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DIRECT REDUCED ORDER DISCRETIZATION OF CONTINUOUS-TIME CONTROLLER

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SUMMARY

In this paper, we investigate the problem of closed-loop approximation of a continuous-time controller by a low-order discrete-time controller with sample and hold devices and an anti-aliasing filter. The problem is first reduced to an open-loop problem of sampled-data controller reduction with continuous-time weights. The operator representing the error in approximation can be approximated (arbitrarily closely) by a time-invariant discrete-time system resulting from applying fast sampling and lifting. We propose a method for obtaining a low-order discrete-time controller which makes small the approximation error based on recasting the approximation problem as a four-block $\mathscr{H}_\infty$ problem. The latter can be solved efficiently by the method proposed by Glover *et al.*[1] We also give a numerical example to verify the effectiveness of the method. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: sampled-data system; controller reduction; discretization; four-block problem

1. INTRODUCTION

Sampled-data feedback control has been paid much attention in the area of control system design, where a continuous-time plant is controlled by a discrete-time controller with sample and hold devices. Especially, there has been much research on analysis and design taking into account intersample behaviour, see e.g. References 2–7 and the references therein. In practice, a low-order discrete-time controller is preferred for implementation over a complex or continuous-time controller. However, there are some reasons that the continuous-time controller is first designed. First, it does not require a predetermination of the sampling time, which is best obtained after considering closed-loop behaviour including bandwidth. Second, physical insight is better retained. It such a continuous-time controller is obtained via H_2 or H_∞ design methods, it will have order comparable to that of the plant, which may be high. The designer must then replace the (possibly) high order continuous-time controller by a (preferably) low-order discrete-time controller.

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In this paper, we consider the problem of obtaining a low-order discrete-time controller from a given high-order continuous-time controller, with the important additional requirement of maintaining desirable control performance by taking into account intersample behaviour.

Figure 1 shows various combinations of discretization and order reduction procedure to obtain a low-order discrete-time controller from a high-order continuous-time controller. At present, there is no direct method to obtain a low-order discrete-time controller. However, there are two possible indirect methods; they are (1) to reduce the order of the controller first and then discretize to get a low-order discrete-time controller, and (2) to discretize first and then apply a controller reduction procedure. We note here that in both methods, the intersample behaviour should and can be taken into consideration.

The work on discretization of continuous-time controllers by taking into consideration the closed loop was first done by Rattan and Yeh.[8] Keller and Anderson[9] proposed a new discretization which guarantees closed-loop stability based on satisfaction of a sufficiency condition. Their work gives an indirect upper-bound for the sampling time of the discretization in which stability can be guaranteed. However, the order of the discrete-time controller is usually high (possibly higher in fact than the order of the continuous-time controller) since the solution is obtained by solving a standard H_∞ problem.

The problem of digital controller reduction (obtaining a low-order discrete-time controller from the designed high-order discrete-time controller that controls a continuous-time plant) was first discussed by Madievski and Anderson.[10] A low-order digital controller which gives good intersample behaviour was found by fast sampling and lifting of the closed-loop sampled-data system followed by discrete-time frequency weighted balanced truncation. Chongsrid and Hara[11] proposed an alternative method based on hybrid state-space representation and sampled-data balanced truncation. The method is exact and is not an approximation, as is the case with the fast sampling and lifting method.

In obtaining a low-order discrete-time controller by the first indirect method, continuous-time controller reduction is first applied. Then we obtain the low-order discrete-time controller by, for example, introducing Keller and Anderson's discretization which considers intersample behaviour. In order to obtain an rth-order reduced order discrete-time controller ($r < n$, where n is the order of the high-order continuous-time controller) by this method, a continuous-time

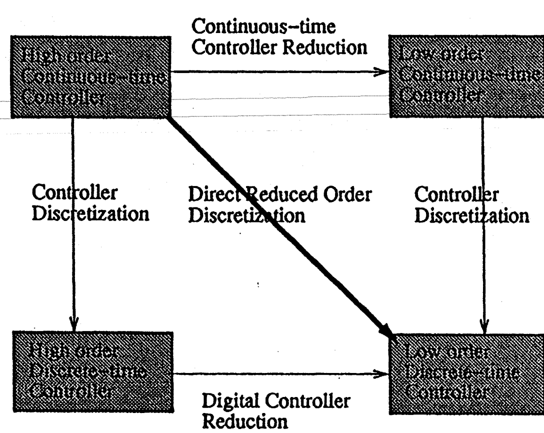

Figure 1. Discretization and order reduction of continuous-time controller

controller of order lower than r may have to be obtained first by the controller reduction procedure since the discretization procedure of Keller and Anderson often produces a discrete-time controller with order greater than the continuous-time controller to be discretized (often, the order of the resulting discrete-time controller is equal to the order of the continuous-time controller plus the order of the plant). This 'extra' order reduction in the first step results in propagation of error which later may become large. In the second indirect method, the high-order controller is first discretized, say by Keller and Anderson's method, and then the low-order discrete-time controller is obtained by the sampled-data controller reduction procedure.[10]

The problem of *simultaneous* discretization and order reduction of a continuous-time controller was considered by Chongsrid and Hara[12] by formulating it as a problem of Hankel-norm approximation of the continuous-time controller by a low-order digital controller. However, the problem is reduced to a problem of discrete-time Hankel-norm approximation with some constraints, and still cannot be solved in general.

In this paper, we propose *a direct reduced order discretization procedure that allows us to do discretization and order reduction of the continuous-time controller at the same time.* We first show in Section 2.1 that the problem can be reduced to the hybrid model approximation problem. The operators representing the error in approximation can be approximated (arbitrary closely) by time-invariant discrete-time systems resulting from applying fast sampling and lifting. We give a partial summary of Keller's result in Section 2.2, as background for the later ideas. In Section 3, we give a procedure for recasting the direct reduced order discretization as a four-block $\mathscr{H}_\infty$ problem. A numerical example is given in Section 4. Section 5 provides conclusions and comments.

Throughout this paper, $\mathscr{S}_\tau$ and $\mathscr{H}_\tau$ denote a τ-periodic sampler and synchronized zero-order hold, respectively. $\|\cdot\|_{L_2/L_2}$ denotes the L_2 induced norm. $RH_\infty^-(r)$ denotes the space of (discrete-time) rational transfer functions with finite infinity norm (i.e. bounded on $|z| = 1$), and with precisely r poles inside the unit disc. The conjugate system of $G[s]$ is denoted by $G^\sim[z] = G^{\mathrm{T}}[z^{-1}]$.

2. PRELIMINARIES

2.1. *Sampled-data controller reduction*

In this section, we will show how sample-data controller reduction problems can be reduced to open-loop sampled-data approximation problems by taking into account of intersample behaviour.

2.1.1. Stability margin consideration. Let $P(s)$ be a continuous-time plant and $C(s)$ be a known continuous-time controller. We assume that $C(s)$ is stable and it stabilizes the plant $P(s)$. Suppose our task is to find a low-order discrete-time controller denoted by $C_{\mathrm{d}}[z]$ which together with sampler, hold and an anti-aliasing filter $F_{\mathrm{a}}(s)$ approximate well $C(s)$. Replacing $C(s)$ by a discrete-time controller $C_{\mathrm{d}}[z]$ results in a closed-loop sampled-data system in Figure 2.

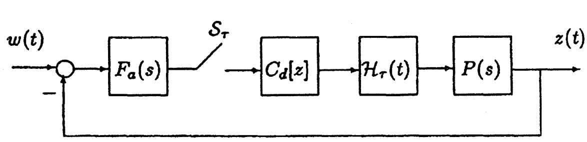

Figure 2. Sampled-data feedback system

The problem is to find a stable reduced order discrete-time controller $C_d[z]$ of order $r < n$ which stabilizes the closed-loop and approximates $C(s)$ in a sense. Applying the small-gain theorem to the rearranged system depicted in Figure 3, we can see that $C_d[z]$ stabilizes the closed-loop system if

$$\|J_s\|_{L_2/L_2} := \|(C - \mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)(I + PC)^{-1}P\|_{L_2/L_2} < 1 \tag{1}$$

holds. Moreover, the smaller the index is, the better is the approximation. The problem is then reduced to an open-loop sampled-data model reduction with input weight $(I + PC)^{-1}P$.

2.1.2. Closed-loop error consideration. Consider again the sampled-data system shown in Figure 2 in which the discrete-time controller $C_d[z]$ with sampler, hold and anti-aliasing filter $F_a(s)$ approximate the continuous-time controller $C(s)$. With the assumptions as in the previous subsection, we can consider the task of finding a low-order discrete-time controller $C_d[z]$ that ensures that the closed-loop operator defining the sampled-data system approximates the closed-loop transfer function

$$T = PC(I + PC)^{-1} \tag{2}$$

Let $\mathscr{T}$ be the operator from $w(t)$ to $z(t)$ of the system in Figure 2. Then, (formally) we have

$$\mathscr{T} = P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a (I + P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)^{-1} \tag{3}$$

The error between the two operators T and $\mathscr{T}$ is expressed as

$$\begin{aligned} T - \mathscr{T} &= PC(I + PC)^{-1} - P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a (I + P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)^{-1} \\ &= (I + P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)^{-1} - (I + PC)^{-1} \\ &= (I + P\mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)^{-1} P(C - \mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)(I + PC)^{-1} \end{aligned} \tag{4}$$

We consider a problem of finding a low-order discrete-time controller $C_d[z]$ which minimizes the norm of $T - \mathscr{T}$. Note that $T - \mathscr{T}$ can be approximated by neglecting a second-order term involving $C - \mathscr{H}_\tau C_d \mathscr{S}_\tau F_a$ as follows:

$$T - \mathscr{T} \approx (I + PC)^{-1} P(C - \mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)(I + PC)^{-1} \tag{5}$$

This implies that we have to approximate $C(s)$ by a low-order controller $C_d[z]$ with input weight $(I + PC)^{-1}$ and output weight $(I + PC)^{-1}P$. In other words, the norm to be minimized is

$$\|J_p\|_{L_2/L_2} := \|(I + PC)^{-1} P(C - \mathscr{H}_\tau C_d \mathscr{S}_\tau F_a)(I + PC)^{-1}\|_{L_2/L_2} \tag{6}$$

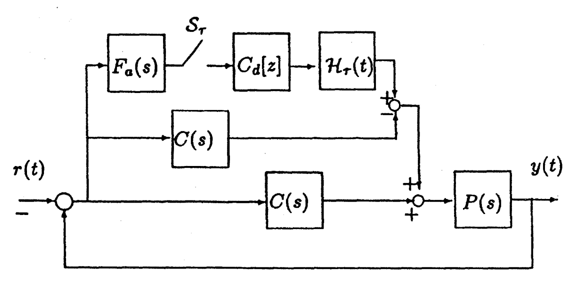

Figure 3. Rearrangement of the system in Fig. 2

2.2. Fast sampling and lifting

In this subsection, we summarize the results on fast sampling and lifting the approximation error of J_s defined in (1) as discussed in References 9 and 13. The results presuppose $C(s)$ is stable. (The ideas for J_p are very similar except that $C(s)$ does not have to be stable.) The continuous-time operator J_s is first sampled with sampling interval τ/N, where N is chosen so that samples covering at the fast sampling rate of N/τ reflect well the underlying continuous-time signal of which they are samples. This results in a multirate N-periodic discrete-time system J_{sd} as depicted in Figure 4, where the input weight $W(s)$ is given by $W = (I + PC)^{-1}P$.

Next, we apply the lifting procedure as described in Reference 12. The $p \times m$ multirate N-sampled system J_{sd} is replaced by an equivalent $pN \times mN$ discrete-time linear shift-invariant system $\tilde{J}_s$. Given state-space representations of $C(s)$, $W(s)$ and $F_a(s)$ as

$$C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right], \qquad W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right]$$

$$F_a(s) = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right]. \tag{7}$$

The lifted system $\tilde{J}_s$ can be written by

$$\tilde{J}_s[z] := \left(\tilde{C}[z] - \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} C_d[z]\,[I\ 0 \ldots 0]\tilde{F}_a[z]\right)\tilde{W}[z] \tag{8}$$

where the discrete-time lifted controller $\tilde{C}[z]$, filter $\tilde{F}_a[z]$ and input weight $\tilde{W}[z]$ are given by fast-sampling with the following realizations:

$$\tilde{C}(z) = \left[\begin{array}{c|c} \mathscr{A}_c & \mathscr{B}_c \\ \hline \mathscr{C}_c & \mathscr{D}_c \end{array}\right], \qquad \tilde{W}(z) = \left[\begin{array}{c|c} \mathscr{A}_w & \mathscr{B}_w \\ \hline \mathscr{C}_w & \mathscr{D}_w \end{array}\right]$$

$$\tilde{F}_a(z) = \left[\begin{array}{c|c} \mathscr{A}_f & \mathscr{B}_f \\ \hline \mathscr{C}_f & \mathscr{D}_f \end{array}\right]. \tag{9}$$

The matrices $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$; $i = \{c, w, f\}$ appear in (9) are given by

$$\mathscr{A}_i = a_i^N, \ \mathscr{B}_i = [a_i^{N-1} b_i \ \cdots \ a_i b_i \ b_i]$$

$$\mathscr{C}_i = [C_i^T \ a_i^T C_i^T \ldots (a_i^T)^{N-1} C_i^T]^T$$

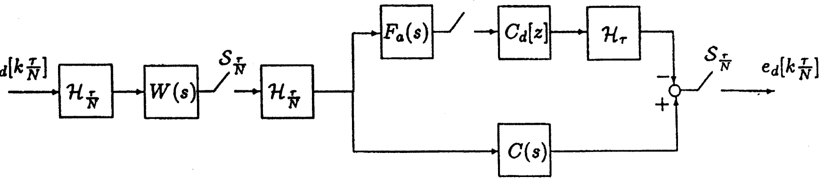

Figure 4. J_{sd}

$$\mathscr{D}_i = \begin{bmatrix} D_i & 0 & \cdots & 0 \\ C_i b_i & D_i & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ C_i a_i^{N-2} b_i & C_i a_i^{N-3} b_i & \cdots & D_i \end{bmatrix}$$

where

$$a_i = \mathrm{e}^{A_i \tau/N}, \qquad b_i = \int_0^{\tau/N} \mathrm{e}^{A_i t} B_i \, \mathrm{d}t.$$

Lemma 2.1 (Reference 9)

For the continuous-time operator J_s, the discrete-time N-periodic system J_{sd} and the discrete-time system $\tilde{J}_s$ defined earlier, we have

$$\|J_s\|_{L_2/L_2} = \lim_{N\to\infty} \|J_{\mathrm{sd}}\|_{l_2/l_2} = \lim_{N\to\infty} \|\tilde{J}_s\|_\infty \tag{10}$$

Minimization of $\|\tilde{J}_s[z]\|_\infty$ (for any finite N) is a standard H_∞ problem, when $C(s)$ is stable (by assumption) and there is no constraint on the order of $C_{\mathrm{d}}[z]$. We will show in the next section that the above minimization with constraint on the order of $C_{\mathrm{d}}[z]$ can be reduced to a four-block model-reduction problem.

3. REFORMULATION AS A FOUR-BLOCK PROBLEM

For simplicity, we will consider only the stability margin criterion. Direct reduced order discretization of a continuous-time controller which matches the closed-loop operators can be easily treated by extending the result derived in this section. We will reduce the problem of finding a stable $C_{\mathrm{d}}[z]$ of order r which minimizes $\tilde{J}_s$ defined in (8) into the problem of finding an unstable transfer function matrix $X[z]$ with precisely r poles in the unit disc which minimizes

$$\left\| \begin{bmatrix} R_{11,s} - X & R_{12,s} \\ R_{21,s} & R_{22,s} \end{bmatrix} \right\|_\infty$$

with $R_{ij,s}; i,j = 1,2$ stable. The stable controller $C_{\mathrm{d}}[z]$ which is an approximated solution for the original problem is then found by back substitution in the final step. The ideas are based on References 1 and 14.

Step 1. The norm of $\tilde{J}_s[z]$ in (8) can be rewritten as

$$\|\tilde{J}_s\|_\infty = \left\| \left(\tilde{C} - U \begin{bmatrix} \sqrt{N} C_{\mathrm{d}}[z] \\ 0 \end{bmatrix} [I \; 0 \ldots 0] \tilde{F}_{\mathrm{a}} \right) \tilde{W} \right\|_\infty \tag{11}$$

where U is a constant unitary matrix given by

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} I & \\ \vdots & \Phi_\perp \\ I & \end{bmatrix} \tag{12}$$

Note that we can select $\Phi_{\perp}$ so that U is unitary. U can be constructed by using the Gram–Schmidt orthonormalization. Since multiplication by a unitary matrix does not change the norm, we have

$$\|\tilde{J}_s\|_{\infty} = \left\| \left(U^{\mathrm{T}}\tilde{C} - \begin{bmatrix} \sqrt{N}C_{\mathrm{d}}[z] \\ 0 \end{bmatrix} [I \ 0 \ldots 0]\tilde{F}_{\mathrm{a}} \right) \tilde{W} \right\|_{\infty} \tag{13}$$

Step 2. Define a transfer matrix $v_1[z]$ as the first block row of $\tilde{F}_{\mathrm{a}}[z]\tilde{W}[z]$, i.e.

$$v_1[z] = [I \ 0 \ \cdots \ 0]\tilde{F}_{\mathrm{a}}[z]\tilde{W}[z] \tag{14}$$

We can see that $v_1[z] \in RH_{\infty}$, is fat (number of columns $\geqslant$ number of rows) and there exists a co-inner–outer factorization

$$v_1[z] = v_{1\mathrm{o}}[z]v_{1,\mathrm{in}}[z] \tag{15}$$

where $v_{1\mathrm{o}}[z]$ is stable with a stable inverse and $v_{1,\mathrm{in}}[z]$ is a co-inner function. By introducing the adjoint transfer matrix $v_{1\mathrm{o}}^{\sim}[z]$ of $v_{1\mathrm{o}}[z]$, we can rewrite the above equation as

$$v_1 = v_{1\mathrm{o}}^{\sim}(v_{1\mathrm{o}}^{\sim})^{-1}v_{1\mathrm{o}}v_{1,\mathrm{in}} \tag{16}$$

We have the following equality:

$$\begin{bmatrix} \sqrt{N}C_{\mathrm{d}} \\ 0 \end{bmatrix} v_1 = \begin{bmatrix} \hat{C}_{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (v_{1\mathrm{o}}^{-1})^{\sim}v_{1\mathrm{o}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_{1,\mathrm{in}} \\ v_{1,\perp} \end{bmatrix} \tag{17}$$

where we define a transfer matrix $\hat{C}_{\mathrm{d}}[z] \in RH_{\infty}^{-}(r)$ as

$$\hat{C}_{\mathrm{d}} = \sqrt{N}C_{\mathrm{d}}v_{1\mathrm{o}}^{\sim} \tag{18}$$

We note here that $\hat{C}_{\mathrm{d}}[z]$ and $C_{\mathrm{d}}[z]$ have the same number of poles in the unit disc. $v_{1,\perp}[z]$ is such that $[v_{1,\mathrm{in}}^{\sim} \ v_{1,\perp}^{\sim}]^{\sim}$ is square and all-pass. Further, $v_{1,\perp}$ and $v_{1,\mathrm{in}}$ live over the same state space, in the sense that a minimal realization for $[\tilde{v}_{1,\mathrm{in}} \ \tilde{v}_{1,\perp}]$ results from a minimal realization of $\tilde{v}_{1,\mathrm{in}}$ by adjustment only of the input to state coupling matrix and the input-to-output direct transmission matrix. We can use the Riccati equation that appears in the inner–outer factorization procedure of v_1 for constructing $v_{1,\perp}$.

Since multiplication by all-pass transfer functions does not change the infinity norm, equation (13) can be rewritten as

$$\begin{aligned} \|\tilde{J}_s\|_{\infty} &= \left\| \begin{bmatrix} \hat{C}_{\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix} - U^{\mathrm{T}}\tilde{C}\tilde{W} \begin{bmatrix} v_{1,\mathrm{in}} \\ v_{1,\perp} \end{bmatrix}^{\sim} \begin{bmatrix} v_{1\mathrm{o}}^{-1}v_{1\mathrm{o}}^{\sim} & 0 \\ 0 & I \end{bmatrix} \right\|_{\infty} \\ &= \left\| \begin{bmatrix} R_{11} - \hat{C}_{\mathrm{d}} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} \end{aligned} \tag{19}$$

Step 3. We can see that finding $\hat{C}_{\mathrm{d}} \in RH_{\infty}^{-}(r)$ which minimizes $\|\tilde{J}_s\|_{\infty}$ given in (19) is a four-block problem. In the procedure for solving the four-block problem, the transfer matrices $R_{ij}[z]$ are required to be stable,[15] whereas in our case we may not have stability. We can see that the unstable part of $R_{11}[z]$ may be absorbed in to $\hat{C}_{\mathrm{d}}[z]$. $R_{12}[z]$, $R_{21}[z]$ and $R_{22}[z]$ can be made stable by pre-multiplication and post-multiplication by suitable (norm-preserving) all-pass transfer function matrices

$$\left\|\begin{bmatrix} R_{11}-\hat{C}_{\mathrm{d}} & R_{12} \\ R_{21} & R_{22}\end{bmatrix}\right\|_{\infty} = \left\|\begin{bmatrix} I & 0 \\ 0 & A_1\end{bmatrix}\begin{bmatrix} R_{11}-\hat{C}_{\mathrm{d}} & R_{12,s} \\ R_{21,s} & R_{22,s}\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_2\end{bmatrix}\right\|_{\infty} \tag{20}$$

where the transfer function matrices $R_{12,s}[z]$, $R_{21,s}[z]$ and $R_{22,s}[z]$ are stable, and $A_1[z]$ and $A_2[z]$ are square unstable all-pass transfer function matrices. An algorithm for the factorization in (20) is given in Appendix A.

The above equation taken with (19) and the task of minimizing $\|\tilde{J}_s\|_\infty$ define a four-block problem, with a restriction that $\hat{C}_{\mathrm{d}}[z]$ lie in $RH_\infty^-(r)$.

Step 4. Let $X[z]$ be the solution of the four-block problem

$$\min_{X\in RH_\infty^-(r)} \left\|\begin{bmatrix} [R_{11}]_+ - X & R_{12,s} \\ R_{21,s} & R_{22,s}\end{bmatrix}\right\|_{\infty}$$

where $[R_{11}]_+$ is the stable part of R_{11} satisfies the following equation:

$$R_{11} = [R_{11}]_+ + [R_{11}]_- \tag{21}$$

Then $\hat{C}_{\mathrm{d}} = X + [R_{11}]_-$ minimizes the norm in (19). However, recall that we require $\hat{C}_{\mathrm{d}}[z]$ to lie in $RH_\infty^-(r)$, so that $C_{\mathrm{d}} = (1/\sqrt{N})\hat{C}_{\mathrm{d}}(v_{10}^{-1})^{\sim} \in RH_\infty$. To resolve this issue, we simply choose a solution for the original problem.

$$C_{\mathrm{d,app}} = \frac{1}{\sqrt{N}}[\hat{C}_{\mathrm{d}}(v_{10}^{-1})^{\sim}]_+ \tag{22}$$

where $[\cdot]_+$ denotes the stable part of the transfer matrix.

The idea of choosing the stable part in this manner parallels what is does in weighted model reduction; in effect, the correct measure of approximation is an L_∞ one, provided that a stable approximant is obtained. Since it is essential to obtain the stable approximant, the problem is modified to allow an H_∞ approximant.

Note that no upper bound for the error is available for the stable approximation of the four-block solution.

Several additional points follow. First, we may utilize an optimization algorithm in order to reduce the error norm by retaining the poles of $C_{\mathrm{d,app}}[z]$ and choosing the numerator to minimize the norm yielding

$$C_{\mathrm{d,app}}[z] = \sum_{i=1}^{r} \frac{c_i^{\mathrm{T}} b_i}{(z+z_i)}$$

For fixed b_i the minimization is convex in c_i, and for fixed c_i the minimization is convex in b_i. Second, solving the four-block problem without the stability constraint in (22) gives us a lower bound for $\|\tilde{J}_s\|_\infty$ which can be reached by $\hat{C}_{\mathrm{d}}[z] \in RH_\infty^-((r)$. This gives a loose upper bound for the τ in which the stability of the closed-loop can be guaranteed with the low order controller. Third, a measure for the impacts of controller discretization (which is affected by sampling time) and order reduction is the value of $\|\tilde{J}_s\|_\infty$. The closed-loop is guaranteed to be stable if $\|\tilde{J}_s\|_\infty < 1$.

4. NUMERICAL EXAMPLE

In this section, we show a numerical example of direct reduced order discretization of a continuous-time controller. The procedure is applied to an example used by Rattan[8] and Keller.[9] The parameters of the plant $P(s)$ and the continuous-time controller $C(s)$ are given by Plant:

$$P(s) = \frac{10}{s(s+1)}$$

Controller:

$$C(s) = \frac{0{\cdot}416s + 1}{0{\cdot}139s + 1}$$

It was reported that most common discretization methods yield non-stabilizing or poor closed-loop performance for sampling times comparable to rise time under a step response. We compare the proposed method with the third-order discrete-time controller obtained by Keller and Anderson and we investigate the following:

- possibility of order reduction given a long sampling time, and requiring retention of system stability.
- step response of the reduced order systems.

For simplicity, $F_a(s)$ was set to the identity which is an approximate of $F_a(s) = a/(s + a)$ with arbitrary large a. First, we compare the discrete-time controllers obtained by

(1) Proposed method, i.e. direct reduced order discretization method (order = 1) and
(2) Indirect method, i.e. apply sampled-data controller reduction (see References 11 and 13 for details) to the third-order discrete-time controller obtained by Keller and Anderson (order = 1).

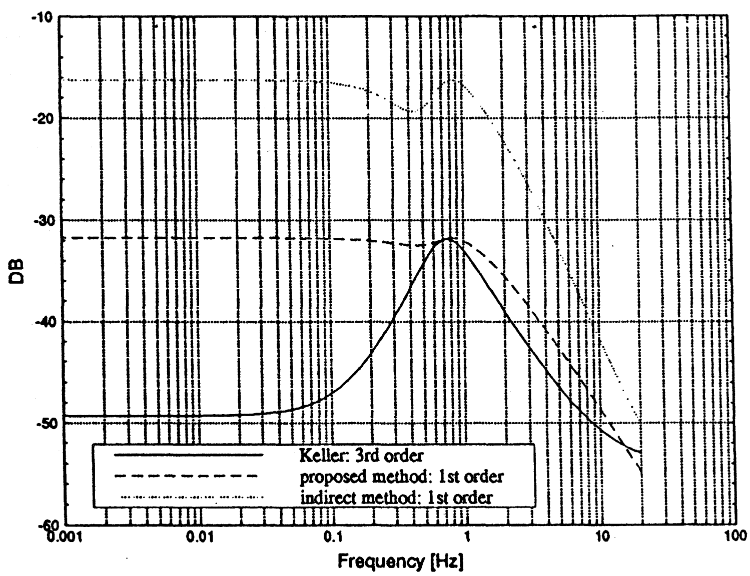

Figure 5. Approximation error

All the discrete-time controllers have a sampling time $\tau = 0{\cdot}0157\,$s. We utilized the optimization algorithm in order to minimize $\|\tilde{J}_s\|_\infty$ by varying C and D of the first-order discrete-time controllers in both cases. The computations in the proposed procedure have been done by using Matlab and simulations for time-responses of the sampled-data system have been done by using Hsys-module in Xmath. Figure 5 shows the approximation error $\bar{\sigma}(\tilde{J}_s)$ as a function of frequency of the discrete-time controller obtained by two methods together with that of the third-order controller discrete-time controller obtained by Keller and Anderson. It is clear that the controller obtained by the proposed method gives a smaller error norm in comparison with the controller obtained by the indirect method.

The step responses of the sampled-data closed-loop with the discrete-time controllers obtained by methods (1) and (2) are illustrated in Figure 6, and compared with the step response of the continuous-time controller. The results show that the reduced-order controller obtained by proposed method approximates the continuous-time controller with error trade-off by discretization and order reduction.

We also compare $\|\tilde{J}_s\|_\infty$ achieved by first-order discrete-time controllers obtained by the above two methods for different τ. In using method (2), $C(s)$ is first discretized by the Keller and Anderson discretization method. This step gives discrete-time controllers of order 3 for $\tau = 0{\cdot}0157\,$s and $0{\cdot}0785\,$s, and gives discrete-time controller of order 2 for $\tau = 0{\cdot}157$ and $0{\cdot}314\,$s. The weighted Hankel singular values are given in Table I. Next, the first-order discrete-time controllers are then obtained by applying the sampled-data controller reduction procedure.

The result is shown in Figure 7. It is clear that controllers obtained by the proposed method give smaller error compared with controllers obtained by the indirect method.

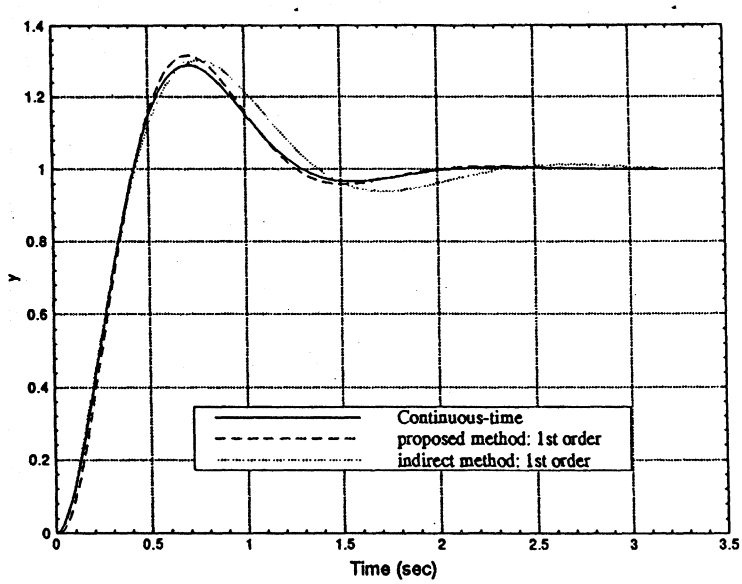

Figure 6. Step response of the closed-loop

Table I

τ (s)	Weighted Hankel singular values		
0·0157	0·6783,	0·0613,	0·0001
0·0785	0·9735,	0·0546,	0·0008
0·1570	1·2343,	0·0075	
0·3140	1·3150,	0·0124	

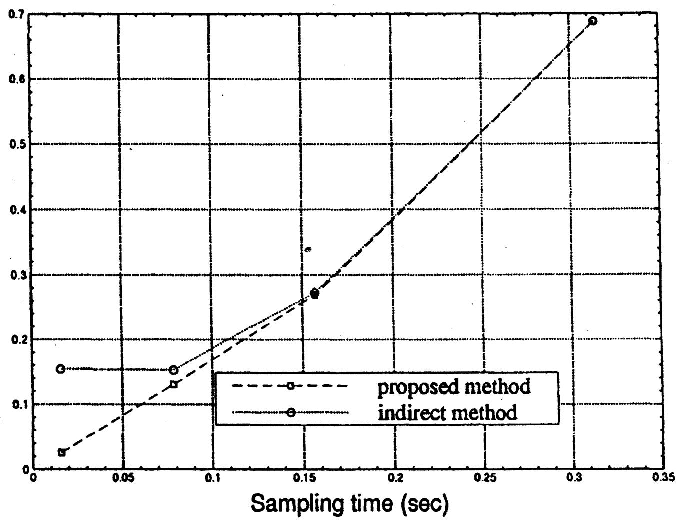

Figure 7. $\|\tilde{J}_s\|_\infty$

5. CONCLUSIONS

In this paper, we have investigated the problem of simultaneous discretization and order reduction of a continuous-time controller. This problem can be reduced to a four-block $\mathscr{H}_\infty$ problem and an approximate solution of the former problem can be analytically given by the stable part of the solution of the four-block problem. The method as presented in detail requires the continuous time controller to be open-loop stable. However, the method can be also applied to the unstable controller case by using stable fractional representations and a different index. A lower bound for the stability margin of a stable rth-order discrete-time controller is given by the stability margin of the unstable controller with r stable poles. However, there is no error bound (*a priori*) given for the proposed approximation; this is an open problem which should be investigated but is likely to be very difficult.

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APPENDIX A. CONVERGE TO A FOUR-BLOCK PROBLEM WITH STABLE TRANSFER FUNCTION MATRICES

In Step 3, we separate the unstable parts of the transfer function matrices to all-pass factors.
First, we assume that $R_{ij} \in RL_\infty$; $i, j = 1, 2$. There exists a left co-prime factorization

$$[R_{21} \; R_{22}] = R_{2,\mathrm{au}}[R_{21,s} \; R_{22,s}] \tag{23}$$

also there exists a right co-prime factorization

$$R_{12} = R_{12,s}R_{12,\mathrm{au}}$$

where $R_{2,\mathrm{au}}$, $R_{12,\mathrm{au}} \in RH_\infty^-$ are all-pass. $R_{12,s}$, $R_{21,s}$ and $R_{22,s} \in RH_\infty$. The state-space formula for the factorization is given in Proposition A.1.

Since multiplication by all-pass transfer matrices does not change the norm, we have the following stable factor of the original four-block formulation:

$$\begin{aligned}\left\|\begin{bmatrix} R_{11} - \hat{C}_\mathrm{d} & R_{12} \\ R_{21} & R_{22}\end{bmatrix}\right\|_\infty &= \left\|\begin{bmatrix} R_{11} - \hat{C}_\mathrm{d} & R_{12,s} \\ R_{21,s} & R_{22,s}\end{bmatrix}\right\|_\infty \\ &= \left\|\begin{bmatrix} R_{11} - \hat{C}_\mathrm{d} & R_{12,s} \\ R_{21,s} & R_{22,s}R_{12,\mathrm{au}}^{\sim}\end{bmatrix}\right\|_\infty\end{aligned}$$

This gives the result.

Proposition A.1

For a transfer function matrix $G[z] \in RL_\infty$ with a minimal realization given by

$$G[z] = \left[\begin{array}{cc|c} A_1 & A_{12} & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{24}$$

such that $|\lambda_i(A_1)| > 0$, $|\lambda_i(A_2)| < 0$, there exist a factorization

$$G = M^{-1}G_s \tag{25}$$

such that M is a square stable all-pass, M, $G_s \in RH_\infty$. Here $M[z]$ is given by

$$M[z] = \left[\begin{array}{c|c} A_1^{-1^\mathrm{T}} & C_1^\mathrm{T} \\ \hline C_M & D_M \end{array}\right] \tag{26}$$

where

$$D_M = (I - C_1A_1Q^{-1}A_1^\mathrm{T}C_1^\mathrm{T})^{1/2}$$
$$C_M = -D_MC_1A_1Q^{-1}$$

where Q is the positive-definite solution of the Lyapunov equation

$$Q - A_1^{-1^\mathrm{T}}QA_1^{-1} = C_1^\mathrm{T}C_1 \tag{27}$$

It can be readily shown that

$$G_s = MG = \left[\begin{array}{cc|c} A_1^{-1^\mathrm{T}} & C_1^\mathrm{T}C_2 - QA_1^{-1}A_{12} & C_1^\mathrm{T}D - QA_1^{-1}B_1 \\ 0 & A_2 & B_2 \\ \hline -C_M & D_MC_2 & D_MD \end{array}\right] \tag{28}$$

We note here that the unstable poles of $G[z]$ are cancelled by a left multiplication of $M[z]$. Further, it is easy to check that $M[z]$ is all-pass, in the light of (27).

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