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Uniform Complete Controllability for Time-Varying Systems

In Kalman[1] a special sort of complete controllability for linear systems is defined, termed uniform complete controllability. This definition has proved useful not only in connection with the quadratic variational problems studied in Kalman, but also in discussing the stability of linear, time-varying systems.[2]

The object of this correspondence is to demonstrate that if certain analyticity conditions are fulfilled, uniform complete controllability may be defined in terms of pointwise properties of functions, as distinct from Kalman's definition involving integrals.[1]

Notational conventions are as follows: small letters denote vectors, capital letters matrices, and for two symmetric matrices A and B, $A \geq B$ will mean that $A-B$ is non-negative definite, $A > B$ that $A-B$ is positive definite.

DEFINITION

The original definition is made in connection with the system

$$\dot{x} = Fx + Gu. \tag{1}$$

With transition matrix $\Phi(.\,,.)$ the *controllability matrix* is defined as

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)G(t)G'(t)\Phi'(t_0, t)dt. \tag{2}$$

By its structure W is non-negative definite. Ordinary complete controllability requires W to be positive definite, while uniform complete controllability requires any two of the following three conditions to hold (any two imply the third).[1]

1) For some σ and all t

$$0 < \alpha_0(\sigma)I < W(t, t+\sigma) < \alpha_1(\sigma)I. \tag{3}$$

2) For some σ and all t

$$\begin{aligned} 0 &< \beta_0(\sigma)I \\ &< \Phi(t+\sigma, t)W(t, t+\sigma)\Phi'(t+\sigma, t) \\ &< \beta_1(\sigma)I. \end{aligned} \tag{4}$$

3) $\|\Phi(t, \tau)\| < \alpha_2(|t-\tau|)$. (5)

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[1] R. E. Kalman, "Contribution to the theory of optimal control," *Bol. Soc. Mat. Mex.*, pp. 102–119, 1960.

[2] L. M. Silverman and B. D. O. Anderson, "Controllability, observability and stability of linear systems," Electronics Research Lab., University of California, Berkeley, Memo. ERL-M210, April 1967.

To deduce a pointwise criterion, begin by setting

$$f(t_0, \lambda) = \Phi(t_0, \lambda)G(\lambda) \quad (6)$$

and regarding f as a function of λ, parametrized by t_0. Suppose $f(t_0, \lambda)$ is analytic in λ and such that there exists $K(t_0)$ with

$$\| f^{(i)}(t_0, \lambda)\| < K(t_0) \quad \text{for all } i. \quad (7)$$

LEMMA

With f defined as in (6), analytic in λ, and with (7) holding, then

$$\det W(t_0, t_0 + t) = [1 + K(t_0)0(t)] \cdot \det Q(t_0)H(t)Q'(t_0). \quad (8)$$

Here $K(t_0)0(t)$ denotes a matrix whose norm approaches zero as t approaches zero, while

$$Q(t_0) = [f(t_0, t_0)\dot{f}(t_0, t_0) \cdots f^{(n-1)}(t_0, t_0)] \quad (9)$$

$$H(t) = \int_0^t \left[1\lambda \frac{\lambda^2}{2!} \cdots \frac{\lambda^{n-1}}{(n-1)!}\right]' \cdot \left[1\lambda \frac{\lambda^2}{2!} \cdots \frac{\lambda^{n-1}}{(n-1)!}\right] d\lambda. \quad (10)$$

Note that the ith column block of $Q(t_0)$ is really

$$\frac{d^{i-1}}{d\lambda^{i-1}} f(t_0, \lambda)$$

evaluated at $\lambda = t_0$.

Proof: From (2) and (6)

$$W(t_0, t_0 + t) = \int_{t_0}^{t_0+t} f(t_0, \lambda)f'(t_0, \lambda)d\lambda \quad (11)$$

and W is analytic in t because f is analytic in λ. Making a power series expansion of W around $t = 0$ yields

$$W(t_0, t_0 + t) = \frac{t}{1!} f(t_0, t_0)f'(t_0, t_0, t_0) + \frac{t^2}{2!}[f(t_0, t_0)\dot{f}'(t_0, t_0) + \dot{f}(t_0, t_0)f'(t_0, t_0)] + \cdots + \frac{t^n}{n!}\left[f^{(n-1)}(t_0, t_0)f'(t_0, t_0) + \binom{n-1}{1} f^{(n-2)}(t_0, t_0) \cdot \dot{f}'(t_0, t_0) + \cdots\right] + \cdots \quad (12)$$

and thus, using (7),

$$\det W(t_0, t_0 + t) = [1 + K(t_0)0(t)] \cdot \det \left\{\frac{t}{1!} f(t_0, t_0)f(t_0, t_0) + \cdots + \frac{t^n}{n!}[f^{(n-1)}(t_0, t_0)f'(t_0, t_0) + \cdots]\right\}. \quad (13)$$

Explicit calculation of $\det Q(t_0)H(t)Q'(t_0)$ using (9) and (10), which involves evaluation of the integral in (10), then yields the desired result.

The pointwise criterion for uniform complete controllability may now be stated.

THEOREM

Consider the system (1), with transition matrix Φ. Let $f(t_0, \lambda)$ be defined as in (6) and analytic in λ. Then sufficient conditions for (1) to be uniformly completely controllable are

1) $\|F(t)\| < K_1$ for all t (14)

2) $\| f^{(i)}(t_0, t_0)\| < K_2$ for all t_0, i, (15)

and where the differentiation operation applies to the second argument,

3) $\det Q(t_0)Q'(t_0) \geq K_3^2 \quad 0$ for all t_0 (16)

where $Q(t_0)$ is defined in (9).

Proof: Inequality (14) ensures by the Gronwall–Bellman lemma that (5) holds. Inequality (15) ensures that, with W defined by (2),

$$W(t, t + \sigma) \leq \alpha_1(\sigma)I. \quad (17)$$

Finally, (8) and (16) yield

$$\det W(t, t + \sigma) \geq [I + 0(t)K_2]K_3^2 \det H(\sigma) \quad (18)$$

and evidently $\det H(\sigma)$ is positive. This means that

$$W(t, t + \sigma) \geq \alpha_0(\sigma)I > 0 \quad (19)$$

see, e.g., Bhatia.[3] Consequently, conditions 1) and 2) of the definition of uniform complete controllability are fulfilled, thus proving the theorem.

EXAMPLE

As an example of the preceding, consider the system already examined from the stability point of view[2,4]

$$\dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \quad (20)$$

Observe that

$$f'(t_0, t_0) = [0, 0 \cdots 0, 1] \quad (21a)$$

$$\dot{f}'(t_0, t_0) = [\Phi(t_0, \lambda)F(\lambda)\Phi(t_0, \lambda)G(\lambda)]' |_{t_0=\lambda} = -[0, 0 \cdots 1, -a_n] \quad (21b)$$

$$\ddot{f}'(t_0, t_0) = [0, 0 \cdots 1, -a_n, -a_{n-1} + \dot{a}_n + a_n^2] \quad (21c)$$

and so on, with

$$\det Q(t_0) = 1. \quad (22)$$

With analyticity conditions on the a_i, it is then true that (14), (15), and (16) all hold, and thus (20) is uniformly completely controllable.

The matrix Q has appeared elsewhere as a controllability matrix Q_c.[2,5] With G a vector, suitable boundedness conditions on F and G in (1) [which merely imply similar conditions on f in (6)], combined with lower and upper bounds on the norm of Q_c, imply the existence of a Liapunov transformation of F and G to phase-variable form. The main theorem thus almost says that a sufficient condition for a pair $[F, G]$ (with G a vector) to be uniformly completely controllable is that there exist a Liapunov transformation taking it to phase-variable form.

A point worth noting concerning the text of the Theorem is that a solution of $\dot{x} = Fx$, or indeed any other differential equation, is not necessary for the checking of the conditions of the Theorem, since the determination of Q involves only the differentiation of F and G. Thus

$$f(t_0, t_0) = \Phi(t_0, t_0)G(t_0) = G(t_0)$$

$$\frac{d}{d\lambda} f(t_0, \lambda)|_{\lambda=t_0} = \left[\frac{d}{d\lambda}\Phi(t_0, \lambda)\right] G(\lambda)|_{\lambda=t} + \Phi(t_0, \lambda)\dot{G}(\lambda)|_{t=t_0} = F(t_0)G(t_0) + \dot{G}(t_0)$$

and so on.

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[5] L. H. Haines and L. M. Silverman, "Internal and external stability of linear systems," *J. Math. Anal. and Appl.* (to be published).