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Brief Paper

Identification of a nonlinear plant under nonlinear feedback using left coprime fractional based representations[1,2]

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Abstract

It has been shown that the set of all nonlinear plants stabilised by a known linear controller, which also stabilises a linear nominal model of the plant, can be parametrised by a stable operator known as the Youla–Kucera parameter. By utilising this description it is possible to convert the closed-loop plant identification problem to one of open-loop identification. This paper extends previous work by allowing the model of the nominal plant and the controller in the above scenario to be nonlinear. The ideas rely on a concept of differential coprimeness for nonlinear fractional system descriptions. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Closed-loop identification; Nonlinear system; Left coprime fractional based representation

1. Introduction

Consider the setting shown in Fig. 1, where P is a nonlinear plant to be identified, C is a nonlinear controller, and H is a linear stable output measurement noise generating system, driven in turn by the zero mean, white, stationary noise process e. It is assumed that C internally stabilises the unknown plant P. While we restrict attention to time-invariant C and P, there would seem to be no difficulty in extending the ideas to the time-varying case, as in Dasgupta and Anderson (1996).

In Dasgupta and Anderson (1996) using coprime factorisations of the plant and controller, the identification of nonlinear time-varying (NLTV) plants operating under linear, possibly time varying feedback is investigated. The authors show that there are left and right coprime based fractional descriptions for the set of all plants stabilised by the linear controller given that the nominal plant model is linear. They use models of the plant that can be based either on the left or right coprime factors of the nominal plant and controller. The plant models are expressed using the nominal plant description, the controller description and a Youla–Kucera parameter. However, the models themselves may not formally be left or right coprime fractions. Identification of the plant is equivalent to identification of the Youla–Kucera parameter, and this observation allows the closed-loop identification problem to be converted to one of open-loop identification.

Verma (1988) has shown that the nominal plant need not be linear to find a right coprime description of the plant, and in Anderson (1996) and Linard and Anderson (1996), results on closed-loop identification were presented that built on the Verma formulation.

This paper endeavours to extend the current theory relating to NLTV plants (which is based on the use of a Youla–Kucera parameter) to enable one to find a left coprime factorisation based description of the set of all plants stabilised by a given controller, given a nominal plant model and a controller that are not necessarily linear. Note that as in Dasgupta and Anderson (1996) not every plant in this set has a left coprime factorisation. When we say that a plant has a "left coprime factorisation based description", we mean we can write it in terms

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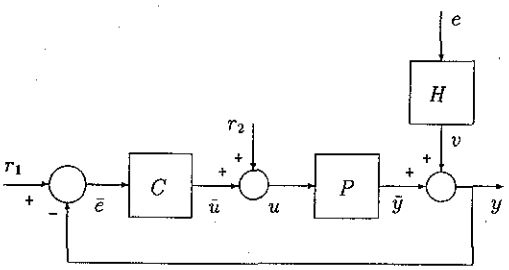

Fig. 1. The closed-loop system.

of the left coprime factors of the controller and a nominal plant model. We can not necessarily express the nonlinear plant input/output operator as a left fraction with coprime factors. This is contrary to the linear case where it is possible to factorise the plant in such a way. Once a plant has been characterised in this manner, this paper shows how the closed-loop identification problem can be turned into one of open-loop identification.

Section 2 contains notations, definitions and assumptions that are used throughout the paper. A notion of differential coprimeness is introduced to help characterise the model set of the plant. A similar method using differential boundedness is contained in Paice and Moore (1990a, b). Section 3 characterises the set of all plants stabilised by a known possibly nonlinear controller. This is done using left fractional descriptions of both the controller and a nominal nonlinear plant stabilised by the controller, as well as a stable operator, known as a Youla–Kucera parameter. It covers the noise free case and then modifies this description to incorporate nonzero measurement noise into the system. The section is concluded by depicting how these model characterisations can be used under a high SNR assumption to identify the system. Appendix A includes stability results that are used in this paper. They are also interesting in their own right as they form a nonlinear version of the Bezout identity.

2. Preliminary background

This section states the definitions and assumptions that later results in the paper are based on. We refer the reader to Vidyasagar (1980) for further details.

2.1. Notations

$L_2^m[0,\infty)$: the vector space of $\mathbb{R}^m$ valued square integrable functions with norm defined by

$$\|u\|^2 = \int_0^\infty |u(t)|^2\,dt.$$

$L_2[0,\infty)$: shorthand for $L_2^m[0,\infty)$ where m is an arbitrary positive integer.

$\mathscr{T}_T$: the truncation operator on the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}^m$ (m an arbitrary positive integer). It is defined by $\mathscr{T}_T u(t) = u(t)$ if $t \le T$, $\mathscr{T}_T u(t) = 0$ if $t > T$.

$L_{2e}^m[0,\infty)$: the vector space of functions f satisfying $\mathscr{T}_T f \in L_2^m[0,\infty)$ for all $T > 0$.

2.2. Definitions

Well-posedness: an operator A: $L_{2e}^p[0,\infty) \to L_{2e}^q[0,\infty)$ is well-posed or causal if $\mathscr{T}_T A \mathscr{T}_T = \mathscr{T}_T A$ for all $T > 0$. It is also assumed that $A(0) = 0$.

Invertible: an operator A is invertible if for all $z \in \mathscr{Y}$, with $z = Ax$, x can be causally and uniquely determined from z.

Bounded operator and gain: an operator A is bounded if it is well-posed and the gain

$$\|A\| = \sup_{u \in L_2^p[0,\infty),\, u \neq 0} \frac{\|Au\|}{\|u\|}$$

is finite. A is also termed BIBO stable.

Unit: an operator W is a unit if W^{-1} exists, and W and W^{-1} are BIBO stable.

Weak Lipschitz continuity: a well-posed operator A is weakly Lipschitz (or weakly Lipschitz continuous) if for every $T > 0$ there exists a finite γ_T such that

$$\|\mathscr{T}_T A\|_L \le \gamma_T$$

where the Lipschitz semi-norm is

$$\|\mathscr{T}_T A\|_L = \sup_{\substack{\mathscr{T}_T x \neq \mathscr{T}_T y,\\ x, y \in L_{2e}[0,\infty)}} \frac{\|\mathscr{T}_T Ax - \mathscr{T}_T Ay\|}{\|\mathscr{T}_T x - \mathscr{T}_T y\|}$$

and obeys $\|\mathscr{T}_T A\|_L \ge \|\mathscr{T}_T A\|$.

Global Lipschitz continuity: a well-posed operator A is globally Lipschitz continuous if there exists a K such that for all $x, y \in L_2^p[0,\infty)$ there holds

$$\|A(x+y) - A(x)\| \le K\|y\|.$$

This means there exists an operator $\partial A_{(x)}$, (causally) dependent on x and bounded independently of x with

$$\partial A_{(x)}(y) = A(x+y) - A(x).$$

When A is linear, $\partial A_{(x)}$ is independent of x.

A nontrivial consequence is that the definition implies $\|A(x+y) - A(x)\| \le K\|y\|$ for any $x \in L_{2e}^p[0,\infty)$, as opposed to $x \in L_2^p[0,\infty)$. This is shown in the following lemma.

Lemma 2.1. *Let a well-posed operator* A: $L_{2e}^p[0,\infty) \to L_{2e}^p[0,\infty)$ *be such that for some* K *and for all* $x, y \in L_2^p[0,\infty)$ *we have*

$$\|A(x+y) - A(x)\| \le K\|y\|.$$

Then this inequality holds for all $x \in L_{2e}^p[0,\infty)$.

Proof. The proof is contained in Appendix B. □

Smoothing (Vidyasagar, 1980): a well-posed operator A is said to be *smoothing*, have no instantaneous direct feedthrough, or have zero uniform instantaneous gain, if it is weakly Lipschitz and for every $T > 0$, $\alpha > 0$, there exists $\delta = \delta(\alpha,T) \in (0,T)$ such that

$$\|\mathscr{T}_{t+\delta}(A\mathscr{T}_{t+\delta} - A\mathscr{T}_t)\|_L \le \alpha, \quad \forall t \in [0, T-\delta].$$

[A linear time-invariant operator has this property if its causal impulse response for $t \ge 0$ is of the form $\sum_{i=1}^{N} \alpha_i \delta(t-\tau_i) + \beta(t)$, $\tau_i > 0$; in discrete time, the equivalent is a strictly causal operator, i.e. $\mathscr{T}_{(k+1)\tau} A \mathscr{T}_{k\tau} = \mathscr{T}_{(k+1)\tau} A$ where τ is the sampling interval.]

Remark 2.1. A system of the form

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) + j(x)u, \tag{2.1}$$

with $f(x)$, $g(x)$, $h(x)$, $j(x)$ smooth functions of x, is smoothing if and only if $j(x) = 0$.

For *a closed-loop* system as shown in Fig. 1, with noise v identically zero:

The closed-loop is well-posed (Vidyasagar, 1980): if

(a) for each $r_1, r_2 \in L_{2e}[0,\infty)$ there exists a unique $\bar{e}, \bar{u}, u, y \in L_{2e}[0,\infty)$ that depend causally on r_1 and r_2;
(b) for each finite T, the dependence of $\mathscr{T}_T\bar{e}$, $\mathscr{T}_T\bar{u}$, $\mathscr{T}_T u$, and $\mathscr{T}_T y$ on $\mathscr{T}_T r_1$ and $\mathscr{T}_T r_2$ is Lipschitz continuous. This means there is a causal weakly Lipschitz closed-loop operator from r_1, r_2 to $\bar{e}, \bar{u}, u, y$.

The smoothing concept of Vidyasagar (1980) is a powerful tool for establishing the well-posedness property for a closed-loop where properties of the individual loop components are specified, and include a smoothing property on one of the components.

The closed-loop is internally stable if it is well-posed and the associated operator has finite gain.

If the noise v is non-zero r_1 and r_2 must be replaced with r_1, r_2 and v in these definitions.

Remark 2.2. We can summarise the following relationships from Vidyasagar (1980).

1. The sum of two weakly Lipschitz operators is also weakly Lipschitz.
2. The sum of two smoothing operators is also smoothing.
3. If the operator A is smoothing and the operator B is weakly Lipschitz then AB is smoothing; BA however may not necessarily be smoothing.
4. Refer to Fig. 2a. If both the operator A and the operator B are weakly Lipschitz and one of them is smoothing then the closed-loop feedback system (A, B) is well-posed.

Remark 2.3. The following scenario will be used in Section 3. Consider Fig. 2b where S_1 and S_2 are smoothing operators. From Item 4 of Remark 2.2 it follows that the loop of Fig. 2b is well-posed. It follows from our definition of well-posedness that the operator from r to $\bar{e}$ is weakly Lipschitz. Using Item 3 of Remark 2.2, it now follows that the operator from r to y is smoothing.

Remark 2.4. As is conventional but not universal in treating BIBO stability of systems, there is no explicit consideration of initial conditions in this paper. They can be introduced in several ways; by postulating they are established using inputs zero in $t < 0$, and acting over an interval [0,1] say, to provide the initial condition at time 1; or one can postulate that operators are indexed by an initial condition; or, in some but not all cases, one can replace an initial condition effect by an extra input or disturbance signal.

The definition of coprimeness in nonlinear fractional representations is not universal. The definitions listed below, based on Bezout identities, have been used in e.g. Chen and de Figueiredo (1992), Danow and Chen (1993) and Verma (1988). However, alternative definitions, based on set theoretic ideas, have been used in e.g. Hammer (1985), Paice et al. (1992) and Tay and Moore (1989). In the linear case, the definitions are equivalent; in the nonlinear case, right coprimeness defined using a Bezout identity is also equivalent to right coprimeness from a set theoretic view point; see Baños (1996). This is not the case for left coprimeness as pointed out in Linard et al. (1998).

Right coprimeness (linear or nonlinear). Let N_r, D_r be a right factorisation for a well-posed P, i.e. $P = N_r D_r^{-1}$ where N_r and D_r are BIBO stable. Then (N_r, D_r) is a right coprime factorisation of P if there exists a BIBO operator $\mathscr{L}_l$ for which

$$\mathscr{L}_l \begin{bmatrix} N_r \\ D_r \end{bmatrix} = I. \tag{2.2}$$

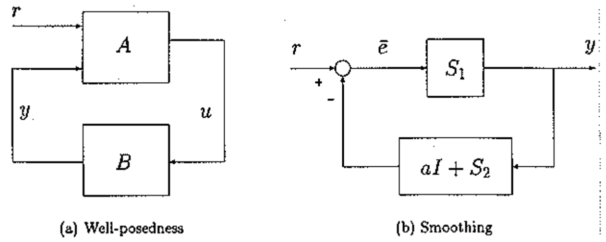

Fig. 2. Closed-loop configuration for (a) well-posedness and (b) smoothing.

Here I denotes the identity operator. The relationship of Eq. (2.2) is known as a Bezout identity. A particular case of this which is always true when using linear operators is $\mathscr{L}_l = [X_l\ Y_l]$ with X_l and Y_l BIBO such that

$$X_l N_r + Y_l D_r = W,$$

where W is a unit. However, $\mathscr{L}_l$ does not, in general, separate into $[X_l\ Y_l]$ in the nonlinear case.

Left coprimeness (linear or nonlinear): let N_l, D_l be a left factorisation for a well-posed P, i.e. $P = D_l^{-1}N_l$, where N_l and D_l are BIBO stable. Then (N_l, D_l) is a left coprime factorisation of P if there exists a BIBO $\mathscr{L}_r$ for which

$$[N_l\ D_l]\mathscr{L}_r = I. \tag{2.3}$$

Again, Eq. (2.3) is a Bezout identity for left coprimeness. It is obvious that $\mathscr{L}_r$ must have the form $\mathscr{L}_r = [X_r Y_r]^{\mathrm{T}}$ for some BIBO operators X_r and Y_r.

Without loss of generality, the right-hand side of Eq. (2.2) or Eq. (2.3) can be replaced by an arbitrary unit W. Appendix E contains a few examples of systems that have left coprime factorisations.

Differential coprimeness: if the pair (N_l, D_l) is left coprime and globally Lipschitz continuous, then we can write

$$N_l U_r + D_l V_r = W, \tag{2.4}$$

where N_l, D_l, U_r and V_r are all nonlinear and W is a unit. Then, one can define well-posed operators

$$\partial N_{l(x)}(.) = N_l(x + .) - N_l(x) \quad \text{and} \quad \partial D_{l(z)}(.) = D_l(z + .) - D_l(z)$$

for all signals x and $z \in L_{2e}$.

If N_l and D_l are linear, $\partial N_{l(x)}(.) = N_l(.)\forall x$ and $\partial D_{l(z)}(.) = D_l(.)\forall z$; then $(\partial N_{l(x)})U_r + (\partial D_{l(z)})V_r$ is a unit, i.e. W.

When N_l and D_l are nonlinear, we shall say that they are differentially coprime if and only if the unit property continues to hold, though now the unit will not usually be W.

Formally, N_l and D_l are left differentially coprime if for all x, $z \in L_{2e}$, there exists BIBO U_r and V_r such that

$$(\partial N_{l(x)})U_r + (\partial D_{l(z)})V_r = W_{x,z},$$

where $W_{x,z}$ is a unit operator. More motivation for the definition of left differential coprimeness is given in Appendix C. Similar definitions hold for right coprime factorisations.

Remark 2.5. We say that N_l and D_l are *uniformly* left differentially coprime if there exists K such that $\|W_{x,z}\| < K$ and $\|W_{x,z}^{-1}\| < K$ independently of x and $z \in L_{2e}$.

Remark 2.6. Note that if N_l and D_l are known to be left differentially coprime in the sense that for some bounded U_r and V_r, $\partial N_{l(x)}U_r + \partial D_{l(z)}V_r$ is a unit for any x and z, then by taking $x = 0$, $z = 0$ we recover the standard coprimeness relation

$$N_l U_r + D_l V_r = W \quad \text{for some unit } W.$$

2.3. Assumptions

In the results which follow, we shall invoke the following assumptions.

Assumption 2.1. *The nonlinear plant P is weakly Lipschitz and well-posed.*

Assumption 2.2. (i) *The controller C is weakly Lipschitz and there exists U_l, V_l, U_r, V_r all stable, well-posed operators with V_l, V_r invertible such that*

$$C = U_r V_r^{-1} = V_l^{-1}U_l. \tag{2.5}$$

(U_r, V_r) are right coprime factors of the controller that are differentially coprime and globally Lipschitz continuous, and *(U_l, V_l) are left coprime factors of the controller that are globally Lipschitz continuous and uniformly differentially coprime.*

(ii) *The nominal plant model P_0 is smoothing, and there exists N_l, D_l, N_r, D_r all stable, well-posed operators with D_l, D_r invertible such that*

$$P_0 = N_r D_r^{-1} = D_l^{-1}N_l. \tag{2.6}$$

(N_r, D_r) are right coprime factors of P_0 that are differentially coprime and globally Lipschitz continuous, and (N_l, D_l) are left coprime factors of P_0 that are globally Lipschitz continuous and uniformly differentially coprime.

It is further assumed that N_l and U_l are smoothing and D_l and V_l are of the form $aI + S$ where aI is the scaled identity operator and S is a smoothing operator.

Remark 2.7. Assumption 2.2 can be quite restrictive; especially the requirement of existence of left coprime factorisations for the nominal plant and the controller. The concept of kernel representation introduced in Paice and van der Schaft (1996) might prove to be useful to alleviate these restrictions. In a sense, left coprime realizations are a special case of kernel representations.

Remark 2.8. The assumption that N_l is smoothing will be fulfilled if there is at least one integration between the input of P and its output. A system writable in the form of Eq. (2.1) with $j(x) = 0$ would have N_l smoothing.

Remark 2.9. Suppose $D_l = aI + S$ where a is a constant, I is the identity operator and S is smoothing. Then

$$D_l^{-1} = (aI + S)^{-1} = a^{-1}I - a^{-1}S(aI + S)^{-1}.$$

Now $a^{-1}S(aI + S)^{-1}$ is smoothing, since it results from a loop with S in the forward path, and a^{-1} in a feedback path; see Remark 2.2. Thus D_l^{-1} is also of the form $\bar{a}I + \bar{S}$ with $\bar{S}$ smoothing.

Remark 2.10. Since C is weakly Lipschitz and P_0 is smoothing it follows that the closed-loop (P_0, C) is well-posed, see Remark 2.2 for further details.

Assumption 2.3. *The controller C stabilises the nominal plant model P_0.*

2.4. A consequence of uniform differential coprimeness

One difficulty in working with nonlinear operators is that $-A(-B)$ does not necessarily equal AB. Thus if for some unit W, coprime pair (N_l, D_l) and BIBO operators U_r, V_r, there holds

$$N_l U_r - D_l(-V_r) = W, \tag{2.7}$$

we cannot say this is equivalent to Eq. (2.4), or even Eq. (2.4) with the unit W replaced by another unit operator. However, with the aid of uniform differential coprimeness, we can show that when (N_l, D_l) is uniformly left differentially coprime, so is $(N_l, -D_l)$.

Lemma 2.2. *Let (N_l, D_l) be a left coprime pair where N_l and D_l are globally Lipschitz continuous and uniformly differentially coprime; i.e. for any x, z, $\beta \in L_{2e}^p$, we have*

$$(\partial N_{l(x)} U_r + \partial D_{l(z)} V_r)\beta = W_{x,z}(\beta) \tag{2.8}$$

where $\|W_{x,z}\| \le K$, $\|W_{x,z}^{-1}\| \le K$ and we assume that N_l and U_r are smoothing and D_l and V_r are of the form $aI + S$ where aI is the scaled identity operator and S is a smoothing operator. Then $(N_l, -D_l)$ is also uniformly left differentially coprime.

Proof. The proof is contained in Appendix D. □

3. Characterisation and identification of nonlinear plants using a left coprime factor based description

This section shows that all nonlinear plants stabilised by a nonlinear controller C can be represented by the setting depicted in Fig. 3, with R a nonlinear BIBO, smoothing, well-posed operator known as the Youla–Kucera parameter. Conversely, if the setup of Fig. 3 defines a well-posed, smoothing P for some BIBO, smoothing, well-posed R, then P is stabilised by C.

As this description involves the left coprime factors of the controller, C, and the nominal plant, P_0, it will be referred to as a left coprime factorisation *based* description. Note that P (as opposed to P_0) does not always have a left coprime factorisation due to the nonlinearity of the operators involved. From Fig. 3, we can write

$$D_l y = N_l u + R(V_l u - U_l(-y)). \tag{3.1}$$

If R, U_l and N_l are linear this reduces to the left coprime factorisation

$$y = (D_l - RU_l)^{-1}(N_l + RV_l)u.$$

However, no such convenient representation may be written down when R, U_l and N_l are allowed to be nonlinear. So we turn to the structure described in Fig. 3. The theorems of Section 3.1 argue that this representation depicts the set of all plants stabilised by a given controller and hence shows how the closed-loop identification problem can be converted to an open-loop problem in the presence of noise. Section 3.2 is broken into two parts. The first part treats the noiseless situation, i.e. $v = 0$. The second part treats the case when the noise is no longer zero, and it describes how the disturbance can be incorporated into the identification algorithm.

3.1. Describing the structure of the set of all plants stabilised by a given controller in a noise free setting

We will use some of the stability and operator existence results set out in Appendix A in proving the following lemma.

Lemma 3.1. *Adopt the assumptions in Section 2.3. Suppose that R is a well-posed, bounded operator. Then if R is smoothing, P is smoothing. Also the closed-loop of Fig. 4 is well-posed and internally stable.*

Proof. Firstly, we want to show that if R is smoothing, then P is smoothing. Observe from Fig. 3 that if R is smoothing then, as U_l and V_l are weakly Lipschitz, the operator $(u, y) \to R(V_l u - U_l(-y))$ is guaranteed to be smoothing; see Remark 2.2. Since N_l is smoothing then

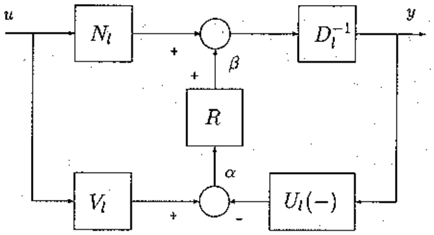

Fig. 3. Left coprime factorisation based description of P.

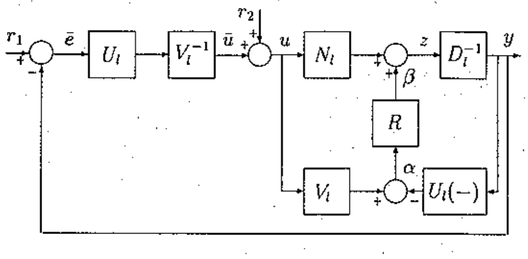

Fig. 4. Closed-loop of Fig. 1 with plant P as in Fig. 3 and $v = 0$.

the operator $(u, y) \to N_l u + R\alpha$ is smoothing; i.e. the operator Z: $[u\,y]^{\mathrm{T}} \to z$ is smoothing and Fig. 3 can be redrawn as in Fig. 5. Since D_l^{-1} has the form $(aI + S)$ where S is smoothing, it follows from Remark 2.3 that P defined in Fig. 3 is a smoothing operator.

Next we wish to show that the closed-loop of Fig. 4 is well-posed and internally stable. If we redraw Fig. 1 as shown in Fig. 6 then the closed-loop will be well-posed if A_C is weakly Lipschitz and P is smoothing; see Remark 2.2. We have just shown that P is smoothing, thus we only need to show that A_C is weakly Lipschitz. Note that

$$u = A_C(r_1, r_2, y) = r_2 + C(r_1 - y). \tag{3.2}$$

Since C is weakly Lipschitz it follows that A_C is also weakly Lipschitz. Hence the closed-loop in Fig. 4 is well-posed.

Finally we wish to show that the closed-loop is internally stable. We have already demonstrated that it is well-posed so we only need to show that the associated loop gain is finite.

Considering Fig. 3 alone, it is evident that

$$\begin{bmatrix} \alpha \\ R\alpha \end{bmatrix} = \begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}. \tag{3.3}$$

Considering Fig. 4, we see that

$$\begin{aligned} V_l u - U_l(-y) &= V_l(r_2 + \bar{u}) - U_l(-y) \\ &= V_l(\bar{u}) + \partial V_{l(\bar{u})}(r_2) - U_l(-y) \\ &= U_l(r_1 - y) + \partial V_{l(\bar{u})}(r_2) - U_l(-y) \\ &= \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2 \end{aligned} \tag{3.4}$$

i.e. in (Eq. (3.3)), $\alpha = \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2$ is bounded. Since R is bounded, $R\alpha$ is bounded also. By Combination A.4 of Appendix A, stability of the closed-loop system (P_0, C) ensures that u, y exists and are bounded. Hence $\bar{u} = u - r_2$ and $\bar{e} = r_1 - y$ are also bounded and internal stability follows. □

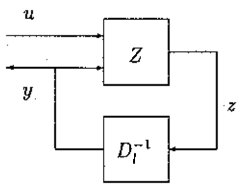

Fig. 5. Redrawn Fig. 3.

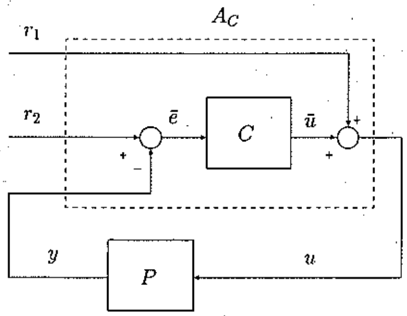

Fig. 6. Closed-loop system of Fig. 1 redrawn as in Fig. 2a with $v = 0$.

It now remains to show the converse result, namely that for a stable (P, C) interconnection, there exists an operator R which is bounded.

Lemma 3.2. *Adopt the assumptions in Section* 2.3 *and suppose the closed-loop in Fig.* 1 *is well*-posed and internally stable. *Then there exists a well*-posed, bounded *R given by*

$$R = (D_l P - N_l)(V_l - U_l(-P))^{-1}, \tag{3.5}$$

such that in Fig. 3

$$y = Pu.$$

Further, if *P is smoothing then R is smoothing.*

Proof. To show that R is a well-posed operator we first have to show that the operator $(V_l - U_l(-P))^{-1}$ exists. From Fig. 7, we have

$$V_l(u - r_2) = U_l(r_1 - Pu)$$

$$\text{whence } V_l u + \partial V_{l(u)}(-r_2) = U_l(-Pu) + \partial U_{l(-y)} r_1$$

which implies

$$[V_l - U_l(-P)]u = \partial U_{l(-y)} r_1 - \partial V_{l(u)}(-r_2).$$

From Combination A.2, $[V_l D_r - U_l(-N_r)]^{-1}$ exists. Now

$$[V_l D_r - U_l(-N_r)] = [V_l - U_l(-P)]D_r$$

or

$$[V_l D_r - U_l(-N_r)]D_r^{-1} = [V_l - U_l(-P)].$$

Hence, $(V_l - U_l(-P))^{-1}$ exists. Thus in Eq. (3.5), for each $\alpha \in L_{2e}[0, \infty)$, there exists a unique $R\alpha$ that depends causally on α. Further, as N_l, D_l, U_l, V_l and P are well-posed operators, R is also well-posed.

Next, we must show that R is bounded. With r_1, r_2 as above and from Fig. 4, we have that $D_l y - N_l u$ is bounded since the closed-loop is internally stable. Further

$$\begin{aligned} D_l y - N_l u &= (D_l P - N_l)u \\ &= (D_l P - N_l)(V_l - U_l(-P))^{-1}\alpha = R\alpha. \end{aligned}$$

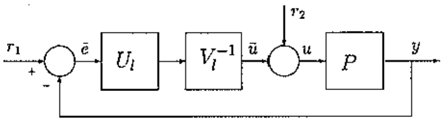

Fig. 7. Closed-loop system of Fig. 1 with true plant P and $v = 0$.

From (3.4) we have

$$\alpha = \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2. \tag{3.6}$$

As U_l and V_l are left uniformly differentially coprime there exist M_r, N_r such that

$$\partial U_{l(-y)} M_r + \partial V_{l(\bar{u})} N_r = W_{y,\bar{u}} \tag{3.7}$$

where $W_{y,\bar{u}}$ is a unit. Thus choosing $r_1 = M_r r$ and $r_2 = N_r r$, Eq. (3.6) becomes $\alpha = W_{y,\bar{u}} r$. From r_1 and r_2 we can make arbitrary $\alpha \in L_2$ with $R\alpha \in L_2$, hence R is bounded.

It remains to show that if the R defined by Eq. (3.5) is inserted in Fig. 4, the plant P is obtained. To this end, observe from Eq. (3.5) that

$$(D_l P - N_l)u = R(V_l - U_l(-P))u$$

$$\text{or } D_l P u = N_l u + R(V_l u - U_l(-P)u). \tag{3.8}$$

In comparison, from Fig. 4 we have the relationship

$$D_l y = N_l u + R(V_l u - U_l(-y)).$$

As the plant in Fig. 4 is well-posed, each u must give rise to a unique y. From Eq. (3.8), Pu, constitutes a possible output; hence by uniqueness $y = Pu$.

Lastly, we will show that in Eq. (3.5), if P is smoothing, R is smoothing, see Remark 2.2. Since $D_l = aI + S$ with S smoothing, $D_l P - N_l$ is smoothing. Also, the operator $[V_l - U_l(-P)]^{-1}$ can be constructed as shown in Fig. 8. It follows from the assumptions and Item 4 of Remark 2.2 that the operator $[V_l - U_l(-P)]^{-1}$: $x \to z$ is well-posed, i.e. it follows from the definition of well-posedness that the operator $[V_l - U_l(-P)]^{-1}$: $x \to z$ is weakly Lipschitz. Therefore it follows from Item 3 of Remark 2.2 that R is smoothing. □

In summary, we now have

Theorem 3.1. *Adopt the assumptions in Section 2.3. Then the closed-loop in Fig. 1 is well-posed and internally stable if and only if P has a description of the form of Fig. 3, with R a well-posed, stable, smoothing operator. Further P is smoothing if and only if R is smoothing.*

Proof. Lemmas 3.1 and 3.2 provide the proof for Theorem 3.1. □

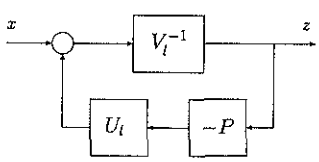

Fig. 8. The operator $[V_l - U_l(-P)]^{-1}$.

3.2. Conversion to open-loop identification and incorporation of measurement noise

This section demonstrates how the measurement noise can be incorporated in order to enable identification. The conversion to open-loop identification requires a small noise assumption (high SNR) so that R may be linearised around its operating trajectory. As in Dasgupta and Anderson (1996), it is shown that instead of identifying the plant P, we can identify the Youla–Kucera parameter, R.

Refer first to Fig. 4. In the noise free case, i.e. with $v = 0$, there holds

$$\alpha = V_l u - U_l(-y) = \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2.$$

Also, if $\beta = R\alpha$, then

$$\beta = -N_l u + D_l y. \tag{3.9}$$

Stability of the closed-loop ensures that α and β are (in principle) computable (boundedly) from r_1, r_2 and u, y, respectively. It is now possible to identify R in a standard open-loop fashion. The next paragraph examines how to take measurement noise into account.

(a) *Incorporation of measurement noise.* When $v \neq 0$, similar equations hold provided we replace r_1 and y by $r_1 - v$ and $y - v$ respectively in determining the input and output of R. This can be seen by examining Fig. 9. Put another way, we now have

$$\beta_v = R\alpha_v$$

with

$$\begin{aligned} \alpha_v &= V_l u - U_l(-y+v) = V_l(r_2 + \bar{u}) - U_l(-y+v) \\ &= V_l \bar{u} + \partial V_{l(\bar{u})} r_2 - U_l(-y) - \partial U_{l(-y)} v \\ &= U_l[(r_1 - y)] + \partial V_{l(\bar{u})} r_2 - U_l(-y) - \partial U_{l(-y)} v \\ &= \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2 - \partial U_{l(-y)} v \\ &= \alpha - \partial U_{l(-y)} v \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \beta_v &= -N_l u + D_l(y - v) = -N_l u + D_l y + \partial D_{l(y)}(-v) \\ &= \beta + \partial D_{l(y)}(-v). \end{aligned} \tag{3.11}$$

As before, α and β are given by Eq. (3.2) and (3.9) and are effectively measurable. That is, the closed-loop identification problem has been transformed into a nonstandard

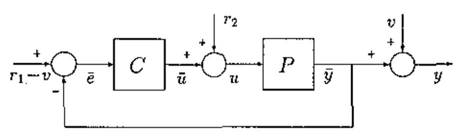

Fig. 9. Rearrangement of the closed-loop system of Fig. 1.

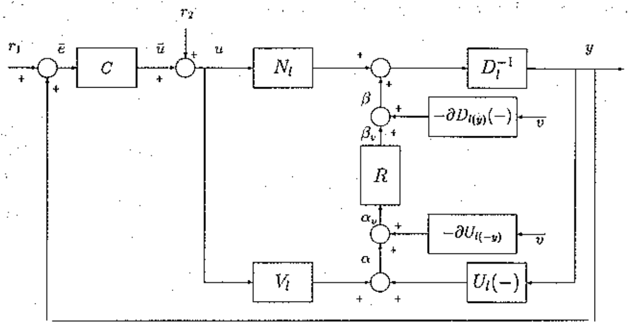

Fig. 10. Incorporation of the noise in the left coprime factorisation based description and conversion to a nonstandard open-loop identification problem.

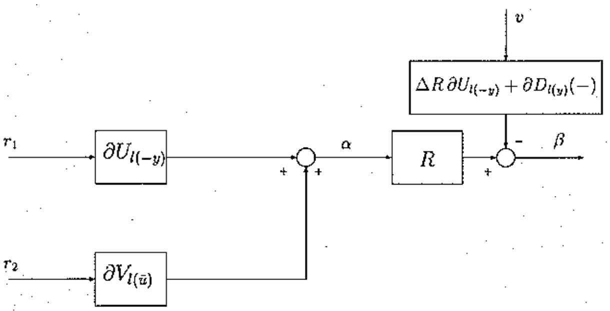

Fig. 11. Conversion to a standard open-loop identification problem.

open-loop identification problem. From $\beta_v = R\alpha_v$ we now obtain Fig. 10.

As in Dasgupta and Anderson (1996) where the conversion process is considered for a nonlinear plant with a linear nominal plant model and a linear controller, the noise enters the structure in two places. This is opposed to the case where the plant, nominal plant model and controller are all linear. In such a case the noise enters in only one place.

(b) *Conversion to a standard open-loop identification problem.* Again assuming a high SNR, there exists a linearisation ΔR of R around the trajectory produced by the input signal α which yields

$$\beta = R\alpha + \Delta R(-\partial U_{l(-y)}v) - \partial D_{l(y)}(-v)$$

$$= R\alpha - (\Delta R\,\partial U_{l(-y)} + \partial D_{l(y)}(-))v.$$

The closed-loop identification problem has been transformed into a standard nonlinear open-loop identification problem as shown in Fig. 11. This method requires both reference signals to be non-zero; see Dasgupta and Anderson (1996) for further details.

4. Conclusion

This paper has considered the identification of a nonlinear plant operating in closed-loop with a nonlinear controller. Factorisation based structures have been derived to help convert the underlying closed-loop identification problem to one that is essentially that of open-loop identification.

Many of the results are analogous to those in Anderson (1996), Dasgupta and Anderson (1996) and Linard and Anderson (1996). In particular, the requirement for a high signal to noise ratio is still present. However, the findings of this paper rest on new stability results. This paper extends the forays into this area by allowing for nonlinearity of the controller and initial plant model given that the actual nonlinear plant need not have a left coprime factorisation.

Appendix A. Stability and operator existence

In order to establish the main result, we need several characterisations of stability. Adopt the assumptions of Section 2.3. We consider that the nominal plant is connected in closed-loop with the stabilising controller. We have two coprime representations for the controller and two for the nominal plant. In this appendix we consider the four combinations of these representations to find a nonlinear form of the double Bezout Identity.

Combination A.1. $(P_0, C) = (N_r D_r^{-1}, U_r V_r^{-1})$. *The closed-loop system* (P_0, C) *is internally stable if and only if*

$$\begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix}^{-1} \tag{A.1}$$

exists and is bounded.

This result is effectively inherent in the definition, and was established in Verma (1988).

Combination A.2. $(P_0, C) = (N_r D_r^{-1}, V_l^{-1} U_l)$. *The closed-loop system* (P_0, C) *is internally stable if and only if*

$$(V_l D_r - U_l(-N_r))^{-1}$$

exists and is bounded.

Proof. Recall the definition of internal stability. We need to show that the closed-loop is well-posed and that the associated operator has finite gain. The closed-loop system is well-posed by assumption. Thus in proving the "if" part of this combination statement, we only have to show that the closed-loop gain is finite. Consider Fig. 12, then from differential coprimeness

$$m = V_l[D_r n - r_2] = V_l(D_r n) + \partial V_{l(D_r n)}(-r_2) \tag{A.2}$$

and also

$$m = U_l[r_1 - N_r n] = U_l(-N_r)n + \partial U_{l(-N_r n)} r_1. \tag{A.3}$$

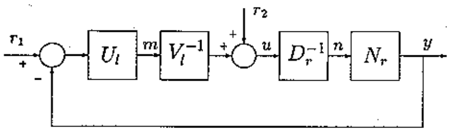

Fig. 12. Closed-loop diagram with the nominal plant $P_0 = N_r D_r^{-1}$ and $C = V_l^{-1} U_l$.

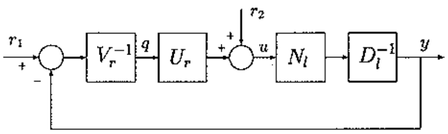

Fig. 13. Closed-loop diagram with the nominal plant $P_0 = D_l^{-1} N_l$ and $C = U_r V_r^{-1}$.

Combining Eqs (A.2) and (A.3) gives

$$(V_l D_r - U_l(-N_r))n = \partial U_{l(p)} r_1 - \partial V_{l(q)}(-r_2) \tag{A.4}$$

where $p = -N_r n$ and $q = D_r n$. Here $\partial U_{l(p)}(.)$ and $\partial V_{l(q)}(.)$ are, respectively, bounded independently of p and q. These operators exist because of the global Lipschitz continuity of U_l and V_l. If the inverse $[V_l D_r - U_l(-N_r)]^{-1}$ exists and is bounded for all bounded r_1, r_2, it follows that n is bounded. Then $y = N_r n$ and $u = D_r n$ are bounded, i.e. the system is internally stable.

Conversely, if the system is stable, y and u are bounded for all bounded r_1 and r_2. By the right coprimeness of N_r, D_r there exists a bounded $\mathscr{L}_l$ with Eq. (2.2) holding. Now

$$n = \mathscr{L}_l \begin{bmatrix} N_r n \\ D_r n \end{bmatrix} = \mathscr{L}_l \begin{bmatrix} y \\ u \end{bmatrix}.$$

Thus n is always bounded for all bounded r_1 and r_2 if the system is stable. As V_l and U_l are left differentially coprime, there exist bounded operators A_r and B_r that satisfy

$$\partial U_{l(q)} A_r - \partial V_{l(p)}(-B_r) = W_{p,q},$$

where $W_{p,q}$ is a unit. Note that we have used Lemma 2.2. With an arbitrary bounded r_3 and with A_r and B_r, define bounded signals

$$r_1 = A_r r_3 \quad \text{and} \quad r_2 = B_r r_3.$$

and observe that (A.4) becomes

$$\begin{aligned}(V_l D_r - U_l(-N_r))n &= [\partial U_{l(p)} A_r - \partial V_{l(q)}(-B_r)] r_3 \\ &= W_{p,q} r_3.\end{aligned}$$

Since r_3 is bounded but arbitrary, $W_{p,q}$ is a unit and n is bounded, it follows that the inverse $[V_l D_r - U_l(-N_r)]^{-1}$ exists and is bounded. □

The next combination appears to be of independent interest. It is used to describe the structure of the set of all plants stabilised by a given controller.

Combination A.3. $(P_0, C) = (D_l^{-1} N_l, U_r V_r^{-1})$ *(Fig. 13). The closed-loop system* (P_0, C) *is internally stable and* $q = V_r^{-1}(r_1 - y)$ *is bounded if and only if*

$$[D_l(-V_r) - N_l U_r]^{-1} \tag{A.5}$$

exists and is bounded.

Proof. Observe first that the closed-loop system is well-posed by assumption. Thus we only need to show that the closed-loop gain is finite to prove internal stability. Note

$$D_l y = N_l(r_2 + U_r q) = N_l U_r q + \partial N_{l(p)}(r_2),$$

where $p = U_r q$ and $\partial N_{l(p)}(.)$ is an operator bounded independently of q, existing because of the global Lipschitz continuity of N_l. Similarly,

$$D_l y = D_l(-V_r q + r_1) = D_l(-V_r q) + \partial D_{l(t)}(r_1),$$

where $t = -V_r q$ and $\partial D_{l(t)}(.)$ is an operator bounded independently of q. The two expressions for $D_l y$ yield

$$(D_l(-V_r) - N_l U_r) q = \partial N_{l(p)}(r_2) - \partial D_{l(t)}(r_1). \tag{A.6}$$

If $[D_l(-V_r) - N_l U_r]^{-1}$ exists and is bounded, then q is obviously bounded when r_1, r_2 are bounded. It follows that $y = r_1 - V_r q$ and $u = r_2 + U_r q$ are bounded, i.e. internal stability holds.

To prove the converse, suppose the loop is stable in the sense that for all bounded r_1 and r_2, q is bounded. We have to prove that $[D_l(-V_r) - N_l U_r]^{-1}$ is also bounded. Let r_3 be an arbitrary bounded signal. Let L_r, M_r be bounded operators existing by virtue of the left differential coprimeness property and satisfying

$$\partial N_{l(p)} M_r - \partial D_{l(t)}(-L_r) = W_{p,t}, \tag{A.7}$$

where $W_{p,t}$ is a unit. Note that we have used Lemma 2.2. Using r_3, L_r and M_r define bounded signals

$$r_1 = -L_r r_3 \quad \text{and} \quad r_2 = M_r r_3$$

and observe that (A.6) becomes

$$\begin{aligned}(D_l(-V_r) - N_l U_r) q &= [\partial N_{l(p)} M_r - \partial D_{l(t)}(-L_r)] r_3 \\ &= W_{p,t} r_3.\end{aligned}$$

Now in this equation, r_3 is bounded and arbitrary; further, q is bounded (because the boundedness of r_3 implies boundedness of r_1 and r_2, which by hypothesis implies q is bounded). Since $W_{p,t}$ is a unit, this means that $[D_l(-V_r) - N_l U_r]^{-1}$ is necessarily a bounded operator. □

The final stability result is found by constructing a nonlinear form of the double Bezout identity; the actual nonlinear form of the double Bezout identity is displayed in Eq. (A.10).

Combination A.4. $(P_0, C) = (D_l^{-1}N_l, V_l^{-1}U_l)$. *The closed-loop system* (P_0, C) *is internally stable if and only if*

$$\begin{bmatrix} V_l & U_l \\ -N_l & D_l \end{bmatrix}^{-1} \tag{A.8}$$

exists and is bounded.

Proof. Consider the following operator $[p\ q]^T \to [\alpha\ \beta]^T$:

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix}\begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{A.9}$$

We want to show that the operator in Eq. (A.8) is bounded. We shall show that

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix}\begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} X_{(\beta)} & 0 \\ 0 & Y_{(\alpha)} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{A.10}$$

where $X_{(\beta)}$ is a unit that depends on β and $Y_{(\alpha)}$ is a unit that depends on α. From the Lipschitz continuity of N_l, D_l, U_l and V_l, Eq. (A.9) becomes

$$\begin{aligned} \begin{bmatrix} p \\ q \end{bmatrix} &= \begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix}\begin{bmatrix} D_r\alpha + U_r\beta \\ N_r\alpha - V_r\beta \end{bmatrix} \\ &= \begin{bmatrix} V_l(D_r\alpha + U_r\beta) - U_l(-N_r\alpha + V_r\beta) \\ -N_l(D_r\alpha + U_r\beta) + D_l(N_r\alpha - V_r\beta) \end{bmatrix} \\ &= \begin{bmatrix} \partial V_{l(U_r\beta)}D_r\alpha - \partial U_{l(V_r\beta)}(-N_r)\alpha \\ -\partial N_{l(D_r\alpha)}(U_r\beta) + \partial D_{l(N_r\alpha)}(-V_r)\beta \end{bmatrix} \\ &= \begin{bmatrix} X_{(\beta)}\alpha \\ Y_{(\alpha)}\beta \end{bmatrix}. \end{aligned} \tag{A.11}$$

To obtain these equalities, we have used $V_lU_r\beta - U_lV_r\beta = 0$ and $-N_lD_r\alpha + D_lN_r\alpha = 0$. These relations are obvious from the definition of P_0 and C. Now suppose the system (P_0, C) is stable. By Combination A.3 and as (U_l, V_l) is *uniformly* differentially coprime, $X_{(\beta)}$ is a unit uniformly over β. By Combination A.2 and as (N_l, D_l) is *uniformly* differentially coprime, $Y_{(\alpha)}$ is a unit uniformly over α. It follows that for all arbitrary bounded p and q, there exists bounded α and β such that

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix}^{-1}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix}\begin{bmatrix} X_{(\beta)}^{-1} & 0 \\ 0 & Y_{(\alpha)}^{-1} \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}. \tag{A.12}$$

The right-hand side is bounded because $X_{(\beta)}^{-1}$ and $Y_{(\alpha)}^{-1}$ have finite gain uniformly over α and β. Therefore the inverse on the left exists and is bounded.

Conversely, suppose that the inverse exists and is bounded. By assumption the closed-loop system (P_0, C) is well-posed so we only need to demonstrate that if Eq. (A.8) exists and is bounded then the closed-loop gain is finite. Let p be arbitrary but of finite norm and $q = 0$. Then, we have $p = X_{(\beta)}\alpha$ and $q = Y_{(\alpha)}\beta$ which implies $\beta = 0$ and $\alpha = X_{(0)}^{-1}p = X^{-1}p$ where $X = [V_lD_r - U_l(-N_r)]$. Let $\mathscr{L}_l$ be such that

$$\mathscr{L}_l\begin{bmatrix} D_r \\ N_r \end{bmatrix} = I.$$

It follows from (A.12) that, for all p

$$\mathscr{L}_l\begin{bmatrix} V_l & U_l \\ N_l & D_l \end{bmatrix}^{-1}\begin{bmatrix} p \\ 0 \end{bmatrix} = \mathscr{L}_l\begin{bmatrix} D_r \\ N_r \end{bmatrix}X^{-1}p = X^{-1}p$$

is bounded. Hence $X = V_lD_r - U_l(-N_r)$ is a unit, and so by Combination A.2, the closed-loop system (P_0, C) is internally stable. □

Appendix B. Proof of Lemma 2.1

Suppose the conclusion were not true. Let $\bar{x} \in L_{2e}^p[0, \infty)$ and $\bar{y} \in L_2^p[0, \infty)$ be such that

$$\|A(\bar{x} + \bar{y}) - A(\bar{x})\| > (K + \varepsilon)\|\bar{y}\|$$

for some $\varepsilon > 0$. Then, with $z = A(\bar{x} + \bar{y}) - A(\bar{x})$, there exists T such that

$$\left[\int_0^T |z(t)|^2\, dt\right]^{1/2} > \left(K + \frac{\varepsilon}{2}\right)\|\bar{y}\|. \tag{B.1}$$

Set $z_T = \mathscr{T}_T z$, $\bar{x}_T = \mathscr{T}_T\bar{x}$, $\bar{y}_T = \mathscr{T}_T\bar{y}$. Then by causality, $z_T = \mathscr{T}_T[A(\bar{x}_T + \bar{y}_T) - A(\bar{x}_T)]$. By hypothesis,

$$\|z_T\| \le K\|\bar{y}_T\| \le K\|\bar{y}\|.$$

However, inequality (B.1) implies

$$\|z_T\| > \left(K + \frac{\varepsilon}{2}\right)\|\bar{y}\|$$

which is a contradiction. □

Appendix C. Motivation for the definition of differential coprimeness

This paper utilises a notion of differential coprimeness. In this appendix, we give a motivation for this definition.

C.1. Left differential coprimeness

Consider Fig. 14 with $r_1 = 0$, $r_2 = 0$, $C = U_rV_r^{-1}$ and $P = D_l^{-1}N_l$, where (N_l, D_l) is left coprime. Thus,

$$p - N_lU_rq = D_lV_rq \quad \text{or} \quad (N_lU_r + D_lV_r)q = p.$$

As (N_l, D_l) is left coprime, $(N_lU_r + D_lV_r)$ is a unit. So for any bounded input p, q is bounded, and conversely. Now let us introduce the signal r_2 into the loop and examine its effect. Refer to Fig. 14 with $r_1 = 0$. This yields

$$p - N_l(U_r q + r_2) + N_l r_2 = D_l V_r q$$

or

$$(\partial N_{l(r_2)} U_r + D_l V_r) q = p.$$

Thus, for any r_2, we would like q to be bounded if p is bounded. This is the first step towards differential coprimeness. We will further introduce the signal r_1 as shown in Fig. 14. This gives

$$p - N_l(U_r q + r_2) + N_l r_2 = D_l(V_r q + r_1) - D_l r_1 = \partial D_{l(r_1)} V_r q$$

or

$$(\partial N_{l(r_2)} U_r + \partial D_{l(r_1)} V_r) q = p.$$

Left differential coprimeness means that for any inputs r_1 and r_2, q is bounded if p is bounded, with the gain from p to q depending on r_1 and r_2.

Left uniform differential coprimeness goes further and requires that q be bounded if p is bounded, independently of r_1 and r_2.

C.2. Right differential coprimeness

Similarly, consider Fig. 15 with $C = V_l^{-1} U_l$ and $P = N_r D_r^{-1}$, where (N_r, D_r) is right coprime. We have

$$q = D_r^{-1}(D_r r_2 + s) - r_2 \quad \text{or} \quad \partial D_{r(r_2)} q = s.$$

Also

$$V_l s = p - U_l(N_r(r_1 + q) - N_r r_1)$$

or

$$V_l \partial D_{r(r_2)} q = p - U_l \partial N_{r(r_1)} q,$$

i.e.

$$(V_l \partial D_{r(r_2)} + U_l \partial N_{r(r_1)}) q = p.$$

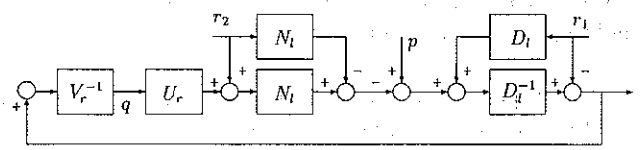

Fig. 14. Left differential coprimeness.

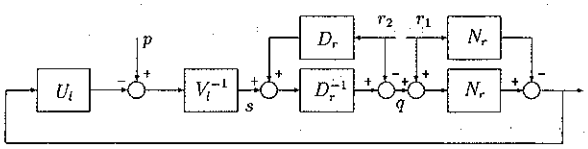

Fig. 15. Right differential coprimeness.

Right differential coprimeness of (N_r, D_r) means that for any inputs r_1 and r_2, q is bounded if p is bounded, with the gain from p to q depending on r_1 and r_2.

Right uniform differential coprimeness goes further and requires that q be bounded if p is bounded, independently of r_1 and r_2.

Appendix D. Proof of Lemma 2.2

In this appendix, we show that when (D_l, N_l) is uniformly left differentially coprime, so is $(-D_l, N_l)$. In the course of the proof, we need to appeal to a preliminary result. Let z_i, $i = 1, 2 \in L^p_{2e}$ and set $z = z_1 + z_2$. Then we claim

$$\partial D_{l(z_1)}(z_2) = -\partial D_{l(z_1+z_2)}(-z_2).$$

This follows because on the one hand

$$D_l(z) = D_l(z_1) + \partial D_{l(z_1)}(z_2)$$

and on the other hand

$$D_l(z_1) = D_l(z) + \partial D_{l(z)}(-z_2) = D_l(z) + \partial D_{l(z_1+z_2)}(-z_2).$$

Now since (D_l, N_l) is uniformly left differentially coprime, for any x, z, $\beta \in L^p_{2e}$, we have

$$(\partial N_{l(x)} U_r + \partial D_{l(z)} V_r)\beta = W_{x,z}\beta \tag{D.1}$$

where $\|W_{x,z}\| \le K$, $\|W^{-1}_{x,z}\| \le K$. Then,

$$(\partial N_{l(x)} U_r - \partial D_{l(\bar{z})}(-V_r))\beta = W_{x,z}\beta \tag{D.2}$$

where $\bar{z} = z + V_r\beta$. For arbitrary x, $\bar{z}$, $\beta \in L^p_{2e}$, define an operator $\bar{W}_{x,\bar{z}}$ by

$$\bar{W}_{x,\bar{z}}(\beta) = W_{x,z}(\beta), \tag{D.3}$$

where $z = \bar{z} - V_r\beta$. It is trivial to see from this that if $\|W_{x,z}\| < K \forall x, z \in L_{2e}$, then $\|\bar{W}_{x,z}\| < K$, $\forall x, z \in L_{2e}$.

To show that $(N_l, -D_l)$ is also left differentially coprime we need to show that $\bar{W}_{x,\bar{z}}$ is a unit. That is, that $\bar{W}^{-1}_{x,\bar{z}}$ is well-posed and bounded independently of x and $\bar{z}$.

Evidently, we have $\bar{W}^{-1}_{x,\bar{z}}(\gamma) = \beta$ or $\gamma = \bar{W}_{x,\bar{z}}(\beta)$ if and only if $W^{-1}_{x,z}(\gamma) = \beta$ or $\gamma = W_{x,z}(\beta)$ and $z = \bar{z} - V_r\beta$. Refer to Fig. 16, where we have represented the operator $\bar{W}^{-1}_{x,\bar{z}}: \gamma \to \beta$. The operator $\bar{W}^{-1}_{x\bar{z}}: \gamma \to \beta$ will be well-posed provided the loop of Fig. 16, is well-posed. Fig. 16 can be redrawn as shown in Fig. 17. Here we have used the results of Appendix C. In turn, Fig. 17 can be redrawn

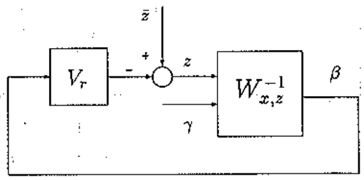

Fig. 16. Construction of $\bar{W}^{-1}_{x,\bar{z}}$.

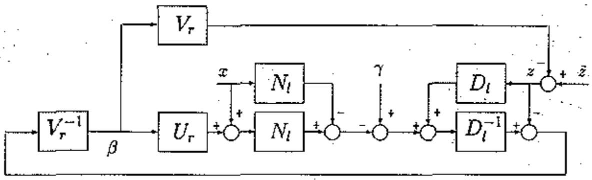

Fig. 17. Construction of $\bar{W}^{-1}_{x,\bar{z}}$.

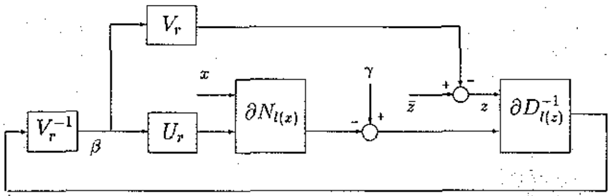

Fig. 18. Construction of $\bar{W}^{-1}_{x,\bar{z}}$.

as shown in Fig. 18. Now D_l^{-1} has the form $aI + S$ for smoothing S means that

$$\partial D^{-1}_{l(z)} = \partial S_{(z)}.$$

Now $\partial S_{(z)}(v) = S(v + z) - S(z)$ and for fixed v, $\partial S_{(z)}(v)$ is an operator on z. It is clear that for fixed v, we can talk about weak Lipschitz continuity with respect to z, and indeed the smoothing property; i.e. since S is smoothing, $\partial S_{(z)}(v)$ for fixed v is smoothing in z. Similarly, it follows from the assumptions that $\partial N_{l(x)}$ is a smoothing operator. Using Item 4 of Remark 2.2, it now follows that Figs. 16–18 are well-posed. Indeed it is clear from Fig. 18 that there is a smoothing element in each subloop while all elements in these loops are weakly Lipschitz.

Since $W^{-1}_{x,\bar{z}}(.)$ is an operator bounded independently of x, z, $\bar{W}^{-1}_{x,\bar{z}}$ must also be bounded independently of x, and $\bar{z}$, i.e. $\bar{W}_{x,\bar{z}}$ is a unit uniformly over x and $\bar{z}$. Hence it follows from (D.2) and (D.3) that $(N_l, -D_l)$ is uniformly left differentially coprime. □

Appendix E. Examples

It has been claimed that there are no plants worth considering that have left coprime factorisations. This appendix contains a few examples of plants that do have left coprime factorisations.

E.1. Stable plant

If the plant P is stable we can choose the left fractional representation

$$P = D_l^{-1} N_l,$$

where $N_l = P$ and $D_l = I$.

E.2. Differential equation representation

Consider

$$\dot{y} + a(y) = bu.$$

Then

$$y = D_l^{-1} N_l u,$$

with

$$D_l = \left(1 + \frac{1}{s+1}(a(.) - 1)\right) \quad \text{and} \quad N_l = \frac{b(.)}{s+1}$$

is a left coprime representation. Similar results hold for matrix differential equations of the type

$$\dot{Y} + A(Y) = BU.$$

E.3. Input nonlinearity

Consider an input nonlinearity ϕ connected to a linear plant with left factorisation $D_l^{-1} N_l$. We can write this as a left factorisation

$$D_l^{-1}(N_l \phi).$$

Note that if we were to try and write this as a right factorisation we would encounter more difficulties. That is, if we had an input nonlinearity ϕ followed by a right factorisation of a linear plant $N_r D_r^{-1}$ we would have

$$N_r(D_r^{-1}\phi).$$

However, we really want

$$\mathcal{D}_r^{-1} = D_r^{-1}\phi.$$

Thus

$$\mathcal{D}_r = \phi^{-1}(D_r^{-1}(.)).$$

This would rule out non-invertible nonlinearities.

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