geqslant 0$:

$$Y_r(\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^{\mathrm{T}}E_{21}^{-1}\hat{C}_{r2})^{\mathrm{T}} + (\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^{\mathrm{T}}E_{21}^{-1}\hat{C}_{r2})Y_r$$
$$+ Y_r(\tilde{C}_{r1}^{\mathrm{T}}\tilde{C}_{r1} - \hat{C}_{r2}^{\mathrm{T}}E_{21}^{-1}\hat{C}_{r2})Y + \tilde{B}_{r1}(I - \hat{D}_{21}^{\mathrm{T}}E_{21}^{-1}\hat{D}_{21})\tilde{B}_{r1}^{\mathrm{T}} = 0 \tag{24}$$

where $E_{21} := \hat{D}_{21}\hat{D}_{21}^{\mathrm{T}}$, and

$$\tilde{A}_r := \tilde{U}_1^{\mathrm{T}} A \tilde{L}_1^{\mathrm{T}}, \qquad \tilde{B}_{r1} := \tilde{U}_1^{\mathrm{T}} B_1, \tag{25}$$

$$\hat{C}_{r2} := \hat{C}_2 \tilde{L}_1^{\mathrm{T}}, \qquad \tilde{C}_{r1} := C_1 \tilde{L}_1^{\mathrm{T}}, \tag{26}$$

$$[\tilde{L}_1^{\mathrm{T}} \quad \tilde{L}_2^{\mathrm{T}} \quad \tilde{L}_3^{\mathrm{T}}]^{\mathrm{T}} := \tilde{S}^{-1} = [\tilde{U}_1 \quad \tilde{T}_1 \quad \tilde{T}]^{-1} \tag{27}$$

where $\tilde{T}$, $\hat{C}_2$ and $\hat{D}_{21}$ are defined in (12), $\tilde{T}_1$ is defined in (14), and $\tilde{U}_1$ is defined in (18), respectively.

(iii) $\rho(XY) < 1$, where

$$X := S^{-\mathrm{T}} \operatorname{diag}\{X_r, 0, 0\} S^{-1} = L_1^{\mathrm{T}} X_r L_1 \tag{28}$$

$$Y := \tilde{S}^{-\mathrm{T}} \operatorname{diag}\{Y_r, 0, 0\} \tilde{S}^{-1} = \tilde{L}_1^{\mathrm{T}} Y_r \tilde{L}_1 \tag{29}$$

(iv) Define the following new plant as:

$$P_n(s) = \begin{bmatrix} P_{n11}(s) & P_{n12}(s) \\ P_{n21}(s) & P_{n22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_n & B_{n1} & B_{n2} \\ \hline C_{n1} & 0 & N_{12} \\ C_{n2} & N_{21} & 0 \end{array}\right] \tag{30}$$

with its matrices in turn defined by

$$A_n := A + B_1 B_1^{\mathrm{T}} X + ZYF_\infty^{\mathrm{T}} F_\infty, \quad Z = (I - YX)^{-1} \tag{31}$$

$$B_{n1} := -ZL_\infty, \quad B_{n2} := B_2 - ZYF_\infty^{\mathrm{T}} N_{12} \tag{32}$$

$$C_{n1} := -F_\infty, \quad C_{n2} := (C_2 - N_{21} L_\infty^{\mathrm{T}} XZ) Z^{-1} \tag{33}$$

$$F_\infty := -E_{12}^{-1/2}(\hat{B}_2^{\mathrm{T}} X + \hat{D}_{12}^{\mathrm{T}} C_1), \quad L_\infty := -(Y\hat{C}_2^{\mathrm{T}} + B_1 \hat{D}_{21}^{\mathrm{T}}) E_{21}^{-1/2} \tag{34}$$

$$N_{12} := E_{12}^{-1/2} \hat{D}_{12}^{\mathrm{T}} D_{12}, \quad N_{21} := D_{21} \hat{D}_{21}^{\mathrm{T}} E_{21}^{-1/2} \tag{35}$$

Then for any jω-axis invariant zero of the realization of either $P_{n12}(s)$ or $P_{n21}(s)$ as induced from (30), call it jω_i, there holds

$$U_{12i}^{*} U_{12i} > X_{12i}^{*} B_{n1} B_{n1}^{\mathrm{T}} X_{12i} \tag{36}$$

or

$$U_{21i}^* U_{21i} > X_{21i}^* C_{n1}^{\mathrm{T}} C_{n1} X_{21i} \tag{37}$$

where

$$\begin{bmatrix} -\mathrm{j}\omega_i I + A_n^{\mathrm{T}} & C_{n1}^{\mathrm{T}} \\ B_{n2}^{\mathrm{T}} & N_{12}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} X_{12i} \\ U_{12i} \end{bmatrix} = 0 \tag{38}$$

or

$$\begin{bmatrix} -\mathrm{j}\omega_i I + A_n & B_{n1} \\ C_{n2} & N_{21} \end{bmatrix} \begin{bmatrix} X_{21i} \\ U_{21i} \end{bmatrix} = 0 \tag{39}$$

where in (38) and (39), the columns of

$$\begin{bmatrix} X_{12i} \\ U_{12i} \end{bmatrix} \text{ and } \begin{bmatrix} X_{21i} \\ U_{21i} \end{bmatrix}$$

form a minimal spanning set for the null-space of the matrices in the relevant equations.

In case these conditions hold, the H^∞ control problems for the 4-block plant $P(s)$ in (1) and the 1-block plant $P_n(s)$ in (30) are solvable with the same controller $K(s)$; moreover, for $P_n(s)$ in (30), the triple (A_n, B_{n2}, C_{n2}) is stabilizable and detectable.

Proof. See Appendices A and B. □

Remark 3.1

The Riccati equation (20) is of size $n - (n_\infty - p) - n_j$ where $n_\infty := \sum_{j=1}^{p} k_j \geqslant p$, with the fact that $k_j \geqslant 1$ $(j = 1, \ldots, p)$ in (7) having been used, and n_j are the dimensions of the infinite and finite jω-axis eigenspaces of $-sP_E + P_A$, respectively. A similar analysis can be given for the size of the Riccati equation (24). The jω_i satisfying (38) and/or (39) are the invariant zeros of the realization of $P_{n12}^{\mathrm{T}}(s)$ and/or $P_{n21}(s)$ obtained from (30), respectively. It is claimed in Lemma A3 in Appendix A that the jω_i are the invariant zeros of the realization of $P_{12}(s)$ and/or $P_{21}(s)$ induced from (1). Therefore, all conditions in Theorem 1 can be checked easily by solving the stabilizing solutions of two reduced-order Riccati equations and checking static conditions related to finite jω-axis zeros of the realization of $P_{12}(s)$ and/or $P_{21}(s)$ induced from (1).

Remark 3.2

If assumption A2 holds, V_r in (8) and $\tilde{V}_r$ in (11) are vacuous, and we can choose $V_\infty = V_h = [0 \quad I_p]^{\mathrm{T}}$ and $\tilde{V}_\infty = \tilde{V}_h = [0 \quad I_q]^{\mathrm{T}}$. Hence, we obtain from (10) and (12) that $\hat{B}_2 = B_2$, $\hat{D}_{12} = D_{12}$, $\hat{C}_1 = C_1$, $\hat{D}_{21} = D_{21}$. If assumption A3 holds, A4 holds trivially, and condition (*iv*) is vacuous. If assumptions A2 and A3 hold, we can choose $S = U_1 = \tilde{S} = \tilde{U}_1 = I_n$, Theorem 1 then reduces to the result for the standard H^∞ control problem, see e.g. Reference [2].

Remark 3.3

If conditions (i)–(iii) in Theorem 1 hold, $P_n(s)$ can be defined as in (30). It should be noted that $P_n(s)$ is equal to Σ_{PQ} in Reference [10]. If condition (iv) also holds, in Appendix B, we show a new property of $P_n(s)$ in (30) that the triple (A_n, B_{n2}, C_{n2}) is stabilizable and detectable. Such a property is necessary to extend the result of the characterization of all closed-loop transfer function matrices generated by all suitable H^∞ controllers in Reference [19] to generalized plants with zeros on the imaginary axis. See Section 4.3 for details.

4. PARAMETERIZATION OF ALL H^∞ CONTROLLERS

4.1. A formula

To present the parameterization of all H^∞ controllers, we need to consider the model matching problem related to $P_n(s)$ in (30). To begin with, bring in a doubly coprime factorization of $P_{n22}(s)$ as Reference [27]:

$$P_{n22}(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \tag{40}$$

and

$$\begin{bmatrix} \tilde{U}(s) & -\tilde{V}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}\begin{bmatrix} M(s) & V(s) \\ N(s) & U(s) \end{bmatrix} = I \tag{41}$$

where the 8 transfer functions matrices in (41) are in RH^∞. Now define

$$\begin{aligned} T_1(s) &:= P_{n11}(s) + P_{n12}(s)M(s)\tilde{V}(s)P_{n21}(s) \\ T_2(s) &:= P_{n21}(s)M(s) \\ T_3(s) &:= \tilde{M}(s)P_{n21}(s) \end{aligned} \tag{42}$$

Next, from Theorem 1, we have the first implicit characterization of the set of H^∞ controllers:

Theorem 2

Suppose that the H^∞ control problem is solvable for the generalized plant (1) under assumptions A1 and A4. Let $P_n(s)$ be defined as in the statement of Theorem 1. Then all H^∞ controllers are given by

$$K(s) = \mathrm{HM}(\Pi(s), S(s)) := (\Pi_{11}(s)S(s) + \Pi_{12}(s))(\Pi_{21}(s)S(s) + \Pi_{22}(s))^{-1} \tag{43}$$

where

$$\Pi(s) = \begin{bmatrix} \Pi_{11}(s) & \Pi_{12}(s) \\ \Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \tag{44}$$

$$:= \begin{bmatrix} P_{n12}^{-1}(s) & -P_{n12}^{-1}(s)P_{n11}(s) \\ P_{n22}(s)P_{n12}^{-1}(s) & P_{n21}(s) - P_{n22}(s)P_{n12}^{-1}(s)P_{n11}(s) \end{bmatrix} \tag{45}$$

$$= \left[\begin{array}{cc|cc} -sI + A_n & B_{n2} & 0 & B_{n1} \\ C_{n1} & N_{12} & -I_p & 0 \\ \hline 0 & I_p & 0 & 0 \\ C_{n2} & 0 & 0 & N_{21} \end{array}\right] \tag{46}$$

and $S(s) \in BH^{\infty}$ is such that

$$Q(s) = -T_2^{-1}(s)(S(s) - T_1(s))T_3^{-1}(s) \in RH^{\infty} \tag{47}$$

holds, where $T_i(s)$ $(i = 1, 2, 3)$ are defined in (42). The inverses of $T_2(s)$ and $T_3(s)$ are guaranteed to exist.

Proof. Since the realizations of $P_{n12}(s)$ and $P_{n21}(s)$ induced from (30) have no invariant zeros in C_+ by Lemma A3, at every point in C_+, the transfer function matrices are invertible. From (42), we know that the inverses of $T_2(s)$ and $T_3(s)$ exist.

Necessity: Suppose that $P_n(s)$ defined in Theorem 1 is a transfer function matrix for which

$$\begin{bmatrix} z_n \\ y \end{bmatrix} = P_n(s) \begin{bmatrix} w_n \\ u \end{bmatrix} \tag{48}$$

Now suppose that $K_0(s)$ $(u = K_0(s)y)$ is an H^{∞} controller. Then $K_0(s)$ stabilizes the closed loop such that the transfer function from w_n to z_n satisfies

$$S_0(s) := F_l(P_n(s), K_0(s)) \in BH^{\infty} \tag{49}$$

Rewriting (48) as follows:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \Pi(s) \begin{bmatrix} z_n \\ w_n \end{bmatrix} \tag{50}$$

where $\Pi(s)$ is defined in (45), we obtain $K_0(s) = \mathrm{HM}(\Pi(s), S_0(s))$ (see the definition of 'HM' in (43) and (44)). In other words, any controller solving the H^{∞} problem can be expressed by (43) with $S_0(s) \in BH^{\infty}$. Now, as to the descriptor-form realization of $\Pi(s)$ in (46), it can be derived easily in the following way: Let x_n be a state of $P_n(s)$ in (30), and choose u as another state x_u. Then from (48) and (30), we obtain

$$\begin{bmatrix} \dot{x}_n \\ 0 \\ u \\ y \end{bmatrix} = \begin{bmatrix} -sI + A_n & B_{n2} & 0 & B_{n1} \\ C_{n1} & N_{12} & -I_p & 0 \\ 0 & I_p & 0 & 0 \\ C_{n2} & 0 & 0 & N_{21} \end{bmatrix} \begin{bmatrix} x_n \\ x_u \\ z_n \\ w_n \end{bmatrix} \tag{51}$$

which completes the derivation of (46).

Next, since $K_0(s)$ is a stabilizing controller, owing to the Youla parameterization in Reference [27], with the quantities defined by (41), there must exist a $Q_0(s) \in RH^{\infty}$ such that

$$K_0(s) = (-MQ_0 + V)(-NQ_0 + U)^{-1} = HM\left(\begin{bmatrix} -M & V \\ -N & U \end{bmatrix}, Q_0\right) \tag{52}$$

Again from Francis [27], it follows from (42) that the transfer function from w_n to z_n satisfies

$$S_0(s) = T_1(s) - T_2(s)Q_0(s)T_3(s) \tag{53}$$

which follows that $-T_2^{-1}(s)(S_0(s) - T_1(s))T_3^{-1}(s) = Q_0(s) \in RH^{\infty}$ holds.

Sufficiency: Suppose that $K(s)$ is given by (43) with (44)–(47) holding. Then from (48) and (50) at once the transfer function from w_n to z_n satisfies

$$S(s) = F_l(P_n(s), K(s)) \in BH^\infty \tag{54}$$

Now (47) can be rewritten as

$$S(s) = \mathrm{HM}(\Pi_Y, Q) \tag{55}$$

where

$$\Pi_Y = \begin{bmatrix} -T_2 & T_1 T_3^{-1} \\ 0 & T_3^{-1} \end{bmatrix} \tag{56}$$

Therefore, from (43), we obtain

$$K(s) = \mathrm{HM}(\Pi, \mathrm{HM}(\Pi_Y, Q)) = \mathrm{HM}(\Pi\Pi_Y, Q) \tag{57}$$

From (45), (42) and (41), a direct calculation yields

$$\Pi\Pi_Y = \begin{bmatrix} -M & V \\ -N & U \end{bmatrix} \tag{58}$$

Therefore, $K(s)$ in (43) is proper and stabilizes the closed-loop system, being given by the Youla parameterization of (52) with $Q(s)$ replacing $Q_0(s)$. Hence, together with (54), $K(s)$ is a desired H^∞ controller. □

From Theorem 2, note that if $\Pi_{22}(s)$ in (45) is invertible, or equivalently $P_n(s)$ in (30) is invertible as a rational function, by direct calculation, the controllers in (43) can be presented by a *lower* linear fractional transformation as follows:

Theorem 3

With the quantities as defined in Theorem 2, if $P_n(s)$ is invertible as a rational function, then all H^∞ controllers as given by (43) are also given by

$$K(s) = F_l(M^\infty(s), S(s)) \tag{59}$$

where $S(s) \in BH^\infty$ is such that (47) holds, and

$$M^\infty(s) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} P_n^{-1}(s) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A_n & B_{n2} & B_{n1} \\ \hline C_{n2} & 0 & N_{21} \\ C_{n1} & N_{12} & 0 \end{array}\right]^{-1} \tag{60}$$

Proof. If $K(s)$ in (43) holds, then we obtain (54). Based on this equation, $K(s)$ can be obtained as (59).

Remark 4.1

Under assumptions A1 and A4, if the H^∞ control problem is solvable, we can only present the parameterization of all H^∞ controllers as described in (43). If both assumptions A2 and A3 hold, for all $S(s) \in BH^\infty$, $K(s)$ in (43) or (59) is an H^∞ controller as $K(s)$ now also achieves stability constraint. This is equivalent to the result for the standard H^∞ control problem, as in e.g. Reference [2].

The rather implicit specification of all solutions in Theorem 2 will be replaced by conditions involving interpolation constraints on $S(s)$ in the next subsection.

4.2. Interpolation conditions

The following steps are taken to find explicit constraints on $S(s)$ equivalent to the implicit constraint of (47), with Steps 1 and 2 being according to Hara *et al.* [11].

Step 1: Obtain the Smith form of $T_i(s)$ $(i = 2, 3)$ over RH^∞ as

$$T_2(s) = U_{2L}(s)\Lambda_2(s)U_{2R}(s) = U_{2L}(s)V_2(s)W_2(s)U_{2R}(s) \tag{61}$$

$$T_3(s) = U_{3L}(s)\Lambda_3(s)U_{3R}(s) = U_{3L}(s)W_3(s)V_3(s)U_{3R}(s) \tag{62}$$

where $U_{iL}(s)$ and $U_{iR}(s)$, $i = 2, 3$, are unimodular matrices, with $\Lambda_i(s) \in RH^\infty$, and with V_i, W_i diagonal matrices such that $V_i \in RH^\infty$ have all zeros in C_- and $W_i \in RH^\infty$ have all zeros on Ω_e.

Step 2: Let

$$\hat{T}_2(s) := U_{2L}(s)V_2(s), \qquad \hat{T}_3(s) := V_3(s)U_{3R}(s) \tag{63}$$

Then $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are unimodular in RH^∞. Evidently, $Q(s)$ in (47) reduces to

$$Q = -U_{2R}^{-1}W_2^{-1}\hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1}W_3^{-1}U_{3L} \tag{64}$$

Note that the matrices W_2, W_3 have zeros on Ω_e. With U_{2R}, U_{3L} unimodular over RH^∞ and $Q \in RH^\infty$ it is necessary that the zeros on Ω_e of W_2, W_3 are those of $\hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1}$.

Step 3: Suppose, in particular, that z_2 is a zero of the ith diagonal entry of W_2 of order j, but is not a zero of any entry of W_3. Let e_i be a column vector with its ith entry 1 and remaining entries 0. Using the unimodularity over RH^∞ of $\hat{T}_3^{-1}$, it follows that if $z_2 \in \Omega$, then

$$\frac{d^k}{ds^k}[e_i^T\hat{T}_2^{-1}(S - T_1)]|_{z_2} = 0, \quad k = 0, 1, \dots, j-1 \tag{65}$$

If $z_2 = \infty$, then

$$\lim_{s\to\infty} s^{j-1}e_i^T\hat{T}_2^{-1}(S - T_1) = 0 \tag{66}$$

Step 4: Similarly, if z_3 is a zero of the pth diagonal entry of W_3 of order q, but is not a zero of any entry of W_2, and if $z_3 \in \Omega$, then

$$\frac{d^k}{ds^k}[(S - T_1)\hat{T}_3^{-1}e_p]|_{z_3} = 0, \quad k = 0, 1, \dots, q-1 \tag{67}$$

If $z_3 = \infty$, then

$$\lim_{s\to\infty} s^{q-1}(S - T_1)\hat{T}_3^{-1}e_p = 0 \tag{68}$$

Step 5: If z_1 is a jth-order zero of the ith diagonal entry of W_2 and a qth-order zero of the pth diagonal entry of W_3, then the (i, p) entry of $\hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1}$ must have a zero z_1 of order $j + q$.

Therefore, if $z_1 \in \Omega$, in addition to (65) and (67), we have

$$\frac{d^k}{ds^k}[e_i^T \hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1}e_p]|_{z_1} = 0, \quad k = 0, 1, \ldots, j + q - 1 \tag{69}$$

If $z_1 = \infty$, in addition to (66) and (68), we obtain

$$\lim_{s \to \infty} s^{j+q-1} e_i^T \hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1}e_p = 0 \tag{70}$$

We have now proved the 'only if' part of the following theorem. The proof of the 'if' part follows the theorem statement.

Theorem 4

With the quantities defined in Theorem 2, $S(s) \in BH^\infty$ such that (47) holds if and only if $S(s) \in BH^\infty$ satisfies interpolation conditions of the type depicted as in (65)–(70).

Proof. ('If' statement). Suppose that the interpolation constraints hold. Then it is not difficult to see that necessarily $\hat{T}_2^{-1}(S - T_1)\hat{T}_3^{-1} = W_2 R W_3$ for some $R \in RH^\infty$. It follows that $Q = U_{2R}^{-1} R U_{3L}^{-1} \in RH^\infty$ owing to the fact that U_{2R}, U_{3L} are unimodular. This completes the proof. □

4.3. Connection to the characterization of all closed-loop transfer function matrices

We discuss the characterization of all closed-loop transfer function matrices generated by all suitable H^∞ controllers in relation to the results developed in Reference [19].

First, we show that the closed-loop transfer function matrices generated by all suitable H^∞ controllers can be directly parameterized by $S(s)$ introduced in Theorems 2 and 4. Let $\mathbf{K}$ be the set of all suitable H^∞ controllers. Define two sets as

$$\mathbf{\Phi} = \{\Phi \mid \Phi = F_l(P, K), K \in \mathbf{K}\}, \qquad \mathbf{\Phi}_n = \{\Phi_n \mid \Phi = F_l(P_n, K), K \in \mathbf{K}\} \tag{71}$$

Owing to two lossless factorizations as described in (A1) and (A2), we know that $\mathbf{\Phi} = \mathbf{\Phi}_n$ holds. Suppose that $K(s)$ is given by (43) with (45)–(47) holding. From (54), $\Phi_n = S = F_l(P_n, K)$ holds. Therefore, we obtain

$$\mathbf{\Phi} = \mathbf{\Phi}_n = \{S \mid S \in BH^\infty \text{ satisfies}(47)\} = \{S \mid S \in BH^\infty \text{ satisfies}(65)\text{–}(70)\} \tag{72}$$

Next, we discuss the connection between the above results and those in Reference [19]. To begin, we recall the following results from Stoorvogel *et al.* [19]. Suppose that the H^∞ control problem for the generalized plant (1) without zeros on the imaginary axis is solvable. Let $\mathscr{X} \in BH^\infty$ with a realization

$$\left[\begin{array}{c|c} A_x & B_x \\ \hline C_x & D_x \end{array}\right]$$

where A_x is stable. Using $\mathscr{X}$ and P_n, Stoorvogel *et al.* [19] constructs a system named Σ_a (see (4.2) of the cited paper with the related notation Σ_{PQ} being used rather than P_n) as follows:

$$\dot{x}_a = \begin{bmatrix} A_x & 0 \\ 0 & A_n \end{bmatrix} x_a + \begin{bmatrix} B_x \\ B_{n1} \end{bmatrix} w + \begin{bmatrix} 0 \\ B_{n2} \end{bmatrix} u$$

$$z = [C_x \quad -C_{n1}]\ x_a + D_x w - N_{12} u \tag{73}$$
$$y = [0 \quad C_{n2}]\ x_a + N_{21} w$$

Then Theorem 4.1 in Reference [19] shows that $\mathbf{\Phi}$ in (71) can be parameterized via $\mathscr{X} \in \mathscr{BH}^{\infty}$ satisfying two other implicit conditions in terms of subspace inclusions. These two conditions are necessary and sufficient to guarantee that the disturbance decoupling problem with measurement feedback and internal stability (DDPMS) for system Σ_a in (73) is solvable. Here, DDPMS for system Σ_a is to find $u = Ky$ such that the internal stability is guaranteed and the transfer function matrix from w to z is zero, i.e. $\mathscr{X} - F_l(P_n, K) = 0$.

Now, we can claim that the above result of Stoorvogel *et al.* [19] can be extended to the generalized plant (1) satisfying assumptions A1 and A4. Indeed, in this case, if the H_∞ control problem is solvable, then we can obtain P_n in (30) with (A_n, B_{n2}, C_{n2}) being stabilizable and detectable. From Stoorvogel and van der Woude [28] and Stoorvogel *et al.* [19], these conditions combined with the above-mentioned two implicit conditions involving subspace inclusion guarantee that DDPMS for system Σ_a in (73) is solvable. In this way, we can extend the result of Stoorvogel *et al.* [19] to the generalized plants with zeros on the imaginary axis.

Moreover, using Youla parameterization in (42) for P_n, we know that the DDPMS for system Σ_a in (73) is solvable if and only if there exists $Q_x \in RH^{\infty}$ such that $\mathscr{X} - (T_1 - T_2 Q_x T_3) = 0$, i.e., $Q_x = -T_2^{-1}(\mathscr{X} - T_1)T_3^{-1} \in RH^{\infty}$. Therefore, $\mathbf{\Phi}$ in (71) can be parameterized by $\mathscr{X} \in \mathscr{BH}^{\infty}$ satisfying (47) with $\mathscr{X}$ replacing S. Hence, the characterization of $\mathbf{\Phi}$ given in Reference [19] can be obtained by using Theorem 2. Note that the implicit condition (47) has been replaced by the explicit ones given in Theorem 4.

5. AN EXAMPLE

Consider the 4-block plant $P(s)$ in (1) as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{74}$$

$$C_2 = [1 \quad 2 \quad 1], \quad D_{21} = [0 \quad 0], \quad D_{22} = 0$$

It is easy to check that assumptions A1 and A4 of Section 1 hold for the plant in (74). However, the realizations of both $P_{12}(s)$ and $P_{21}(s)$ induced from (1) with matrices given in (74) have invariant zeros at infinity and $s = 0$, respectively.

First, we calculate the infinite eigenspace of $-sP_E + P_A$ defined in (4) with matrices given in (74). From $P_E v_1^1 = 0$, we obtain $v_1^1 = [0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}}$. From $P_E v_1^2 = P_A v_1^1$, we get $v_1^2 = [1 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}$. Since $P_A v_1^2 \notin \operatorname{Im} P_E$, we know that v_1^2 is the last (highest) infinite eigenvector. Then from (8), we obtain $V_h = v_1^2$ and $V_r = v_1^1$. From (10), we obtain

$$T = [1 \quad 1 \quad 1]^{\mathrm{T}}, \quad \hat{B}_2 = [1 \quad 3 \quad 4]^{\mathrm{T}}, \quad \hat{D}_{12} = [1 \quad 1]^{\mathrm{T}}$$

Since $s = 0$ is an invariant zero of $-sP_E + P_A$, from (13) we obtain $T_1 = [4 \quad 0 \quad 1]^T$. Calculating (17), we obtain that $\Lambda_{12} = -4$ and $U_1 = [1 \quad 2/3 \quad 5/8]^T$. Therefore, it follows that the Riccati equation (20) has a stabilizing solution $X_r = 0.1112$. Hence, from (28),

$$X = \begin{bmatrix} 0.4449 & 1.3348 & -1.7798 \\ 1.3348 & 4.0044 & -5.3393 \\ -1.7798 & -5.3393 & 7.1190 \end{bmatrix}$$

Dually, we can see that the Riccati equation (24) has a semi-positively stabilizing solution for the plant in (74), and from (29),

$$Y = \begin{bmatrix} 0.0178 & 0.0089 & -0.0356 \\ 0.0089 & 0.0044 & -0.0178 \\ -0.0356 & -0.0178 & 0.0712 \end{bmatrix}$$

then $\rho(YX) = 0.8731 < 1$. Thus, we have established that conditions (i)–(iii) of Theorem 1 hold for the plant (74).

Now we will check condition (iv) of Theorem 1. To begin with, we obtain that the realization of $P_n(s)$ defined in (30) has constituent matrices

$$A_n = \begin{bmatrix} 6.0673 & 11.4123 & -20.2692 \\ 3.5292 & 7.6928 & -10.1168 \\ -10.1168 & -22.7713 & 44.4673 \end{bmatrix}, \quad B_{n1} = \begin{bmatrix} -2.7399 \\ -1.4407 \\ 5.7627 \end{bmatrix}, \quad B_{n2} = B_2 \tag{75}$$

$$C_{n1} = [-1.8877 \quad -4.2489 \quad 7.5509], \quad C_{n2} = C_2, \quad N_{12} = N_{21} = 0 \tag{76}$$

Note that $s = 0$ and -4 are invariant zeros of $P_{n12}(s)$ and $P_{n21}(s)$ induced from (30) with the matrices in (75) and (76). Then, as to $s = 0$, we obtain from (38) and (39) that $X_{12} = [-0.0174 \quad 0.1559 \quad -0.1385]^T$, $U_{12} = 0.9779$, $X_{21} = [0.0625 \quad 0.0207 \quad -0.1038]^T$, $U_{21} = 0.9924$. Hence, we can verify that (36) and (37) hold. Therefore, the H^∞ control problem for plant (74) is solvable.

Next, we will find the parameterization of all controllers by using Theorems 2 and 4. Since all H^∞ controllers in (43) are readily written out (the detailed form of $\Pi(s)$ in (45) is omitted for brevity), we will only work out the explicit constraints on $S(s)$ described as in Section 4.2. To this end, we obtain $T_i(s)$ $(i = 1, 2, 3)$ in (42) by using (41) or using the equivalent state-space method given in Reference [27], i.e. with F and H being any matrices which stabilize $A_{nF} := A_n + B_{n2}F$ and $A_{nH} := A_n + HC_{n2}$, then

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} := \left[\begin{array}{cc|cc} A_{nF} & -B_{n2}F & B_{n1} & B_{n2} \\ 0 & A_{nH} & B_{nH} & 0 \\ \hline C_{nF} & -N_{12}F & 0 & N_{12} \\ 0 & C_{n2} & N_{21} & 0 \end{array}\right]$$

where $C_{nF} := C_{n1} + N_{12}F$, $B_{nH} := B_{n1} + HN_{21}$. In this example, we find F and H such that $\sigma(A_{nF}) = \sigma(A_{nH}) = \{-1, -2, -4\}$ using the Matlab function *acker.m*. We obtain

$$T_1(s) = \frac{54.8065(s+60.94)(s-0.0364)(s+0.03281)}{(s+2)^2(s+1)^2} \tag{77}$$

$$T_2(s) = \frac{1.4142s}{(s+2)(s+1)}, \qquad T_3(s) = \frac{0.14142s}{(s+2)(s+1)} \tag{78}$$

Since $T_2(s)$ and $T_3(s)$ are already in their Smith forms, it follows that $\hat{T}_2(s) = \hat{T}_3(s) = 1$. Since $s = 0$ and ∞ are zeros of both $T_2(s)$ and $T_3(s)$ of order 1, respectively, it follows from (65)–(70) that

$$S(0) = T_1(0) = -0.9972, \qquad \left.\frac{d}{ds}S(s)\right|_{s=0} = \left.\frac{d}{ds}T_1(s)\right|_{s=0} = -0.0223 \tag{79}$$

$$\lim_{s\to\infty} S(s) = \lim_{s\to\infty} T_1(s) = 0, \qquad \lim_{s\to\infty} sS(s) = \lim_{s\to\infty} sT_1(s) = 54.8065 \tag{80}$$

Equation (80) implies that the relative degrees of $S(s)$ is 1.

Let us now look for a particular $S(s)$. It is easy to see that there exists no first or second-order stable system $S(s)$ satisfying (79) and (80). Therefore, considering the third-order system, we find an $S(s) \in BH^\infty$ satisfying interpolation conditions (79) and (80) as

$$S(s) = \frac{54.8065(s^2 - 20s - 1)}{s^3 + 80s^2 + 1097.9778s + 54.9604}, \qquad \|S(s)\|_\infty = 0.9972 \tag{81}$$

To summarize the above results, the parameterization of all H^∞ controllers is given by (43) with $S(s) \in BH^\infty$ satisfying the interpolation conditions (79) and (80), a particular $S(s)$ is given in (81).

6. CONCLUSIONS

The contributions of this paper are twofold. First, using eigenstructures related to the infinite and finite jω-zeros of the realizations of $P_{12}(s)$ and $P_{21}(s)$ induced from (1), we have presented necessary and sufficient conditions for the solvability of an H^∞ control problem under no more conditions than stabilizability and detectability (which cannot be dispensed with) and a full rank condition on $P_{12}(s)$ and $P_{21}(s)$. The conditions involve Riccati equations (and stabilizing solutions thereof) together with some matrix inequalities linked to the jω-axis zeros of the realizations of $P_{12}(s)$ and $P_{21}(s)$ induced from (1), and are easily obtained by modifying the DGKF theory.

Second, we characterize all controllers, in terms of the satisfaction of some interpolation conditions. We have no simple state-space formulae for the set of all controllers, and our characterization is not one where setting a parameter to zero, one recovers something like a central or minimum entropy controller. Indeed, finding any controller could be quite tedious, and calculating all H^∞ controllers in a numerically stable way must be the subject of future work.

APPENDIX A. PROOF OF NECESSARY PART OF THEOREM 1

The proof of the necessity of Theorem 1 can be summed up in the following steps.

Step 1: Prove condition (i) via the solvability of the full information problem corresponding to $P(s)$ in (1).

Step 2: Show the existence of the lossless factorization

$$P(s) = \Theta(s) * P_{\text{tmp}}(s) \tag{A1}$$

where $\Theta(s)$ is an inner matrix and $P_{\text{tmp}}(s)$ is a 2-block plant.

Step 3: Prove the conditions (ii) and (iii) via the solvability of the full information problem corresponding to $P_{\text{tmp}}^{\text{T}}(s)$.

Step 4: Show the existence of the lossless factorization

$$P_{\text{tmp}}^{\text{T}}(s) = \Psi^{\text{T}}(s) * P_n^{\text{T}}(s) \tag{A2}$$

where $\Psi^{\text{T}}(s)$ is an inner matrix, $P_n(s)$ is the 1-block plant defined as in (30).

Step 5: Prove condition (iv) via the static solvability conditions related to the jω-axis zeros of the realization of $P_{n12}(s)$ or $P_{n21}(s)$ induced from (30).

The details of Steps 1–3, which are not exactly the same as those in the references, are given. Since Step 4 is just a copy of Step 2 and Step 5 is a direct application of Theorem 6 in Reference [11], the details of Steps 4 and 5 are omitted.

Proof of condition (*i*) *of Theorem 1.* Let $\bar{x}$ be the state of system (1). Define

$$[x_1^{\text{T}} \quad x_2^{\text{T}} \quad x_3^{\text{T}}]^{\text{T}} = S^{-1}\bar{x} \tag{A3}$$

Perform the similarity transformation S on (1), use (23) and consider just x_1 and z:

$$\begin{aligned} \dot{x}_1 &= L_1 A U_1 x_1 + L_1 A T_1 x_2 + L_1 A T x_3 + L_1 B_1 w + L_1 B_2 u \\ z &= C_1 U_1 x_1 + C_1 T_1 x_2 + C_1 T x_3 + D_{12} u \end{aligned} \tag{A4}$$

From Lemma 3 in Reference [25], there exists a matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $A_{11} \in R^{n_r \times n_r}$ and $A_{22} \in R^{p \times p}$ such that

$$\begin{bmatrix} -sI + A & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix} \begin{bmatrix} -sI + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{A5}$$

$$\begin{vmatrix} -sI + A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \neq 0, \quad \forall s \in C \tag{A6}$$

Based on these two equations and (13), there follows

$$\begin{aligned} \dot{x}_1 &= L_1 A U_1 x_1 + L_1 B_1 w + L_1 \hat{B}_2 \hat{u} \\ z &= C_1 U_1 x_1 + \hat{D}_{12} \hat{u} \end{aligned} \tag{A7}$$

where $\hat{u} := -A_{22}T_2x_2 + A_{21}x_3 + A_{22}u$. Evidently, if the full information H^∞ control problem for $P_{\mathrm{FI}}(s)$ is solvable with control u, then the full information H^∞ control problem for $P_{\mathrm{FI}r}(s)$ is solvable with control $\hat{u}$, where $P_{\mathrm{FI}r}(s)$ is defined by using notations in (21), (22) and (10) as

$$P_{\mathrm{FI}r}(s) := \left[\begin{array}{c|cc} A_r & B_{r1} & \hat{B}_{r2} \\ \hline C_{r1} & 0 & \hat{D}_{12} \end{array}\right] \tag{A8}$$

Next, by using (A5), (A6), (14) and (17), we can show that the realization of

$$P_{12r}(s) := \left[\begin{array}{c|c} A_r & \hat{B}_{r2} \\ \hline C_{r1} & \hat{D}_{12} \end{array}\right] \tag{A9}$$

is a stabilizable realization and has no finite $j\omega$-axis invariant zeros.

Therefore, condition (i) of Theorem 1 is a direct consequence of the standard H^∞ control problem [1] owing to the fact that $P_{\mathrm{FI}r}(s)$ in (A8) is a standard plant. □

Note that the Riccati equation (20) for X_r is of size less than n. For later use, we can also construct the following Riccati equation of size n, whose proof is straightforward and is omitted here.

Lemma A1

Suppose that the 4-block H^∞ control problem is solvable for the generalized plant (1) under assumptions A1 and A4. Then, with quantities as defined in condition (i) of Theorem 1 and Lemma 1,

$$\begin{aligned} X(A - \hat{B}_2E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_1) + (A - \hat{B}_2E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_1)^{\mathrm{T}}X \\ + X(B_1B_1^{\mathrm{T}} - \hat{B}_2E_{12}^{-1}\hat{B}_2^{\mathrm{T}})X + C_1^{\mathrm{T}}(I - \hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}})C_1 = 0 \end{aligned} \tag{A10}$$

has a solution X as defined by (28) which yields

$$XT_1 = 0, \qquad XT = 0 \tag{A11}$$

Further, with

$$A_X := A - \hat{B}_2E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_1 + (B_1B_1^{\mathrm{T}} - \hat{B}_2E_{12}^{-1}\hat{B}_2^{\mathrm{T}})X \tag{A12}$$

there holds

$$S^{-1}A_XS = \begin{bmatrix} A_{Xr} & 0 & 0 \\ * & \Lambda_j & 0 \\ * & * & A_{11} \end{bmatrix} \tag{A13}$$

where Λ_j is defined in (13) and A_{11} is defined in (A5), and

$$A_{Xr} = L_1A_XU_1 = A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^{\mathrm{T}}C_{r1} + (B_{r1}B_{r1}^{\mathrm{T}} - \hat{B}_{r2}E_{12}^{-1}\hat{B}_{r2}^{\mathrm{T}})X_r \tag{A14}$$

is stable.

Based on the result of Lemma A1, we have the following result for the lossless factorization of $P(s)$.

Lemma A2

Suppose that the 4-block H^∞ control problem is solvable for the generalized plant (1) under assumptions A1 and A4. With quantities as defined in condition (i) of Theorem 1, and X in Lemma A1, $P(s)$ can be factorized as (A1) with

$$\Theta(s)=\begin{bmatrix}\Theta_{11}(s) & \Theta_{12}(s)\\ \Theta_{21}(s) & \Theta_{22}(s)\end{bmatrix}=\left[\begin{array}{c|cc} A_r+\hat{B}_{r2}\hat{F}_r & B_{r1} & \hat{B}_{r2}E_{12}^{-1/2}\\ \hline C_{r1}+\hat{D}_{12}\hat{F}_r & 0 & \hat{D}_{12}E_{12}^{-1/2}\\ -B_{r1}^{\mathrm{T}}X_r & I_r & 0\end{array}\right] \tag{A15}$$

$$P_{\mathrm{tmp}}(s)=\left[\begin{array}{c|cc} A+B_1B_1^{\mathrm{T}}X & B_1 & B_2\\ \hline -F_\infty & 0 & N_{12}\\ C_2+D_{21}B_1^{\mathrm{T}}X & D_{21} & 0\end{array}\right] \tag{A16}$$

where $\hat{F}_r := -E_{12}^{-1}(\hat{B}_{r2}^{\mathrm{T}}X_r+\hat{D}_{12}^{\mathrm{T}}C_{r1})$. In this case, $\Theta(s)$ is a lossless matrix, i.e. $\Theta^{\sim}(s)\Theta(s)=I$, $\Theta(s)\in RH^\infty$ and $\Theta_{21}^{-1}(s)\in RH^\infty$. Moreover, the triple $(A+B_1B_1^{\mathrm{T}}X, B_2, C_2+D_{21}B_1^{\mathrm{T}})$ is stabilizable and detectable.

Proof. Since ARE (28) takes the same form as that in Reference [1], it then follows from a fairly standard calculation, see e.g. [26], that the properties of $\Theta(s)$ hold. We can show that (A1) holds by using the star product formula of the LFT and a similarity transformation.

Next, we will show that $(A+B_1B_1^{\mathrm{T}}X, B_2, C_2+D_{21}B_1^{\mathrm{T}}X)$ is stabilizable and detectable. Suppose that $(A+B_1B_1^{\mathrm{T}}X, B_2)$ is uncontrollable at λ_0 with $\mathrm{Re}[\lambda_0]\geqslant 0$. Since the H^∞ control problem for $P_{\mathrm{tmp}}(s)$ is solvable, then λ_0 must be an uncontrollable or unobservable mode of realization (A16) of $P_{\mathrm{tmp}}(s)$, i.e. $(A+B_1B_1^{\mathrm{T}}X, [B_1 \quad B_2])$ is uncontrollable or $(A+B_1B_1^{\mathrm{T}}X, [-F_\infty^T \quad (C_2+D_{21}B_1^{\mathrm{T}}X)^{\mathrm{T}}]^{\mathrm{T}})$ is unobservable at λ_0. Otherwise, there does not exist a stabilizing controller for $P_{\mathrm{tmp}}(s)$. For the former case, there exists ξ_0 such that

$$\xi_0^{\mathrm{T}}(A+B_1B_1^{\mathrm{T}}X)=\lambda_0\xi_0^{\mathrm{T}},\quad \xi_0^{\mathrm{T}}[B_1 \quad B_2]=0 \tag{A17}$$

from which it follows that $\xi_0^{\mathrm{T}}A=\lambda_0\xi_0^{\mathrm{T}}$. This implies that (A, B_2) is not stabilizable, which contradicts assumption A1. For the latter case, there exists ξ_1 such that

$$(A+B_1B_1^{\mathrm{T}}X)\xi_1=\lambda_0\xi_1,\quad -F_\infty\xi_1=0,\quad (C_2+D_{21}B_1^{\mathrm{T}}X)\xi_1=0 \tag{A18}$$

Then from (A12), it follows that $A_X\xi_1=(A+B_1B_1^{\mathrm{T}}X+\hat{B}_2E_{12}^{-1/2}F_\infty)\xi_1=\lambda_0\xi_1$, which yields with (A13) that

$$\begin{bmatrix}A_{Xr} & 0 & 0\\ * & \Lambda_j & 0\\ * & * & A_{11}\end{bmatrix}S^{-1}\xi_1=\lambda_0S^{-1}\xi_1 \tag{A19}$$

Since A_{Xr} is stable, then $S^{-1}\xi_1=[0 \quad * \quad *]^{\mathrm{T}}$. Hence, $\xi_1\in\mathrm{Im}\,[T_1 \quad T]$. From (A11), we obtain that $X\xi_1=0$. Inserting this into (A18), we have $A\xi_1=\lambda_0\xi_1$, $C_2\xi_1=0$, which contradicts the fact that (A, C_2) is detectable. Hence, $(A+B_1B_1^{\mathrm{T}}X, B_2)$ is stabilizable.

Very similarly, we can show that $(A+B_1B_1^{\mathrm{T}}X, C_2+D_{21}B_1^{\mathrm{T}}X)$ is detectable. □

According to Lemma A2 and Lemma 15 in Reference [1], the H^∞ control solvability questions for $P(s)$ in (1) and $P_{\mathrm{tmp}}(s)$ in (A16) are equivalent, both being solvable with the same controller. This is the key to the next step of the proof.

Proof of conditions (ii) *and* (iii) *of Theorem 1.* Above we established the necessity of condition (i) of Theorem 1; we now use this fact in relation to $P_{\text{tmp}}^{\text{T}}(s)$, for which the H^{∞} control problem is solvable owing to Lemma A2. With (12) and (14), we can determine the Ω_e eigenstructure of the system matrix pencil of the 1–2 part of $P_{\text{tmp}}^{\text{T}}(s)$ with the realization induced from (A16). Then by applying Lemma A1 to $P_{\text{tmp}}^{\text{T}}(s)$, we know that

$$W(A + B_1B_1^{\text{T}}X - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\bar{C}_2)^{\text{T}} + (A + B_1B_1^{\text{T}}X - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\bar{C}_2)W$$
$$+ W(F_{\infty}^{\text{T}}F_{\infty} - \bar{C}_2^{\text{T}}E_{21}^{-1}\bar{C}_2)W + B_1(I - \hat{D}_{21}^{\text{T}}E_{21}^{-1}\hat{D}_{21})B_1^{\text{T}} = 0 \tag{A20}$$

has a solution $W \geqslant 0$ with

$$W\tilde{T}_1 = 0, \quad W\tilde{T} = 0 \tag{A21}$$

where $\bar{C}_2 := \hat{C}_2 + \hat{D}_{21}B_1^{\text{T}}X$. Similar to Reference [5], it follows that

$$Y = W(I + XW)^{-1} = (I + WX)^{-1}W \geqslant 0 \tag{A22}$$

is a solution of

$$Y(A - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\hat{C}_2)^{\text{T}} + (A - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\hat{C}_2)Y$$
$$+ Y(C_1^{\text{T}}C_1 - \hat{C}_2^{\text{T}}E_{21}^{-1}\hat{C}_2)Y + B_1(I - \hat{D}_{21}^{\text{T}}E_{21}^{-1}\hat{D}_{21})B_1^{\text{T}} = 0 \tag{A23}$$

From (A22), we obtain $I + WX = (I - YX)^{-1} = Z > 0$; it follows that $\rho(XY) < 1$ and

$$Y = Z^{-1}W \tag{A24}$$

hold. Then we obtain from (A21) that $Y\tilde{T}_1 = 0$ and $Y\tilde{T} = 0$. Then Y in (A22) can be represented as

$$\tilde{S}^{\text{T}}Y\tilde{S} = \text{diag}\{Y_r, 0, 0\} \tag{A25}$$

where $Y_r \geqslant 0$ owing to the fact that $Y \geqslant 0$. Moreover, dually using Lemma A1, we can see that Y_r as defined by (A25) is a solution of (24). Consider the matrices

$$A_W := A + B_1B_1^{\text{T}}X - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\bar{C}_2 + W(F_{\infty}^{\text{T}}F_{\infty} - \bar{C}_2^{\text{T}}E_{21}^{-1}\bar{C}_2) \tag{A26}$$

$$A_Y := A - B_1\hat{D}_{21}^{\text{T}}E_{21}^{-1}\hat{C}_2 + Y(C_1^{\text{T}}C_1 - \hat{C}_2^{\text{T}}E_{21}^{-1}\hat{C}_2) \tag{A27}$$

for which there holds, see e.g. [26],

$$A_Y = Z^{-1}A_WZ \tag{A28}$$

Based on this equation and applying the dual statement of (A13) to A_W, we can claim that Y_r as defined by (A25) is a stabilizing solution of (24). This completes a demonstration of the necessity for conditions (ii) and (iii) of Theorem 1. □

Based on the two lossless factorizations (A1) and (A2), we know that the H^{∞} control problem for the 4-block plant $P(s)$ in (1) is solvable with the controller $K(s)$ if and only if the H^{∞} control problem for the 1-block plant $P_n(s)$ in (30) is solvable with the controller $K(s)$.

By applying Lemma A2 with $P(s)$ replaced by $P_{\text{tmp}}^{\text{T}}(s)$ and $P_{\text{tmp}}(s)$ replaced by $P_n^{\text{T}}(s)$, we know from the final conclusion in the lemma statement that the triple (A_n, B_{n2}, C_{n2}) is stabilizable and detectable.

Also, we can show in a straightforward way that $P_n(s)$ in (30) has the following properties alluded to in Remark 3.1.

Lemma A3

Consider the generalized plant (1) under the assumptions A1 and A4. If conditions (i)–(iii) in Theorem 1 hold, and $P_n(s)$ with its realization is defined in (30), then

(i) the realization of $P_{n12}(s)$ induced from (30) has no invariant zeros in C_+ and has the same finite jω-axis invariant zeros as those of the realization of $P_{12}(s)$ induced from (1);
(ii) the realization of $P_{n21}(s)$ induced from (30) has no invariant zeros in C_+ and has the same finite jω-axis invariant zeros as those of the realization of $P_{21}(s)$ induced from (1).

APPENDIX B. PROOF OF SUFFICIENT PART OF THEOREM 1

To begin with, we need the following result to show that assumption A1 holds for the realization of $P_n(s)$ defined in Theorem 1.

Lemma B1

Consider the generalized plant (1) under assumptions A1 and A4. If conditions (i)–(iv) in Theorem 1 hold, and $P_n(s)$ is defined as in Theorem 1, then the triple (A_n, B_{n2}, C_{n2}) is stabilizable and detectable.

Proof. We will just prove that (A_n, B_{n2}) is stabilizable. Dually we can prove that (A_n, C_{n2}) is detectable. From Lemma A3, since the realization of $P_{n12}(s)$ induced from (30) has no invariant zeros in C_+, it follows that

$$\text{rank}[-sI + A_n \quad B_{n2}] = n, \quad \forall s \in C_+ \tag{B1}$$

holds. Now, to obtain a contradiction, suppose that (A_n, B_{n2}) is not stabilizable. Then there exist jω_i with ω_i real and $\eta \neq 0$ such that

$$\begin{bmatrix} -\mathrm{j}\omega_i + A_n^{\mathrm{T}} \\ B_{n2}^{\mathrm{T}} \end{bmatrix} \eta = 0 \tag{B2}$$

From (38), we have

$$\begin{bmatrix} X_{12i} \\ U_{12i} \end{bmatrix} = \begin{bmatrix} \eta & * \\ 0 & * \end{bmatrix} \tag{B3}$$

which yields from (36) that $0 > \eta^* B_{n1} B_{n1}^{\mathrm{T}} \eta$. Then

$$B_{n1}^{\mathrm{T}} \eta = 0 \tag{B4}$$

Now taking the equivalent definition of (A_n, B_{n1}, B_{n2}) from (30), and using (B2) and (B4), and using the pattern of the proof of the stabilizability of $(A + B_1 B_1^{\mathrm{T}} X, B_2)$ in Lemma A2, we obtain that

$$\eta \in \text{Im}[\tilde{T}_1 \quad \tilde{T}] \tag{B5}$$

Thus, from (A21), $W\eta = 0$ which yields together with (B2)–(B4) that

$$(A^{\mathrm{T}} + XB_1B_1^{\mathrm{T}})\eta = \mathrm{j}\omega_i\eta, \quad \hat{D}_{21}B_1^{\mathrm{T}}\eta = 0, \quad B_2^{\mathrm{T}}\eta = 0 \tag{B6}$$

On the other hand, from (12) and (14), we can conclude that $B_1^{\mathrm{T}}[\tilde{T}_1 \quad \tilde{T}] \in \mathrm{Im}\,\hat{D}_{21}^{\mathrm{T}}$. Together with the second term in (B6), we obtain that $B_1^{\mathrm{T}}\eta = 0$. It implies from (B6) that $(A^{\mathrm{T}}, B_2^{\mathrm{T}})$ is not observable at $\mathrm{j}\omega_i$. This contradicts assumption A1. Thus, (A_n, B_{n2}) is stabilizable. □

Proof of Sufficient conditions of Theorem 1. From Lemma B1, we note that the triple (A_n, B_{n2}, C_{n2}) is stabilizable and detectable. Therefore, according to Theorem 6 in Reference [11], first we know that the matrix norm conditions related to $\mathrm{j}\omega$-axis zeros are satisfied owing to condition (iv) of Theorem 1. Also since in this paper, $D_{11} := P_{n11}(\infty) = 0$ holds, from Hara *et al.* [11], the matrix norm conditions related to infinite zeros hold trivially. Next, two generalized Riccati equations in Theorem 6 in Reference [11] have solutions of zero matrices owing to the fact that $P_{n12}(s)$ and $P_{n21}(s)$ with their realizations induced from (30) have no invariant zeros in C_+. Therefore, the H^∞ control problem for the plant $P_n(s)$ is solvable, and so also then for the plant $P(s)$. □

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