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Approximation of frequency response for sampled-data control systems[1]

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Abstract

This paper proves that the frequency response gains of fast-sample/fast-hold approximations of a sampled-data system converge to that of the original system as the sampling rate gets faster. While this may appear to hold trivially, there is a serious technical difficulty, and the proof is indeed nontrivial. It is also guaranteed that this convergence is uniform on the total frequency range. The latter property is necessary to guarantee that a single approximant can be used for frequency response computation for the overall frequency range.

Keywords: Sampled-data systems; Frequency response; Approximation; Fast sampling; Convergence

1. Introduction

The modern notion of frequency response for sampled-data systems (Araki et al., 1996; Yamamoto and Khargonekar, 1996) has proven to be very effective in characterizing various features of such systems; see, e.g., Chen and Francis (1995). The major difference from the classical notion is that the new notion takes intersampling behavior into account, and the continuous-time characteristic can be well captured through this notion. For example, an H^{∞} design objective can be naturally understood with frequency response. On the other hand, since the obtained gain function is derived as an operator norm, its computation is a nontrivial problem. While it can be characterized as a solution to the determination of a finite-dimensional rank condition (Yamamoto and Khargonekar, 1996) or as a limit of maximal singular values of matrices of increasing dimension (Araki et al., 1996), these theoretical methods tend to give a computationally burdensome route. In particular, the relevant system matrices must be computed anew at each frequency, and this is quite time consuming.

Instead of going through these theoretical methods, one can take an approximation via fast sampling/fast hold. By subdividing the sampling interval into N subintervals, we can approximate inputs in one sample period by step functions. The outputs are likewise approximated by taking sampled values on these subintervals. The approximated system then becomes a finite-dimensional discrete-time system, and its frequency response is expected to approach that of the original system as $N \to \infty$. While this appears to be quite natural to expect, its proof of convergence induces various technical difficulties. For example, for sampled outputs to approximate the original outputs, they should not oscillate too wildly. However, we do not know for which outputs this can be guaranteed to begin with, so there must be some kind of uniformity to guarantee this. Furthermore, in order that this approximated system can be used for gain computation, the convergence must be uniform in frequency ω. This raises another technical issue.

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The objective of the present paper is to give a proof for this convergence. After giving a setup in Section 2, we introduce the fast-sampling formula in Section 3. We then give the convergence proof in Section 4. The crucial fact is that the frequency response operator is a compact perturbation of a multiplication operator given by the direct feedthrough term in the plant. This gives a generalization of our early result in Yamamoto et al. (1997) where the feedthrough term is assumed to be zero. We also prove that this convergence is uniform in ω. This property is crucial in guaranteeing that a single approximant gives a satisfactory approximation over all the frequency range, and hence can be used as a uniform model for gain computation.

2. System description

Consider the sampled-data system depicted in Fig. 1. The continuous-time plant $G(s)$ is described by

$$\dot{x}(t) = A_c x(t) + B_1 w(t) + B_2 u(t)$$
$$z(t) = C_1 x(t) + D_{11} w(t), \tag{1}$$
$$y(t) = C_2 x(t)$$

while the discrete-time controller $K(z)$, with sampling period h, obeys the equation

$$x_{d,k+1} = A_d x_{d,k} + B_d y(kh),$$
$$v_k = C_d x_{d,k} + D_d y(kh),$$
$$u_k(\theta) \equiv v_k, \quad 0 \le \theta < h.$$

Note that the direct feedthrough term D_{11} in $G(s)$ is not assumed to be zero.

It is now well known (Bamieh and Pearson, 1992; Bamieh et al., 1991; Toivonen, 1992; Yamamoto, 1994)

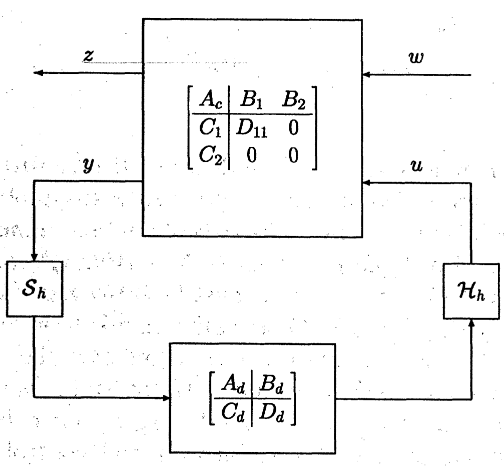

Fig. 1. Sampled feedback system.

that the lifting operator

$$\mathscr{W}: L^p_{\text{loc}}[0, \infty) \to \ell_{Lp[0,h)}: \varphi \mapsto \{\varphi_k\}_{k=0}^{\infty},$$
$$\varphi_k(\theta) := \varphi(kh + \theta) \tag{2}$$

enables us to describe the closed-loop system (Fig. 1) by time-invariant discrete-time transition equations. We follow the notation in Yamamoto and Khargonekar (1996):

$$\begin{bmatrix} x_{c,k+1} \\ x_{d,k+1} \end{bmatrix} = \begin{bmatrix} A_{cs} & A_{cd} \\ A_{ds} & A_d \end{bmatrix} \begin{bmatrix} x_{c,k} \\ x_{d,k} \end{bmatrix} + \begin{bmatrix} \int_0^h e^{A(h-\tau)} B_1 w(\tau) d\tau \\ 0 \end{bmatrix}$$
$$=: \mathbf{A} x_k + \mathbf{B} w_k, \tag{3}$$

$$z_k(\theta) = [C_1(\theta) \quad C_2(\theta)] \begin{bmatrix} x_{c,k} \\ x_{d,k} \end{bmatrix} + D_{11} w_k(\theta)$$
$$+ \int_0^\theta C_1 e^{A(\theta-\tau)} B_1 w_k(\tau)\, d\tau$$
$$=: \mathbf{C} x_k + \mathbf{D} w_k, \tag{4}$$

where $x_{c,k} = x_c(kh)$ and $x_{d,k}$ denote, respectively, the continuous and discrete state variables and belong to $\mathbf{C}^{n_c}$ and $\mathbf{C}^{n_d}$; matrices A_{cs}, A_{cd}, A_{ds}, $C_i(\theta)$, are given as follows:

$$A_{cs} = e^{A_c h} + \int_0^h e^{A_c(h-\tau)} B_2 D_d C_2\, d\tau,$$
$$A_{cd} = \int_0^h e^{A_c(h-\tau)} B_2 C_d\, d\tau,$$
$$A_{ds} = B_d C_2, \tag{5}$$
$$C_1(\theta) = C_1 \left(e^{A_c \theta} + \int_0^\theta e^{A_c(h-\tau)} B_2 D_d C_2\, d\tau \right),$$
$$C_2(\theta) = C_1 \int_0^\theta e^{A_c(\theta-\tau)} B_2 C_d\, d\tau.$$

System (3), (4) induces the (operator) transfer function with obvious definition $\mathscr{F}_l(G, K)(z) := \mathbf{D} + \mathbf{C}(zI - \mathbf{A})^{-1}\mathbf{B}$, where $\mathscr{F}_l(\cdot,\cdot)$ denotes the lower linear fractional transformation. Note here that the "A" operator is a matrix, and we assume from here on that A is power stable, i.e. $A^n \to 0$ as $n \to \infty$. This is equivalent to the eigenvalues of A lying all inside the unit circle.

Since the closed-loop system is stable, the substitution $z = e^{j\omega h}$ is possible, and this gives rise to a continuous-linear operator

$$\mathscr{F}_l(G, K)(e^{j\omega h}): L^2[0, h] \to L^2[0, h]. \tag{6}$$

This is the frequency response operator we are concerned with here (Yamamoto and Khargonekar, 1996). The *gain* of this operator at ω is defined to be its induced norm

$$\|\mathscr{F}_l(G, K)(e^{j\omega h})\| := \sup_{u \in L^2[0,h], u \neq 0} \frac{\|\mathscr{F}_l(G, K)(e^{j\omega h})u\|}{\|u\|}. \tag{7}$$

Unlike the more conventional notion of frequency response where only the sampled behavior is considered, this new notion takes into account the intersampling bahavior, which fails to be captured by the conventional concept. See Yamamoto and Araki (1994) for the relationship with the concept of aliasing, and Hara et al. (1996) for discussions with various numerical examples.

3. Fast-sampling approximation

It is known that the gain computation of frequency response is reducible to a generalized eigenvalue problem (Yamamoto and Khargonekar, 1996). Unlike the H^∞ norm computation, however, this must be repeated at each frequency, and is computationally quite burdensome. In particular, the relevant system matrices must be computed anew at each frequency. Furthermore, it is only recently that a bisection search algorithm was obtained (Hara et al., 1995). On the other hand, in the approach proposed by Araki and co-workers (1996), one has to give a high-dimensional approximating expansion of the transfer operator, and giving an a priori estimate for an appropriate order of expansion appears difficult.

Instead of going over these "exact" procedures, we here employ an approximation approach (Anderson and Keller, 1998). We subdivide the kth sampling interval $[kh, (k+1)h)$ into N subintervals $[kh + \ell h/N, kh + (\ell+1)h/N)$, $\ell = 0, \ldots, N-1$, and approximate the input w by step functions and output by sampled values of z at these points. In other words, we consider the following fast-sampling/fast-hold operators:

$$\mathscr{H}_{h/N}: \{w(\ell h/N)\}_{\ell=0}^{\infty} \mapsto w(t), \quad w(t) = w(\ell h/N), \quad \ell h/N \le t < (\ell+1)h/N.$$

$\mathscr{S}_{h/N}$ is the sampler that reads out the function values every h/N seconds:

$$\mathscr{S}_{h/N} y := \{y(\ell h/N)\}_{\ell=0}^{\infty},$$

For this to be well defined, we assume that y is right continuous. We then compose these operators with $\mathscr{F}_l(G,K)(z)$ as $\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}$ as in Fig. 2, and then compute its ℓ^2-induced norm. One might expect that we could trivially establish convergence to the gain of $\mathscr{F}_l(G,K)(z)$ as $N \to \infty$; there are, however, a number of technical difficulties and the proof is quite nontrivial as we will see in the next section.

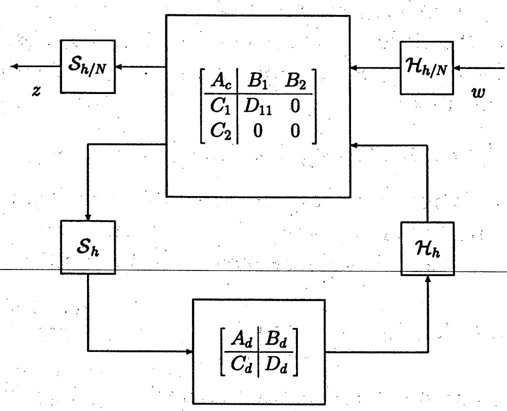

Fig. 2. Fast sample/hold approximation.

Let us first give concrete formulae for $\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}$. Note first that there are two sampling periods in $\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}$, thus it is not a time-invariant system. To remedy this, we invoke discrete-time lifting, and stack the sequence $\{w(0), w(h/N), w(2h/N), \ldots\}$ into a blocked sequence of N-vectors $[w_k(0), w_k(1), \ldots, w_k(N)]^{\mathrm{T}}$ with

$$w_k(\ell) := w(kh + \ell h/N), \quad \ell = 0, 1, \ldots, N-1, \ k = 0, 1, \ldots.$$

A straightforward calculation yields the following proposition.

Proposition 3.1. *The lifted transfer operator of $\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}$ is given by $\mathscr{F}_l(G_d, \mathscr{K})(z)$, where $G_d(z)$ is the discrete-time system*

$$G_d(z) = \left[\begin{array}{c|c} e^{A_c h} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right] \tag{8}$$

with

$$\bar{B} := \int_0^{h/N} \exp(A_c t)[B_1, B_2]\,\mathrm{d}t,$$

$$\mathscr{B} := [\mathrm{e}^{A_c(N-1)h}\bar{B}, \ldots, \mathrm{e}^{A_c h}\bar{B}, \bar{B}],$$

$$\bar{C} := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

$$\mathscr{C} := [\bar{C}^{\mathrm{T}}, \mathrm{e}^{A_c^{\mathrm{T}} h}\bar{C}^{\mathrm{T}}, \ldots, \mathrm{e}^{A_c^{\mathrm{T}}(N-1)h}\bar{C}^{\mathrm{T}}]^{\mathrm{T}},$$

$$\mathscr{D} := \begin{bmatrix} D_{11} & 0 & \cdots & \\ \bar{C}\bar{B} & D_{11} & \cdots & \\ \cdots & \cdots & \cdots & \\ \bar{C}\mathrm{e}^{A_c(N-2)h}\bar{B} & \bar{C}\mathrm{e}^{A_c(N-3)h}\bar{B} & \cdots & D_{11} \end{bmatrix}$$

and

$$\mathscr{K}(z) := \begin{bmatrix} I_{p1} \\ \vdots \\ I_{p1} \end{bmatrix} K(z)[I_{m_1}, 0, \ldots, 0].$$

Sketch of Proof. In this realization the control input u and measured output y are also fast-sampled and stacked, hence we must also execute discrete-time lifting on $K(z)$. For $[y(kh), \ldots, y(kh + (N-1)h/N)]^{\mathrm{T}}$, sampling occurs at $t = kh$, hence we must multiply $K(z)$ from the right by $[I_{m1}, 0, \ldots, 0]$. On the other hand, the hold device is $\mathscr{H}_h$ so that the input u is constant over the interval $[kh, kh + (N-1)h/N]$. Thus $K(z)$ must be multiplied from the left by $[I_{p_1}, I_{p_1}, \ldots, I_{p_1}]^{\mathrm{T}}$. Writing down the effect of fast sampling of the exogenous and control inputs readily yields the formulas above. □

4. Convergence of gain

We can now state our main result.

Theorem 4.1. *Under the same notation as given in Proposition* 3.1, $\|\mathcal{F}_l(G_{d,N}, \mathcal{K})(e^{j\omega h})\| \to \|\mathcal{F}_l(G, K)(e^{j\omega h})\|$ *as* $N \to \infty$. *Furthermore, this convergence is uniform in* $\omega \in [-\pi/h, \pi/h]$.

As we stated already, this theorem appears to hold trivially, but actually this is not so. The approximant $\mathcal{S}_{h/N}\mathcal{F}_l(G, K)\mathcal{H}_{h/N}$ takes the output value by sampling. In order that this approximates the actual output, the output should not oscillate too much. In other words, N should be large enough so that the fast-sampling period h/N is fine enough to capture the behavior of the output. Now this should hold for the worst-case output z_{worst}, but we cannot know z_{worst} in advance, and it can only be approximated. Therefore, to guarantee N to be large enough in the above sense, we need some sort of uniformity that ensures a relatively smooth output for a class of inputs. This is where we need the assumption that all the "D" terms in the plant $G(s)$ apart from D_{11} be zero: this in turn will guarantee that the overall closed-loop transfer operator $\mathcal{F}_l(G, K)(e^{j\omega h})$ (excluding D_{11}) is compact, and assures the desired uniformity. Furthermore, in order that the fast-sampling approximation can be used for gain computation, it is necessary that this convergence be uniform in frequency ω. Otherwise, for each different ω, it would require a higher-order approximation, and then a single approximant cannot be used for gain computation. Uniformity in frequency is also guaranteed in the above theorem.

Another technical problem is that the fast-sampled outputs belong to different spaces for different N's. To remedy this, we embed such discrete-time outputs into $L^2[0, h]$ by composing them with $\mathcal{H}_{h/N}$. Then we can consider these outputs in a common space. The L^2 norm applying before embedding becomes multiplied by a factor $\sqrt{h/N}$ when outputs are considered over. However, this scaling effect can be suitably cancelled by considering the norm of (lifted) inputs $[w_k(0), \ldots, w_k((N-1)h/N)]^{\mathrm{T}}$ in the space $L^2[0, h]$ via $\mathcal{H}_{h/N}$ too.

We start with the following lemma that guarantees the uniform equicontinuity of $[\mathcal{F}_l(G, K)(e^{j\omega h})w](\theta)$ for w in the unit ball of $L^2[0, h]$.

Lemma 4.2. *Let* $\Phi_w(\theta) = [\mathcal{F}_l(G, K)(e^{j\omega h})w](\theta) - D_{11}w$. *Then the family* $\{\Phi_w\}$ *is uniformly equicontinuous for* $U_1 = \{w \in L^2[0, h]: \|w\|_2 \le 1\}$. *That is, for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|t - s| < \delta$ *implies* $|\Phi_w(t) - \Phi_w(s)| < \varepsilon$ *for every* $w \in U_1$. ($|\cdot|$ *denotes the Euclidean norm.*) *In particular, for every* $\varepsilon > 0$ *there exists* N_0 *such that if* $N \ge N_0$ *then*

$$\sup_{0 \le t \le h} |\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_w(t) - \Phi_w(t)| \le \varepsilon \tag{9}$$

for every $w \in U_1$.

Proof. In view of the definition of Φ_w, we may assume $D_{11} = 0$ and consider $\Phi_w = \mathcal{F}_l(G, K)(e^{j\omega h})w$. From Eqs. (3) and (4), $\Phi_w = \mathbf{D}w + \mathbf{C}(e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w$. Since

$$(\mathbf{B}w) = \begin{bmatrix} \int_0^h e^{A(h-\eta)}Bw(\eta)\,d\eta \\ 0 \end{bmatrix},$$

Schwarz's inequality easily implies that there exists $M_1 > 0$ such that

$$|\mathbf{B}w| \le M_1\|w\| \le M_1 \tag{10}$$

for every $w \in U_1$. Now

$$(\mathbf{C}(e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w)(\theta) = [C_1(\theta) \quad C_2(\theta)](e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w.$$

Then

$$\begin{aligned} &|(\mathbf{C}(e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w)(t) - (\mathbf{C}(e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w)(s)| \\ &= |[C_1(t) - C_1(s) \quad C_2(t) - C_2(s)](e^{j\omega h}I - \mathbf{A})^{-1}\mathbf{B}w)| \\ &\le |[C_1(t) - C_1(s) \quad C_2(t) - C_2(s)]| \cdot |(e^{j\omega h}I - \mathbf{A})^{-1}| \cdot |\mathbf{B}w| \end{aligned}$$

and by Eqs. (5) and (10), this is uniformly small when $|t - s|$ is small, irrespective of w in U_1.

Hence we need only establish equicontinuity for $\mathbf{D}w$. Write $W(t)$ for $Ce^{At}B$, and let $t > s$.

$$(\mathbf{D}w)(t) - (\mathbf{D}w)(s) = \int_0^s [W(t-\eta) - W(s-\eta)]w(\eta)\,d\eta + \int_s^t W(t-\eta)w(\eta)\,d\eta.$$

Set $\Xi(t, s, \eta) := W(t-\eta) - W(s-\eta)$. An easy application of Schwarz's inequality yields

$$\begin{aligned} &|(\mathbf{D}w)(t) - (\mathbf{D}w)(s)|^2 \\ &\le \int_0^h \operatorname{trace}[\Xi(t,s,\eta)\Xi(t,s,\eta)^{\mathrm{T}}\}\,d\eta \cdot \int_0^h |w(\eta)|^2\,d\eta \\ &\quad + \int_s^t \operatorname{trace}\{W(t-\eta)W(t-\eta)^{\mathrm{T}}\}\,d\eta \cdot \int_0^h |w(\eta)|^2\,d\eta \\ &\le \int_0^h \operatorname{trace}[\Xi(t,s,\eta)\Xi(t,s,\eta)^{\mathrm{T}}\}\,d\eta \\ &\quad + \int_s^t \operatorname{trace}\{W(t-\eta)W(t-\eta)^{\mathrm{T}}\}\,d\eta. \end{aligned} \tag{11}$$

Since W is uniformly continuous on $[0, h]$ (as a continuous function on a closed interval), for every $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that whenever $|t - s| < \delta_0$, $t, s \in [0, h]$, $|W(t) - W(s)| < \varepsilon_0$ follows. Also, there exists a maximum M of $|W(t)|$ on $[0, h]$. It then follows that

$$\int_0^h \operatorname{trace} \Xi(t,s,\eta)\Xi(t,s,\eta)^{\mathrm{T}}\,\mathrm{d}\eta + \int_s^t \operatorname{trace} W(t-\eta)W(t-\eta)^{\mathrm{T}}\,\mathrm{d}\eta \le \varepsilon_0^2 h + M^2[t-s]. \tag{12}$$

Take any $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $|t-s| < \delta_1$ then $|W(t) - W(s)| < \varepsilon/\sqrt{2h}$. Let $\delta := \min\{\delta_1, \varepsilon^2/2M^2\}$. Then by Eqs. (11) and (12),

$$|(\mathbf{D}w)(t) - (\mathbf{D}w)(s)|^2 \le \varepsilon^2,$$

whenever $|t-s| < \delta$, as claimed. Combining this claim with that for $\mathbf{C}(\mathrm{e}^{\mathrm{j}\omega h}I - \mathbf{A})^{-1}\mathbf{B}$ completes the proof.

Finally, Eq. (9) is a direct consequence of the proved uniform equicontinuity. □

We are now ready to prove our main theorem.

Proof of the Main Theorem. First fix ω; we shall prove that $\|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\|$ converges to $\|\mathscr{F}_l(G, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\|$. In what follows we drop the dependence on ω for simplicity.

Recall $\mathscr{F}_l(G, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})[v] = \Phi_v + D_{11}v$ according to the definition of Φ_v in Lemma 4.2, and let

$$\rho := \|\mathscr{F}_l(G, \mathscr{K})[v_0]\| \le \rho.$$

Without loss of generality, we may assume that this v_0 is continuous on $[0, h]$. On the other hand, by Lemma 4.2, there exists N_0 such that for $N > N_0$

$$\|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_v - \Phi_v\| \le \varepsilon \tag{13}$$

for every $v \in U_1$. Furthermore, since $\mathscr{F}_l(G, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})$ is a continuous linear operator on $L^2[0, h]$, there holds for some $M > 0$

$$\|\mathscr{F}_l(G, \mathscr{K})[v] - \mathscr{F}_l(G, \mathscr{K})[w]\| \le M\|v - w\|, \quad v, w \in L^2[0, h]. \tag{14}$$

Take sufficiently large N_1 such that

$$\|\mathscr{H}_{h/N}\mathscr{S}_{h/N}v_0 - v_0\| \le \varepsilon \tag{15}$$

for all $N > N_1$. Since v_0 is continuous on $[0, h]$ (and hence uniformly continuous), such N_1 exists. Define $v_N := \mathscr{S}_{h/N}v_0$.

Now let $N_2 := \max\{N_0, N_1\}$. It then follows that for every $N > N_2$,

$$\begin{aligned}\|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\mathscr{H}_{h/N}v_N} - \Phi_{v_0}\| &\le \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\mathscr{H}_{h/N}v_N} - \Phi_{\mathscr{H}_{h/N}v_N}\| \\ &\quad + \|\Phi_{\mathscr{H}_{h/N}v_N} - \Phi_{v_0}\| \\ &\le (1+M)\varepsilon \end{aligned}\tag{16}$$

by Eqs. (13)–(15). Observing $\mathscr{F}_l(G, K)(v) = D_{11}v + \Phi_v$, we have

$$\begin{aligned}\|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})\| &= \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}\| \\ &\ge \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\mathscr{F}_l(G,K)\mathscr{H}_{h/N}v_N\|/\|v_N\| \\ &= \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\mathscr{H}_{h/N}v_N} + D_{11}\mathscr{H}_{h/N}v_N\|/\|v_N\| \\ &\quad (\text{note } \mathscr{H}_{h/N}\mathscr{S}_{h/N}D_{11}\mathscr{H}_{h/N}v_N = D_{11}\mathscr{H}_{h/N}v_N) \\ &= \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\mathscr{H}_{h/N}v_N} + D_{11}\mathscr{H}_{h/N}v_N \\ &\quad - D_{11}v_0 - \Phi_{v_0} + D_{11}v_0 + \Phi_{v_0}\|/\|v_N\| \\ &\ge (\|\Phi_{v_0} + D_{11}v_0\| - (1 + M + \|D_{11}\|)\varepsilon)/(1+\varepsilon) \\ &\quad (\text{by Eq. (16)}) \\ &\ge (\rho - (M + 2 + \|D_{11}\|\varepsilon)/(1+\varepsilon)\end{aligned}\tag{17}$$

by definition of v_0. Since this holds for any $N > N_2$,

$$\liminf_{N\to\infty} \|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\| \ge \|\mathscr{F}_l(G, K)(\mathrm{e}^{\mathrm{j}\omega h})\|. \tag{18}$$

Conversely, take any $\bar{v} \in \mathbb{R}^N$ such that $\bar{w}_N := \mathscr{H}_{n/N}\bar{v} \in U_1$. Clearly,

$$\|\Phi_{\bar{w}_N} + D_{11}\bar{w}_N\| = \|\mathscr{F}_l(G, K)\bar{w}_N\| \le \rho. \tag{19}$$

On the other hand, by Eq. (13)

$$\|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\bar{w}_N} - \Phi_{\bar{w}_N}\| \le \varepsilon$$

for $N \ge N_0$. Since $\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\bar{w}_N} + D_{11}\bar{w}_N = \mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})\bar{v}$, this readily implies

$$\begin{aligned}\|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})\bar{v}_N\| &= \|\mathscr{H}_{h/N}\mathscr{S}_{h/N}\Phi_{\bar{w}_N} + D_{11}\bar{w}_N\| \\ &\le \|\Phi_{\bar{w}_N} + D_{11}\bar{w}_N\| + \varepsilon \le \rho + \varepsilon\end{aligned}$$

and hence

$$\liminf_{N\to\infty} \|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})\| \le \|\mathscr{F}_l(G, K)\|. \tag{20}$$

Combining this with Eq. (18), we have the desired convergence for a fixed ω.

We now prove that this convergence is uniform in ω in $[-\pi/h, \pi/h]$. First note that $\|\mathscr{F}_l(G, K)(\mathrm{e}^{\mathrm{j}\omega h})\|$ is a continuous function of ω. Fix any ω and take N_ω such that

$$|\|\mathscr{F}_l(G, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\| - \|\mathscr{F}_l(G_{\mathrm{d},N}, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\|| < \varepsilon \tag{21}$$

for all $N \ge N_\omega$. Then by the continuity of $|\|\mathscr{F}_l(G, \mathscr{K})(\mathrm{e}^{\mathrm{j}\omega h})\|$ in ω, there exists $\delta_\omega > 0$ such that for any ω' with $|\omega - \omega'| < \delta_\omega$, Eq. (21) still holds for $N > N_\omega$. We then have the covering

$$[-\pi/h, \pi/h] = \bigcup_{\omega \in [-\pi/h, \pi/h]} (-\omega - \delta_\omega, \omega + \delta_\omega).$$

Since $[-\pi/h, \pi/h]$ is a compact set, there exists a finite subcovering

$$[-\pi/h, \pi/h] = \bigcup_{n=1}^{L} (-\omega_n - \delta_{\omega_n}, \omega_n + \delta_{\omega_n}).$$

Accordingly, define $N := \max\{N_{\omega_1}, \ldots, N_{\omega_L}\}$. Clearly, for $n > N$, estimated (21) is satisfied for any $\omega \in [-\pi/h, \pi/h]$. □

5. Concluding remarks

We have shown that the fast-sample/fast-hold approximation uniformly approximates the gain function of the sampled-data frequency response. While we gave a proof for the L^2-induced norm, the proof here works equally well for L^p-induced norms, $1 \leq p \leq \infty$, with suitable changes in the estimate involving $\mathbf{D}$ operator as in Eq. (1).

The computational algorithm via fast sampling is implemented in $\mathscr{H}_{\mathrm{SYS}}$ module (Hara et al., 1995), and shown to be very effective. We refer the reader to Yamamoto et al. (1997) and Hara et al. (1995) for a numerical example and some comparison with other methods.

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