Limits to the Fluctuation-Dissipation Theorem for Nonlinear Circuits

Geoffrey J. Coram, *Student Member, IEEE*, Brian D. O. Anderson, *Fellow, IEEE*, and John L. Wyatt, Jr., *Senior Member, IEEE*

***Abstract*—It is well known that the equilibrium thermal noise behavior at the terminals of any linear time-invariant (LTI) RLC circuit can be predicted from knowledge of the driving-point impedance and temperature alone. This paper examines the conjecture that similar results hold if the capacitors and inductors are nonlinear. We refine the conjecture by analyzing the behavior of an RLC bridge circuit with the nonlinear inductor and capacitor carefully matched so the terminal behavior reduces to that of a linear resistor R. We show that the terminal noise current is *not* that predicted by the Nyquist–Johnson model for R if the driving voltage is time dependent or the inductor and capacitor are time varying. This counterexample disproves the conjecture, which does hold, however, for the bridge circuit with nonlinear (but time invariant) devices if the driving voltage is zero or constant. This paper makes exact calculations using techniques from stochastic differential equations and using reversibility arguments.**

***Index Terms*—Bridge circuit, matching condition, noise theory, nonlinear networks.**

I. Introduction

CONSIDER the bridge circuit of Fig. 1. It is a standard result of linear circuit theory that under the matching condition $L = R^2C$, the driving-point impedance reduces to R and the natural frequency of the circuit does not appear as a pole [1], [2]. Regardless of the values of the capacitor and inductor, for high frequencies, the capacitor is essentially a short circuit, whereas the inductor is essentially an open circuit; at low frequencies, the opposite occurs. The matching condition ensures that a balance is preserved for intermediate frequencies: the charging of the capacitor is matched by the fluxing of the inductor. In the language of control theory, the state equations become *nonminimal* in the matched case.

A. Central Questions

Suppose one has two black boxes, one with a matched bridge circuit inside and the other with a single equivalent linear resistor. Is it possible to distinguish the two using the noise behavior? How does the answer change if the inductor and capacitor are nonlinear or time varying?

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G. J. Coram and J. L. Wyatt, Jr. are with the Department of Electrical Engineering and Computer Science and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

B. D. O. Anderson is with the Research School of Information Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia.

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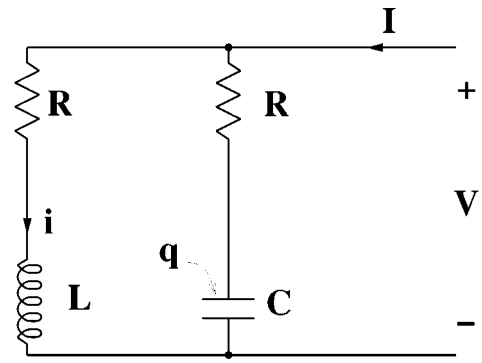

Fig. 1. Linear noise-free bridge circuit is matched and has input impedance R if $L = R^2C$.

B. The LTI Case

It is straightforward to verify directly in the LTI case that if a Nyquist–Johnson noise model [3], [4] (as shown in Fig. 2) is associated with each resistor, then the spectrum of the short-circuit terminal current in a matched bridge circuit is also that of a Nyquist–Johnson noise model for a single resistor of value R. The verification can be done by standard frequency-domain techniques or by stochastic calculus [5]. The high-pass filtering of the RC branch is precisely balanced by the low-pass filtering of the RL branch, so that the terminal noise spectrum is flat. Of course, both resistors must be at the same temperature. As noted in [2], applying a dc voltage to the circuit would result in differential heating of the resistor in the RL branch. If the resistors were not properly connected to thermal reservoirs, one could heat up and become noisier than the other, and the noise spectrum would no longer be flat. This is a trivial exception to the results of this paper, which assumes uniform, constant temperature.

The result above is a particular example of a general circuit theory result, namely, that a one-port network of LTI passive elements with port admittance $Y(j\omega)$ presents a short-circuit thermal noise current with power spectrum $2kT\,\mathrm{Re}\{Y(j\omega)\}$, where k is Boltzmann's constant and T is the absolute temperature [6]. Physicists regard such results as particular cases of the fluctuation-dissipation theorem [7].

C. Generalizations

This paper studies one carefully chosen example, motivated by the question of whether some form of fluctuation-dissipation theorem holds for some class of nonlinear circuits. Our initial formulation appears below as a conjecture for any pair of two-terminal networks, each comprising an interconnection of

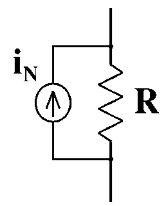

Fig. 2. Nyquist–Johnson thermal noise model (Norton form) is a noiseless linear resistor in parallel with a Gaussian white noise current source i_N with power spectral density $2kT/R$.

LTI resistors at a uniform, constant temperature, described by the Nyquist–Johnson model, and possibly also capacitors and inductors that may be nonlinear or time varying. Two such networks are said to be *zero-state deterministically equivalent* if every applied terminal voltage waveform $v(t), t \geq 0$, produces the same current response $i(t)$ from both networks, provided all capacitor voltages and inductor currents are initially zero and all noise sources in the resistor models are set to zero. (In the LTI case, this just means the two input admittances are identical.)

Preliminary Fluctuation-Dissipation Conjecture for Networks: No two zero-state deterministically equivalent networks can be distinguished by their terminal noise current responses to any applied voltage waveform.

The conjecture just hypothesizes that the deterministic terminal behavior uniquely determines the noise current response for all voltage drives, independent of the details of the network. The conjecture is true in the LTI case. (Closely related formulations for the current-driven and multiport cases [6] also hold true for LTI networks, but we ignore them here for simplicity.)

D. Main Result

An examination of the bridge circuit will show that this preliminary conjecture is wrong when the applied voltage waveform or the circuit elements are time varying.

This paper considers only the Nyquist–Johnson model for noise in a linear resistor. That model does not assume any knowledge of the deterministic current flow mechanism. The results of this paper disprove the existence of such "black-box" noise models for systems with internal nonlinearities when the nonlinearities are in the lossless subsections.

In Section II, we develop the matching condition for the bridge circuit with nonlinear, time-invariant inductor and capacitor under which it becomes deterministically equivalent to a single linear resistor R at the terminals. In Section III, we show that such a matched nonlinear bridge gives a short-circuit port current noise statistically identical to that of the Nyquist–Johnson model for R at thermal equilibrium. We also show that the same result holds for any dc applied voltage once the capacitor and inductor have settled to statistical steady state. In Section IV, we develop the matching condition for the bridge circuit with linear time-varying inductor and capacitor. We show that in this case, however, the current noise is *not* that of the Nyquist–Johnson model for such a resistor, and thus the preliminary fluctuation-dissipation conjecture must be modified. We then apply this result to the nonlinear time-invariant bridge circuit linearized about any trajectory to conclude that the preliminary fluctuation-dissipation conjecture also fails for the nonlinear bridge circuit with time-varying input voltage.

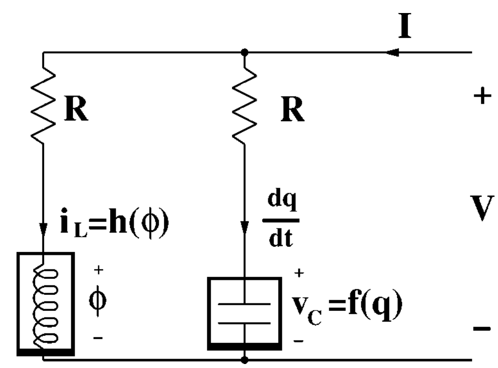

Fig. 3. The nonlinear bridge circuit.

All derivations are exact, involving no approximations, except for the last. Further details, including a stochastic calculus derivation for the LTI bridge, treatment of a dual circuit, and more explicit calculations in some proofs have been omitted here for brevity but can be found in [5]. Other mathematically oriented studies of noise in nonlinear circuits include [8]–[11].

II. Nonlinear, Noise-Free Case

Consider the circuit of Fig. 3. Of course, $R > 0$. In addition, we require the following constraints, drawn essentially from [12] and [13].

Assumption 1: Nonlinear Reactive Element Properties. The mappings h: $\phi \rightarrow i_L$ and f: $q \rightarrow v_C$ obey the following:

1) $h(0) = 0, f(0) = 0$;
2) h and f are continuously differentiable functions, and for all values of the arguments and some fixed $\epsilon > 0$, there holds

$$\frac{dh}{d\phi} \geq \epsilon > 0 \quad \text{and} \quad \frac{df}{dq} \geq \epsilon > 0.$$

This assumption ensures that the circuit is passive, and that $(q, \phi) = (0, 0)$ is a globally asymptotically stable equilibrium point for $V = 0$.

As noted in Section I, in the linear case the condition $L = R^2C$ ensures that the bridge appears as a simple linear resistor at its terminals. In the following theorem, this condition is generalized by finding a condition relating the two nonlinearities which ensures this simple terminal behavior.

Theorem 1: Matching Condition for the Nonlinear Bridge. Consider the circuit of Fig. 3, with Assumption 1 holding. Suppose the circuit is in the zero state at $t = 0$ and is excited by a voltage $V(t)$ for $t > 0$. Then for all $V(t)$ there holds

$$V(t) = RI(t) \tag{1}$$

for all $t \geq 0$, if and only if

$$f(q) = Rh(Rq) \tag{2}$$

for all values of q.

Remark: Since $f'(q) = 1/C(q)$ is the reciprocal of the incremental capacitance and $h'(\phi) = 1/L(\phi)$ is the reciprocal of the incremental inductance, then (2) implies $L(\phi) = R^2 C(q)|_{q=\phi/R}$, a local version of the linear matching condition $L = R^2 C$.

Remark: The above theorem is almost certainly not novel. However, we are unaware of a reference.

Proof: The circuit differential equations are

$$\frac{dq}{dt} = \frac{V - f(q)}{R} \tag{3}$$

$$\frac{d\phi}{dt} = V - Rh(\phi) \tag{4}$$

and the port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R}. \tag{5}$$

First, suppose (2) holds. Observe from (3) and (4) that, irrespective of $V(\cdot)$

$$\begin{aligned}\frac{d(\phi - Rq)}{dt} &= -Rh(\phi) + f(q)\\ &= -Rh(\phi) + Rh(Rq)\\ &= -Rh'(\xi)(\phi - Rq)\end{aligned}$$

where ξ lies between ϕ and Rq, by application of the Mean Value Theorem. It follows that

$$\begin{aligned}\frac{d}{dt}[\phi - Rq]^2 &= -2Rh'(\xi)(\phi - Rq)^2\\ &\leq -2R\epsilon(\phi - Rq)^2\end{aligned} \tag{6}$$

using Assumption 1. Since $\phi(0) = q(0) = 0$, then for all $t \geq 0$, $\phi(t) = Rq(t)$. Thus, the matching condition (2) together with (5) yields $I(t) = V(t)/R$ as required.

Conversely, if we suppose that $I(t) = V(t)/R$ for all t, then from (5)

$$Rh(\phi(t)) = f(q(t)) \tag{7}$$

must hold for all t. In addition, the two parallel branches give two distinct expressions for $V(t)$, also evident from (3) and (4)

$$V = Rh(\phi) + \frac{d\phi}{dt} = f(q) + R\frac{dq}{dt}.$$

In light of (7), the last equality yields

$$\frac{d\phi}{dt} = R\frac{dq}{dt}$$

and with zero initial conditions for ϕ and q, this means that

$$\phi(t) = Rq(t). \tag{8}$$

Hence, in (7), we have for all t, $Rh(R\,q(t)) = f(q(t))$. Since all values of $q(t)$ are clearly attainable by using some appropriate $V(t)$, it follows that $Rh(Rq) = f(q)$ for all q, as required. ■

Remark: The arguments above easily show that if the initial conditions are nonzero, then $\phi(t) - Rq(t)$ decays to zero exponentially fast, and thus (1) holds asymptotically for large t.

III. Nonlinear, Noisy Case: Successful Results

For this section, a Norton-form Nyquist–Johnson noise model is associated with each resistor in the circuit, as in Fig. 4. We would like to show that the terminal current noise of the matched bridge is the same as that for a single linear resistor, when V is constant and the circuit is in steady state. To first order, this result is clear. Recall that the incremental capacitance and inductance satisfy $L(\phi) = R^2 C(q)|_{q=\phi/R}$. A linearization about the noise-free equilibrium operating point (q, ϕ) for a dc applied voltage of a nonlinear matched circuit will yield a matched linear circuit. By superposition, the noise current for the linearized circuit is unaffected by the applied voltage. The point of this section is to show that this equivalence holds *exactly*, even for high temperatures or strong nonlinearities for which the noise could drive the circuit out of the valid region of linearization.

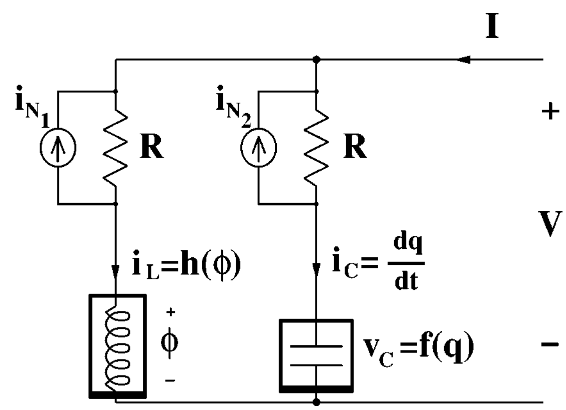

Fig. 4. Nonlinear bridge circuit with Nyquist–Johnson noise sources.

The circuit is described by stochastic differential equations (SDEs):

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \tag{9}$$

$$\frac{d\phi}{dt} = V - Rh(\phi) - Ri_{N_1} \tag{10}$$

where i_{N_1} and i_{N_2} are independent Gaussian white noise processes with power spectral density $2kT/R$. The port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R} - i_{N_2}. \tag{11}$$

One might be tempted to use the matching condition (7) and immediately conclude $I = V/R - i_{N_2}$. However, this condition does *not* hold, because (7) was derived for a different excitation: q and ϕ no longer satisfy $\phi(t) = Rq(t)$, because they are now driven by independent noise sources. So, the power spectrum of I must be calculated more methodically.

A. The $I(t) - V(t)$ Relation in the Presence of Noise

Before proceeding to study the noise power spectrum, we show that the nonlinear inductor and capacitor cannot "rectify" the noise, even with a time-varying $V(t)$. Rectification would cause incorrect "average" behavior, or first-order statistics of the circuit, such that it would be pointless to study the second-order statistic of the power spectral density.

Theorem 2: Terminal Noise Current Is Zero-Mean. Consider the circuit of Fig. 4, described by (9)–(11), with Assumption 1 and the matching condition (2) in force. Let $V(t)$ be an arbitrary excitation, and assume zero initial conditions. Then

$$E\{I(t)\} = \frac{V(t)}{R}. \tag{12}$$

Proof: Taking expectations on both sides of (11)

$$E\{I\} = E\{h(\phi)\} + \frac{V(t)}{R} - \frac{E\{f(q)\}}{R} - 0. \tag{13}$$

In order to compute the expectations of $f(q)$ and $h(\phi)$, we need to know something about the probability densities ρ for q and ϕ. The Fokker–Planck equations [14], [15] for the evolutions of these densities are, for (9) and (10), respectively

$$\frac{\partial \rho_q}{\partial t} = -\frac{\partial}{\partial q}\left[\frac{V(t)-f(q)}{R}\rho_q\right] + \frac{kT}{R}\frac{\partial^2 \rho_q}{\partial q^2} \tag{14}$$

$$\frac{\partial \rho_\phi}{\partial t} = -\frac{\partial}{\partial \phi}[(V(t) - Rh(\phi))\rho_\phi] + kTR\frac{\partial^2 \rho_\phi}{\partial \phi^2}. \tag{15}$$

Using the matching condition (2), these two equations become identical up to a scaling. The reader can verify that a density $\rho_\phi(\phi, t)$ satisfies (15) if and only if the scaled version

$$\rho_q(q,t) = R\rho_\phi(Rq, t) \tag{16}$$

satisfies (14). The densities corresponding to zero initial conditions (δ functions) also satisfy (16) at $t = 0$. Thus, the solutions of (14) and (15) satisfy (16) for all time, and it follows by direct calculation that

$$E\{f(q(t))\} = RE\{h(\phi(t))\}, \quad t \geq 0. \tag{17}$$

Substituting (17) into (13) shows that the desired result (12) holds. ∎

More details are given in [5].

Definition: A *steady-state* density satisfies $d\rho/dt = 0$. *Thermal equilibrium* for this circuit is the steady state with $V = 0$.

Corollary: Theorem 2 remains true if, instead of zero initial conditions, the circuit initially has a steady-state density with $V(0) \neq 0$.

Proof: The densities

$$\rho_q(q) = A_q \exp\left[\frac{1}{kT}\int_0^q (V - f(\tilde{q}))\, d\tilde{q}\right] \tag{18}$$

$$\rho_\phi(\phi) = A_\phi \exp\left[\frac{1}{kT}\int_0^\phi \left(\frac{V}{R} - h(\tilde{\phi})\right) d\tilde{\phi}\right] \tag{19}$$

where A_q and A_ϕ are normalization constants, are the steady-state solutions to (14) and (15). Under the matching condition (2), the steady-state initial densities satisfy (16) at $t = 0$. Thus, the solutions of (14) and (15) again satisfy (16) for all time, and the desired result (12) holds. ∎

B. Thermal Noise Current

This section derives the thermal noise current spectrum at the external terminals of the circuit.

Theorem 3: Terminal Noise Current Is That of a Nyquist–Johnson Resistor. Consider the circuit of Fig. 4, described by (9)–(11) with Assumption 1 and the matching condition (2) in force. Assume the circuit is in steady state at $t = 0$. Denote by R_{nn} the autocorrelation of the terminal noise current $n(t) = I(t) - V(t)/R$. Then for $t, \tau > 0$

(a) $E\{n(t)\} = 0$;
(b) $R_{nn}(t-\tau) = (2kT/R)\delta(t-\tau)$;
(c) $\int_0^t n(s)\, ds$ is a scaled Wiener process

provided that one of the following two sufficient conditions holds:

i) the circuit is LTI, i.e., $f(q) = q/C$ and $h(\phi) = \phi/L$, or
ii) the voltage $V(t)$ is constant.

Proof:

i) The sufficiency of condition i) is an immediate consequence of superposition for linear circuits. The deterministic behavior was shown in Section II, and the noise behavior for linear circuits at equilibrium was shown in [6]. Adding together the results of the independent excitations proves the theorem for this condition.

ii) ⇒ (a) This was shown in Theorem 2.

ii) ⇒ (b) The autocorrelation of $n(t)$ is, from (11)

$$R_{nn}(t,\tau) = E\left\{\left[h(\phi(t)) - \frac{f(q(t))}{R} - i_{N_2}(t)\right] \times \left[h(\phi(\tau)) - \frac{f(q(\tau))}{R} - i_{N_2}(\tau)\right]\right\}.$$

Since $i_L(\cdot)$ is independent of $i_{N_2}(\cdot)$ and the latter has zero mean

$$E\{h(\phi(t)) i_{N_2}(\tau)\} = E\{h(\phi(t))\}E\{i_{N_2}(\tau)\} = 0.$$

Since $i_L(\cdot)$ is also independent of $v_C(\cdot)$, though neither has zero mean

$$\left\{E\{h(\phi(t))\}\frac{f(q(\tau))}{R}\right\} = E\{h(\phi(t))\}E\left\{\frac{f(q(\tau))}{R}\right\}.$$

The proof of Theorem 2 used the similarity of the Fokker–Planck equations (14) and (15) to show

$$E\{f(q(t))\} = RE\{h(\phi(t))\},$$

for all times t (or τ), and similarly there holds

$$E\{h(\phi(t))h(\phi(\tau))\} = E\left\{\frac{f(q(t))}{R}\frac{f(q(\tau))}{R}\right\}.$$

The autocorrelation can thus be simplified to

$$\begin{aligned} R_{nn}(t,\tau) = 2E\left\{\frac{f(q(t))}{R}\frac{f(q(\tau))}{R}\right\} &- 2E\left\{\frac{f(q(t))}{R}\right\}E\left\{\frac{f(q(\tau))}{R}\right\} \\ &+ E\left\{\frac{f(q(t))}{R}i_{N_2}(\tau)\right\} + E\left\{i_{N_2}(t)\frac{f(q(\tau))}{R}\right\} \\ &+ E\{i_{N_2}(t)i_{N_2}(\tau)\}. \end{aligned} \tag{20}$$

If we multiply both sides of the differential equation (9) for $q(t)$ by $f(q(\tau))$ and take expectations, we obtain

$$\begin{aligned} \frac{d}{dt}E\{f(q(\tau))q(t)\} &= \frac{V(t)}{R}E\{f(q(\tau))\} - \frac{1}{R}E\{f(q(t))f(q(\tau))\} \\ &- E\{i_{N_2}(t)f(q(\tau))\}. \end{aligned} \tag{21}$$

The dummy time indexes t and τ may be interchanged, corresponding to writing the SDE in τ and multiplying through by $f(q(t))$, to get

$$\begin{aligned}\frac{d}{d\tau}E\{f(q(t))q(\tau)\} &= \frac{V(\tau)}{R}E\{f(q(t))\} - \frac{1}{R}E\{f(q(\tau))f(q(t))\} \\ &\quad - E\left\{i_{N_2}(\tau)f(q(t))\right\}. \end{aligned} \tag{22}$$

Define

$$F(t,\tau) = E\{f(q(\tau))q(t)\}$$

so that the autocorrelation may be expressed

$$\begin{aligned} R_{nn}(t,\tau) = &\left[\frac{V(t)}{R} - E\left\{\frac{f(q(t))}{R}\right\}\right] E\left\{\frac{f(q(\tau))}{R}\right\} \\ &+ \left[\frac{V(\tau)}{R} - E\left\{\frac{f(q(\tau))}{R}\right\}\right] E\left\{\frac{f(q(t))}{R}\right\} \\ &- \frac{1}{R}\left[\frac{dF(t,\tau)}{dt} + \frac{dF(\tau,t)}{d\tau}\right] \\ &+ E\left\{i_{N_2}(t)i_{N_2}(\tau)\right\}. \end{aligned} \tag{23}$$

For arbitrary time-varying $V(t)$ and strictly nonlinear inductor and capacitor, no further simplification is apparent.

We now require condition ii). Since V is constant and the system is initially at steady state, it remains in steady state for $t \geq 0$, i.e., $q(t)$ and $\phi(t)$ are *stationary* random processes. Taking expectations of both sides of the differential equation (9)

$$E\left\{\frac{dq}{dt}\right\} = 0 = E\left\{\frac{V - f(q(t))}{R}\right\} + E\left\{i_{N_2}(t)\right\}$$

so that

$$V = E\{f(q(t))\}.$$

Since $q(t)$ is stationary, $F(t,\tau) = F(t-\tau)$ depends only on the difference $(t-\tau)$. Further, a consequence of Assumption 1 and (9) is that $q(t)$ is a *reversible* process [8], i.e., for all t_1 and t_2,

$$\begin{aligned}&\Pr[\alpha \leq q(t_1) \leq \alpha + d\alpha, \beta \leq q(t_2) \leq \beta + d\beta] \\ &\quad = \Pr[\beta \leq q(t_1) \leq \beta + d\beta, \alpha \leq q(t_2) \leq \alpha + d\alpha].\end{aligned}$$

As a consequence of reversibility, F is an even function:

$$\begin{aligned} F(t-\tau) &= E\{q(t)f(q(\tau))\} \\ &= \iint af(b)p(q(t)=a, q(\tau)=b)\,da\,db \\ &= \iint af(b)p(q(\tau)=a, q(t)=b)\,da\,db \\ &= E\{q(\tau)f(q(t))\} = F(\tau - t) \end{aligned} \tag{24}$$

where $p(\cdot,\cdot)$ represents the joint probability density of its two arguments, and equality between the first and second lines follows from reversibility. Since $F(\cdot)$ is an even function, $F'(\cdot)$ must be odd, and

$$\begin{aligned}\frac{d}{dt}F(t,\tau) &= \frac{d}{dt}F(t-\tau) = F'(t-\tau) \\ &= -F'(\tau - t) = -\frac{d}{d\tau}F(\tau,t). \end{aligned} \tag{25}$$

Therefore, the autocorrelation reduces to

$$R_{nn}(t,\tau) = E\left\{i_{N_2}(t)i_{N_2}(\tau)\right\} = \frac{2kT}{R}\delta(t-\tau).$$

ii) $\Rightarrow$ (c) It remains to show that $w(t) \triangleq \int_0^t n(s)\,ds$ is a scaled Wiener process, or equivalently, that $n(t)$ is a Gaussian white noise process. From the zero-mean property of $n(t)$ and its covariance, it is trivial to see that $w(t)$ obeys $E\{w(t)\} = 0$ and $E\{w(t)w(s)\} = (2kT/R)\min[t,s]$, and $w(t)$ is a martingale.[1] From (9) and (10), it follows that the sample paths of ϕ and q are continuous with probability 1, by a result of stochastic differential equation theory [15], and accordingly from an integrated version of (11), $w(t)$ also has this property. A theorem of Doob [16] then allows one to conclude that because $w(t)$ is a continuous martingale with covariance equal to that of a scaled Wiener process, it is necessarily itself a scaled Wiener process. ■

It is perhaps somewhat surprising that this analysis holds exactly. There are two noise sources driving nonlinear elements, so one might expect a nonlinear "mixing" under which the two drives interact to produce a colored noise spectrum, but this does not happen in this circuit.

IV. Failures of the Conjecture

As mentioned in the Introduction, there are some situations in which the noise current of the matched bridge circuit is not statistically equivalent to the noise of a single linear resistor. Even if the circuit is kept at constant temperature, the conjecture fails for a time-varying circuit. This failure casts doubts on the hopes of establishing the general nonlinear nonequilibrium result for a time-varying driving voltage.

Suppose the energy storage elements in Fig. 4 are linear, but time varying. This will provide the first nontrivial failure of fluctuation-dissipation hypothesis in the Introduction; it is sufficient to consider the short-circuit (undriven) behavior. The circuit differential equations are

$$\frac{d\phi}{dt} = -\frac{R\phi(t)}{L(t)} - Ri_{N_1}(t) \tag{26}$$

$$\frac{dq}{dt} = -\frac{q(t)}{RC(t)} - i_{N_2}(t) \tag{27}$$

and we assume $E\{q(0)\} = E\{\phi(0)\} = 0$ so that $q(t)$ and $\phi(t)$ are zero mean. The port current is

$$I(t) = \frac{\phi(t)}{L(t)} - \frac{q(t)}{RC(t)} - i_{N_2}(t). \tag{28}$$

The corresponding matching condition is of course

$$L(t) = R^2C(t). \tag{29}$$

The differential equation for $q(t)$ can be solved explicitly in terms of sample paths of the noise process $i_{N_2}(t)$:

$$\begin{aligned} q(t) = &\exp\left[-\int_0^t \frac{ds}{RC(s)}\right] \\ &\times \left(q(0) - \int_0^t i_{N_2}(\sigma)\exp\left[\int_0^\sigma \frac{ds}{RC(s)}\right]d\sigma\right). \end{aligned} \tag{30}$$

[1]A martingale is a random process $w(t)$ such that the conditional expectation for the future, given the entire past, is simply the present value. Symbolically, $E\{w(t_2)\,|\,w(t), 0 \leq t \leq t_1\} = w(t_1)$ for all $t_2 > t_1$.

The autocorrelation function for the port current (which is entirely noise current) for $\tau > t$ is

$$\begin{aligned} R_{nn}(t,\tau) &= E\left\{\left[\frac{\phi(t)}{L(t)} - \frac{q(t)}{RC(t)} - i_{N_2}(t)\right] \right.\\ &\quad \left. \times \left[\frac{\phi(\tau)}{L(\tau)} - \frac{q(\tau)}{RC(\tau)} - i_{N_2}(\tau)\right]\right\} \\ &= \frac{1}{R^2C(t)C(\tau)}E\{q(t)q(\tau)\} + \frac{1}{L(t)L(\tau)} \\ &\quad \times E\{\phi(t)\phi(\tau)\} + \frac{1}{RC(\tau)}E\left\{q(\tau)i_{N_2}(t)\right\} \\ &\quad + E\left\{i_{N_2}(t)i_{N_2}(\tau)\right\} \end{aligned} \tag{31}$$

where the other terms vanish because the variables are uncorrelated as argued previously but now also zero-mean, or by causality in that $i_{N_2}(\tau)$ cannot affect $q(t)$ for $\tau > t$. Again by appeal to the Fokker–Planck equations and the matching condition (29), it can be shown that

$$\frac{1}{R^2C(t)C(\tau)}E\{q(t)q(\tau)\} = \frac{1}{L(t)L(\tau)}E\{\phi(t)\phi(\tau)\}.$$

Thus, in order that the short-circuit current noise have the proper autocorrelation, it must be shown that

$$\frac{2}{R^2C(t)C(\tau)}E\{q(t)q(\tau)\} + \frac{1}{RC(\tau)}E\left\{q(\tau)i_{N_2}(t)\right\} = 0. \tag{32}$$

Two quick calculations from (30) yield

$$\begin{aligned} &E\{q(t)q(\tau)\} \\ &\quad = \exp\left[-\int_0^t \frac{ds}{RC(s)}\right] \exp\left[-\int_0^\tau \frac{1}{RC(s)}\,ds\right] \\ &\qquad \times \left(E\{q^2(0)\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma\right) \end{aligned}$$

and

$$\begin{aligned} &E\left\{q(\tau)i_{N_2}(t)\right\} \\ &\quad = -\frac{2kT}{R}\exp\left[-\int_0^\tau \frac{ds}{RC(s)}\right] \exp\left[\int_0^t \frac{ds}{RC(s)}\right]. \end{aligned}$$

Substituting these into (32) and canceling common factors, the test reduces to

$$\begin{aligned} 0 &\overset{?}{=} E\{q^2(0)\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma \\ &\quad - C(t)kT\exp\left[2\int_0^t \frac{ds}{RC(s)}\right]. \end{aligned} \tag{33}$$

Differentiating by t will yield a necessary condition for the equation to be true

$$\begin{aligned} 0 &\overset{?}{=} \frac{2kT}{R}\exp\left[2\int_0^t \frac{ds}{RC(s)}\right] d\sigma \\ &\quad - \frac{dC(t)}{dt}kT\exp\left[2\int_0^t \frac{ds}{RC(s)}\right] \\ &\quad - C(t)kT\exp\left[2\int_0^t \frac{ds}{RC(s)}\right]\left(\frac{2}{RC(t)}\right) \\ &= -\frac{dC(t)}{dt}kT\exp\left[2\int_0^t \frac{ds}{RC(s)}\right]. \end{aligned}$$

Thus, the time-varying bridge does not have stationary current noise at the terminals as required by the Nyquist–Johnson model, except in the trivial case that C is a constant. In this case, the integrals in (33) can be computed, and if the system starts at equilibrium, i.e., $E\{q^2(0)\} = kTC$, then this condition is sufficient as well as necessary. Of course, if C is a constant, then the bridge is simply the standard linear, time-invariant circuit, for which the result was already known.

Remark: For a driving voltage $V(t)$ significantly larger than the noise, one could solve the deterministic system and then compute an approximation for the noise behavior by linearization about this time-varying solution. This approximation would behave like the time-varying linear system described above. Since the second-order statistics for that system are incorrect, we believe that the second-order statistics for the nonlinear system driven by a time-varying voltage will not match the statistics of a single linear resistor driven by that same voltage.

V. Conclusion

We have explored an extension of the fluctuation-dissipation theorem (or, in circuit theory terms, a result relating impedances to noise spectra) to a nonlinear situation. The spectral calculations have been nontrivial, calling on a reversibility idea and martingale theory. The positive results hold for a specific time-invariant bridge circuit, linear or nonlinear, in thermal equilibrium or at dc steady state.

The negative results in Section IV show that our original fluctuation-dissipation conjecture is not correct as stated and must be limited to exclude time-varying networks and nonlinear networks with time-varying inputs. Is the modified form below correct? This remains an open question in the field, and some of the ideas in [8] may be of assistance.

Modified Fluctuation-Dissipation Conjecture for Networks: No two zero-state deterministically equivalent *time-invariant* networks can be distinguished by the terminal noise currents *at any dc voltage input* when the networks are in statistical steady state.

The assumptions here remain those in the paragraph preceding the initial formulation (see the Introduction), including LTI Nyquist–Johnson resistors and nonlinear inductors and capacitors. Additional assumptions may be required to guarantee reversibility of the charge or flux random processes. The further extensions to include nonlinear resistor noise models or multiterminal circuits remain completely unexplored, so far as we know.

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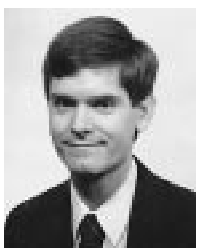
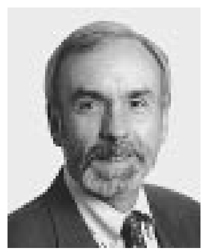


Geoffrey J. Coram (S'97) received the B.A. degree (cum laude) in physics and mathematics and the Master of Electrical Engineering degree from Rice University in 1993. He spent the following year in a laser laboratory at the University of Würzburg, Germany, on a Fulbright grant. He has been developing a doctoral thesis on thermodynamics and noise in the M.I.T. EECS Department since 1994, spending one summer on noise simulations at Analog Devices, Inc. and another on noise measurement at the M.I.T. Lincoln Laboratory.

Brian D. O. Anderson (S'62–M'66–SM'74–F'75) was born in Sydney, Australia, and received his undergraduate education at the University of Sydney, with majors in pure mathematics and electrical engineering. He received the Ph.D. degree in electrical engineering from Stanford University.

Following completion of his education, he worked in industry in Silicon Valley and served as a Faculty Member in the Department of Electrical engineering at Stanford. He was Professor of electrical engineering at the University of Newcastle, Australia from 1967 to 1981 and is now Professor of systems engineering at the Australian National University and Director of the Research School of Information Sciences and Engineering. His interests are in control and signal processing. He is a Fellow of the Royal Society, the Australian Academy of Science, Australian Academy of Technological Sciences and Engineering, and the Institute of Electrical and Electronic Engineers, and an Honorary Fellow of the Institution of Engineers, Australia. He holds doctorates (honoris causa) from the Université Catholique de Louvain, Belgium, Swiss Federal Institute of Technology, Zürich, University of Sydney and University of Melbourne.

He served a term as President of the International Federation of Automatic Control from 1990 to 1993 and is currently President of the Australian Academy of Science. His awards include the IEEE Control Systems Award of 1997.



John L. Wyatt, Jr. (S'75–M'78–SM'95) received the B.S. degree from the Massachusetts Institute of Technology (MIT), Cambridge, in 1968, the M.S. degree from Princeton University, Princeton, NJ, in 1970, and the Ph.D. degree from the University of California, Berkeley, in 1979, all in electrical engineering.

He spent two years in the U.S. Public Health Service developing a cheap, sensitive radiation measuring instrument and one postdoctoral year in the Physiology Department at the Medical College of Virginia modeling membrane transport phenomena in medicine and biology. He joined the M.I.T. faculty in 1979 and has been a Full Professor of electrical engineering since 1990. He codirects a joint project with the Harvard Medical School to develop a retinal implant device for the blind. His theoretical interests are in the area of nonlinear and noisy circuits and systems.