



# On the choice of inputs in identification for robust control<sup>1</sup>

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*Optimization of a specific singular value of the data matrix provides a new tool for the design of inputs which are persistently exciting from the point of view of identification for robust control.*

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## Abstract

The thesis that noisy identification has close ties to the study of the singular-value decomposition of perturbed matrices is investigated. In particular by assuming an upper bound on the norm of the perturbation, one can obtain a convex parametrization of an uncertain family of systems which contains the system generating the data. In this approach, the *second-smallest singular value*  $\sigma_*$  of an appropriately defined data matrix becomes a quantity of importance as it provides an upper bound for the size of the uncertain family. This yields a new tool leading to the design of input functions which are *optimal* or *persistently exciting* from the point of view of identification for robust control. © 1999 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

In this paper, we will be concerned with identification in the sense of estimating the coefficients of the numerator and denominator polynomials of a rational transfer function, given noisy measurements of both the input and output of the system with this transfer function. An important issue in identification is the appropriate choice of the input or excitation function, or its spectrum to the extent that such a choice is possible. After all, injecting a random sequence with spectrum centered in a narrow band around 1 kHz into a system with cut-off frequency approximately 1 Hz will not be of great assistance. Generally speaking, we want an excitation producing measurements that yield a maximum amount of information about the underlying system, despite disturbances

which may affect the measurements or errors which may arise by assuming the wrong model structure.

It is now well known that, depending on the number of parameters defining a system, an excitation used for identification must effectively contain a minimum number of separate frequencies, the number depending on the number of unknown parameters. Such thinking gives rise to the idea of *persistently exciting* inputs. But beyond this, it is intuitively clear that excitation at some frequencies is likely to be much more useful than excitation at other frequencies, and it is this issue that we wish to explore in the paper.

In Goodwin and Payne (1977) for a similar identification problem, the so-called *Fisher information matrix* of the measurements is defined and the optimization criterion consists in maximizing the logarithm of the determinant of this matrix. This is the so-called *D-optimality* criterion. An earlier approach to the problem along the same lines can be found in Arimoto and Kimura (1971). Other optimization criteria can be formulated; for a recent survey see e.g. Schoukens et al. (1993). As we will see later, these optimization criteria do not always accurately measure the amount of information that can be extracted

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from a system. Moreover, some of them are computationally intensive and only approximate solutions can be obtained. Generically, optimal identification signals typically turn out to be either *white noise* or *pseudo-random binary sequences* or *multi-sine signals*. In Mareels et al. (1987), a different criterion is proposed, namely the maximization of the *smallest eigenvalue* of the information matrix. The motivation for the selection of this criterion comes from adaptive control. It can be shown that the rate of convergence of an associated adaptive control law is bounded from below by the smallest eigenvalue of the information matrix (see Kosut et al., 1985).

The identification set-up we work with involves the determination of an eigenvector associated with the smallest eigenvalue of a hermitian matrix; this matrix is non-negative or positive definite and depends on the system inputs and outputs (and thus on the system inputs and the system itself); furthermore, the entries of this matrix are perturbed as a result of measurement errors. The accuracy of the system identification problem is tied to the accuracy of the eigenvector determination. Now for an arbitrary Hermitian matrix  $M$  with any two successive eigenvalues  $\lambda_i$ ,  $i = 1, 2$ , and the corresponding eigenvectors  $v_i$ ,  $i = 1, 2$ , it is well known, see e.g. Stewart and Sun (1990), that the accuracy with which the two eigenvectors can be computed increases with the difference  $|\lambda_1 - \lambda_2|$ . Inspired by this fact, we propose in the sequel an approach to optimal and persistently exciting input signal design, which is based on the maximization of the *second-smallest eigenvalue* of an appropriately defined *data covariance matrix*. This formulation results by considering *system identification* as a problem involving the *singular-value decomposition* (SVD) of a matrix.

In our approach, we adopt the assumption that an upper bound on the norm of the perturbation affecting the data is a priori known. As a consequence, the data covariance matrix used for identification defines an uncertain family of systems with norm bounded uncertainty. The systems belonging to this uncertain family form a convex set. As will be shown in Section 3.1, the maximization of the second-smallest singular value of the data covariance matrix, through choice of the exciting input results in the *minimization* of an upper bound of the uncertainty affecting the system parameters. The consequence of using this optimization criterion for the construction of input signals is that the ensuing *optimal* functions are *different* from those resulting by means of the D-optimization. One set of inputs often used, for instance, is that of pseudo-random sequences. It turns out that such inputs increase the second-smallest eigenvalue in question moderately for *generic* systems, but without maximizing it.

In the considerations to follow the optimal input is determined by maximizing the criterion discussed above (second-smallest singular value); for this the parameters of the underlying system  $\Sigma$  are assumed known. This is

only an apparent paradox (long since recognized in the literature, although this fact is not well-known) which is explored more fully in the sections that follow. The issue is illustrated by means of a simple simulation in Section 7.3, which uses the iterative scheme described in Section 3.3. It should be stressed that such schemes have been used as far back as Goodwin and Payne (1977). Finally, a word on the connection with *errors-in-variables* identification: in this paper, we assume that the noise is bounded in the sense of Eq. (3.7); consequently in our framework, noise may affect both the input and the output or just the output.

The material developed below has close connections, and in some respects complements, the so-called *subspace based state-space system identification* (4SID) methods. As stated in Viberg (1994), the class of 4SID methods is somewhat loosely defined as those techniques that employ a rank-reduction of some quantity estimated from input-output data, arriving at an estimate of an extended observability matrix, and subsequently to an estimate of the  $(A, B, C, D)$  matrices of the system which is identified. The rank reduction is performed using the SVD. In our case the SVD is also central. But in contrast to 4SID methods, our goal is to address accuracy of model issues; thus by manipulating the input — which affects the singular values of the data matrix — we are seeking to identify a class of uncertain models where the size of the uncertainty is reduced.

Subspace-based methods for system identification have a long history. The first contribution was Kalman's realization theory which was put forward in 1965; realization is the problem of recovering the system matrices  $(A, B, C, D)$ , from (exact) measurements of the impulse response. In the 1970s the SVD was introduced as a tool in the realization problem. Subsequently, we note the paper of Willems (1986, 1987) which inspired a lot of activity in this area, and the work of Van Overschee and De Moor (1996) which is exposed in the recent book; note also the paper of the authors Anderson and Antoulas (1990). Further results can also be found in McKelvey (1994). For an extended overview see van der Veen et al. (1993); for a shorter one see Viberg (1994).

The paper is organized as follows: the next section reviews the classical approach to designing optimal inputs for identification. Section 3 reviews the identification problem for robust control; under the assumption that an upper bound on the norm of the perturbation affecting the data is a priori known a new approach is reviewed. It leads to the conclusion that the second-smallest eigenvalue of an appropriately defined data covariance matrix is a quantity which provides an upper bound for the size of the uncertainty affecting the data. In Section 4 the problem of computing optimal inputs is reformulated in the frequency domain. Section 5 shows that this problem is equivalent to a linear matrix inequality problem; it also shows that the class of optimal inputs

includes a multi-sine input signal. In Section 6 the stationary points of the eigenvalues of the data covariance matrix  $\mathcal{D}$  are computed explicitly in terms of the coefficients of the numerator and denominator of the transfer function; this leads to upper and lower bounds for the second-smallest singular value  $\sigma_*$  of  $\mathcal{D}$ . Section 7 treats the case of first-order systems thus giving additional insight into the structure of this optimization problem. Section 7.3 presents the results of a simulation experiment comparing the random input (which is one of the classical persistently exciting signals) with the input obtained by optimizing the criterion discussed in Section 3.3. Section 8 discusses the inverse optimal-input problem. The main result here is an upper bound of  $\sigma_*$  in terms of the  $h_2$  norm of the system. In conclusion, Section 9 summarizes the results of the paper and lists some open problems. Details on two of the results are given in the Appendix.

## 2. Optimal inputs for identification: the classical approach

In this section we will argue that one can only compute an optimal input for identification when one knows the system already.

What type of identification problems are we interested in? At the broadest level, our interest is in identifying linear system parameters using noisy measurements. Though this is a fairly old topic (see e.g. Goodwin and Payne, 1977) there are a number of points that are all too easy to overlook. Because much of the philosophy of this paper is quite similar to ideas of Goodwin and Payne (1977), we shall now review some of these key points.

Suppose there is an underlying linear system, operating for simplicity in open loop, with

$$\mathbf{y}(t) = G_1(z, \theta)\mathbf{u}(t) + \mathbf{n}(t).$$

Here,  $\mathbf{n}(t)$  is a white noise sequence,  $\mathbf{u}(t)$  an input excitation sequence,  $\mathbf{y}(t)$  an output sequence, and  $\theta$  a vector parametrizing the model. The interest is in estimating  $\theta$ , not  $G_1$  evaluated at one or more frequencies. It is presumed the structure of  $G_1$  is such that an asymptotically unbiased and asymptotically efficient estimate of  $\theta$  can be obtained (given suitable excitation). Put another way, let

$$U_N := \{\mathbf{u}(1), \mathbf{u}(2), \dots, \mathbf{u}(N)\},$$

$$Y_N := \{\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(N)\},$$

$$W_N := \{U_N, Y_N\}.$$

Then, there is an algorithm that processes the measurements  $W_N$  to yield an (asymptotically unbiased) estimate  $\hat{\theta}_N$  with the property that for  $N \rightarrow \infty$ ,

$$\|\text{cov } \hat{\theta}_N - M_N^{-1}\| \rightarrow 0$$

where  $M_N$  is the Fisher information matrix:

$$M_N := \mathcal{E}_{W_N|\theta} \left\{ \left[ \frac{\partial \log p(W_N|\theta)}{\partial \theta} \right]^* \left[ \frac{\partial \log p(W_N|\theta)}{\partial \theta} \right] \right\},$$

where  $\mathcal{E}_{W_N|\theta}$ ,  $p(W_N|\theta)$  denotes the expected value, conditional probability density, respectively, of  $W_N$  given  $\theta$ . In practice, a large experiment time interval is used, but for analysis purposes, an infinite one is contemplated, and a per-sample covariance matrix is used in order that finite interval results are obtained:

$$\bar{M} = \lim_{N \rightarrow \infty} \frac{1}{N} M_N.$$

Further  $\mathbf{u}(t)$  is generally assumed to have a spectral representation and, to avoid ill-posed or uninteresting problems, the input power of  $\mathbf{u}(t)$  is constrained.

While  $\bar{M}$  is a matrix measure of estimate quality, a scalar measure of estimation quality is needed; popular choices are

$$J_1 = -\log \det \bar{M} \quad \text{and} \quad J_2 = \text{trace}[Q\bar{M}^{-1}]$$

for some positive definite  $Q > 0$ . The input design problem is then one of choosing  $\mathbf{u}(t)$  to minimize the index  $J_1$  or  $J_2$ .

It is crucial to understand that the optimum  $\mathbf{u}(t)$  (which in most problems turns out to be a finite collection of sinusoids) depends on the unknown  $\theta$ . This is both natural and practically relevant.

Why is it natural? Consider the task of identifying with a power-limited signal the parameters  $\alpha$  and  $\beta$  in the following continuous time model:<sup>2</sup>

$$\mathbf{y} = \frac{\beta}{(s + \alpha)} \mathbf{u} + \mathbf{n}$$

(here,  $\mathbf{n}$  is zero mean very wide bandwidth noise). Suppose one can assign the power in  $\mathbf{u}$  arbitrarily across the frequency spectrum. Where should one put it? Concentrating at low frequencies means that the quotient  $\beta/\alpha$  should be identifiable but not  $\beta$  and  $\alpha$  separately. Concentrating at very high frequencies will lead to nothing being identified. Evidently, one must have some sense of the model bandwidth and then locate at least some energy near the 3 db frequency.

The correct and intuitively appealing answer in this case incidentally, as shown in Goodwin and Payne (1977), is to use a single sinusoid with frequency  $\alpha/\sqrt{3}$ .

If the optimum solution to an identification problem actually depends on the unknown parameter vector to be identified, why does it even make sense to contemplate the problem of optimum selection, let alone why could it

<sup>2</sup> Goodwin and Payne (1977) treat both continuous and discrete time models. Intuition is greater for the former.

be argued to be practically relevant? There are several aspects to an answer:

- There may be partial prior knowledge of a system; a good experiment to improve that knowledge can then be designed, even if the experiment is not optimum.
- Having used rough prior knowledge of a system to obtain an input which is optimal for a nominal parameter vector, one can easily check on the performance of this input for identifying parameters in a range round the rough nominal value; there will be a range in which the performance is acceptable and the true value may well be in this range.
- As the above example shows (through the nondependence of the optimum input on  $\beta$ ) it is not necessarily the case that (rough) estimates need be available of *all* entries of the unknown parameter vector, in order to define a notionally optimal input.
- The results provide a logical basis for recursive re-design of the input and re-identification. An input  $\mathbf{u}^{[0]}$  and output  $\mathbf{y}^{[0]}$  will yield an initial parameter estimate  $\theta^{[0]}$ . Then  $\theta^{[0]}$  is used to determine an input  $\mathbf{u}^{[1]}$  (optimal for  $\theta^{[0]}$ ). This  $\mathbf{u}^{[1]}$  together with the associated output  $\mathbf{y}^{[1]}$  yields  $\theta^{[1]}$ , and so on.

### 3. Identification for robust control

Our goal in this section is to present the framework in which identification is being carried out: linear, finite-dimensional, time-invariant, discrete-time systems, long measurement interval, noise corrupting input and output measurements, with transfer function numerator and denominator coefficients being sought.

We shall argue that the data gives rise to a convex set of possible system models, obtainable in a manner familiar from robust control considerations.

Finally, we shall formulate the input design problem.

#### 3.1. Description of the model and measurements

Given is a string of  $N + 1$  input–output measurements (observations)

$$\hat{\mathbf{u}}(t) \in \mathbf{R}, \hat{\mathbf{y}}(t) \in \mathbf{R}, \quad t = 0, 1, \dots, N. \tag{3.1}$$

The system generating the above data is assumed to be *linear, time-invariant, discrete-time*, and the perturbation affecting the data is assumed to be additive:

$$\hat{\mathbf{u}}(t) = \mathbf{u}(t) + \tilde{\mathbf{u}}(t), \quad \hat{\mathbf{y}}(t) = \mathbf{y}(t) + \tilde{\mathbf{y}}(t), \tag{3.2}$$

where  $\mathbf{u}(t)$ ,  $\mathbf{y}(t)$  is generated by the true system and  $\tilde{\mathbf{u}}(t)$ ,  $\tilde{\mathbf{y}}(t)$  results from the perturbations affecting the measurements. (The presence of both inputs and outputs is the characteristic of an errors-in-variables identification problem.)

Let the system which is described by the pair of polynomials

$$\mathbf{p}(\xi) \in \mathbb{R}[\xi], \mathbf{q}(\xi) \in \mathbb{R}[\xi], \quad n := \deg \mathbf{q} \geq \deg \mathbf{p} =: m$$

for fixed non-negative  $n, m$ , be compatible with  $\mathbf{u}(t)$ ,  $\mathbf{y}(t)$ , i.e.

$$\mathbf{q}(\sigma)\mathbf{y}(t) = \mathbf{p}(\sigma)\mathbf{u}(t), \quad 0 \leq t \leq N - \max\{m, n\},$$

where  $\sigma f(t) := f(t + 1)$  (the backward shift). The latter equation can be written compactly as

$$\begin{bmatrix} \mathbf{p}(\sigma) & -\mathbf{q}(\sigma) \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) \end{bmatrix} = 0, \quad 0 \leq t \leq N - \max\{m, n\}. \tag{3.3}$$

Given a polynomial  $\mathbf{r}(\xi)$  of degree  $k$  we will use the notation:

$$\begin{aligned} \mathbf{r}(\xi) &:= r_0 + r_1\xi + \dots + r_k\xi^k \quad \text{and} \\ \mathbf{r} &:= (r_0 \quad r_1 \quad \dots \quad r_k)^* \in \mathbb{R}^{k+1}, \end{aligned} \tag{3.4}$$

where  $\xi$  is an indeterminate or a complex number and  $(\cdot)^*$  denotes complex conjugation and tranposition. The resulting *data matrix* is

$$\begin{aligned} \hat{\mathcal{M}} &:= \frac{1}{\sqrt{N+1}} \begin{pmatrix} \hat{\mathcal{M}}_1 \\ \hat{\mathcal{M}}_2 \end{pmatrix} \in \mathbb{R}^{(m+n+2) \times (N-n+1)} \\ &:= \frac{1}{\sqrt{N+1}} \begin{pmatrix} \hat{\mathbf{u}}(0) & \hat{\mathbf{u}}(1) & \dots & \hat{\mathbf{u}}(N-n-1) & \hat{\mathbf{u}}(N-n) \\ \hat{\mathbf{u}}(1) & \hat{\mathbf{u}}(2) & \dots & \hat{\mathbf{u}}(N-n) & \hat{\mathbf{u}}(N-n+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\mathbf{u}}(m) & \hat{\mathbf{u}}(m+1) & \dots & \hat{\mathbf{u}}(N-n+m-1) & \hat{\mathbf{u}}(N-n+m) \\ \hat{\mathbf{y}}(0) & \hat{\mathbf{y}}(1) & \dots & \hat{\mathbf{y}}(N-n-1) & \hat{\mathbf{y}}(N-n) \\ \hat{\mathbf{y}}(1) & \hat{\mathbf{y}}(2) & \dots & \hat{\mathbf{y}}(N-n) & \hat{\mathbf{y}}(N-n+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\mathbf{y}}(n) & \hat{\mathbf{y}}(n+1) & \dots & \hat{\mathbf{y}}(N-1) & \hat{\mathbf{y}}(N) \end{pmatrix}. \end{aligned} \tag{3.5}$$

If  $\hat{\mathcal{M}}$  were generated with noise-free data, a moment's reflection shows that it would have a left null vector ( $\mathbf{p}^* - \mathbf{q}^*$ ). With noise perturbing the measurements, following Eq. (3.2),  $\hat{\mathcal{M}}$  can be written as a sum

$$\hat{\mathcal{M}} = \mathcal{M} + \tilde{\mathcal{M}}. \tag{3.6}$$

We will assume that an upper bound  $\varepsilon$  on the norm of  $\tilde{\mathcal{M}}$  is a priori known or can be computed:

$$\|\tilde{\mathcal{M}}\|_2 \leq \varepsilon. \tag{3.7}$$

In order to get a better understanding of this condition, it is useful to examine what happens when the noise is white and the observation interval tends to infinity. In this case the squares of  $m + 1$  singular values of  $\tilde{\mathcal{M}}$  are equal to the variance of the noise affecting the inputs, and the remaining  $n + 1$  singular values are equal to the variance

of the noise affecting the output (see also Remark 3.1.4). Therefore, the norm of  $\tilde{\mathcal{M}}$  is equal to the square root of the largest variance of the noise affecting the data, and  $\varepsilon$  is an upper bound thereof.

Now, the central idea of identification is to determine an approximate null vector of  $\tilde{\mathcal{M}}$ . In order to best do this, one can make use of the singular-value decomposition (SVD) of  $\tilde{\mathcal{M}}$ . With  $r := n + m + 2$ , let

$$\begin{aligned} \tilde{\mathcal{M}} &= \hat{U} \hat{\Sigma}_1 \hat{V}^* \quad \text{where } \hat{U} \hat{U}^* = I_r, \hat{V} \hat{V}^* = I_{N-n+1}, \\ \hat{\Sigma}_1 &= (\hat{\Sigma} \quad 0) \in \mathbb{R}^{r \times (N-n+1)}, \quad \hat{\Sigma} \in \mathbb{R}^{r \times r}. \end{aligned} \quad (3.8)$$

The singular values and left singular vectors will be denoted as follows:

$$\begin{aligned} \hat{U} &= (\hat{u}_r \quad \hat{u}_{r-1} \quad \cdots \quad \hat{u}_2 \quad \hat{u}_1) \quad \text{and} \\ \hat{\Sigma} &= \text{diag}(\hat{\sigma}_r, \dots, \hat{\sigma}_2, \hat{\sigma}_1), \quad \hat{\sigma}_i \leq \hat{\sigma}_{i+1}. \end{aligned} \quad (3.9)$$

The SVDs of  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are  $\mathcal{M} = U \Sigma_1 V^*$  and  $\tilde{\mathcal{M}} = \tilde{U} \tilde{\Sigma}_1 \tilde{V}^*$ , respectively, where the symbols  $U, \tilde{U}, \Sigma_1, \tilde{\Sigma}_1, V, \tilde{V}$  are defined in a way similar to  $\hat{U}, \hat{\Sigma}_1, \hat{V}$ . Notice in particular that the smallest singular value of the noise-free data matrix  $\mathcal{M}$  is zero:  $\sigma_1 = 0$ .

Since  $\varepsilon$  satisfies Eq. (3.7)  $\hat{\sigma}_r > \varepsilon \geq \hat{\sigma}_1$  holds. In the sequel, we will assume that we are in the *low noise* case, which we define as the case when  $\varepsilon$  lies between the smallest and second-smallest singular values of  $\tilde{\mathcal{M}}$ :

$$\hat{\sigma}_2 > \varepsilon \geq \hat{\sigma}_1. \quad (3.10)$$

It should be mentioned that if the order of the system is not accurately known (estimated), the above inequality may not be satisfied, even in the low-noise case.

### 3.2. A convex parametrization of the uncertainty

We will now characterize all systems which could have given rise to the data matrix  $\tilde{\mathcal{M}}$ , under the assumption that the measurement noise is limited as per Eq. (3.7). As already discussed the coefficient vector

$$\theta^* := (\mathbf{p}^* \quad - \mathbf{q}^*)$$

is in the left kernel of the noise-free data matrix  $\mathcal{M}$ , i.e.

$$\theta^* \mathcal{M} = 0 \quad (3.11)$$

holds. It follows from Eq. (3.6) that

$$\theta^* \tilde{\mathcal{M}} = \underbrace{\theta^* \mathcal{M}}_{=0} + \theta^* \tilde{\mathcal{M}} = \theta^* \tilde{\mathcal{M}}$$

and consequently from Eq. (3.7), we obtain

$$\frac{\|\theta^* \tilde{\mathcal{M}}\|_2}{\|\theta\|_2} = \frac{\|\theta^* \tilde{\mathcal{M}}\|_2}{\|\theta\|_2} \leq \varepsilon. \quad (3.12)$$

If we define the *misfit* function  $\mu$  between the coefficient vector  $x$  and the data matrix  $\tilde{\mathcal{M}}$  as

$$\mu(x, \tilde{\mathcal{M}}) := \frac{\|x^* \tilde{\mathcal{M}}\|_2}{\|x\|_2}, \quad x \in \mathbb{R}^r, \quad (3.13)$$

it follows that  $\mu(\theta, \tilde{\mathcal{M}}) \leq \varepsilon$ ; therefore  $\theta$  belongs to the family

$$\mathcal{F} := \{x \in \mathbb{R}^r : \mu(x, \tilde{\mathcal{M}}) \leq \varepsilon\}. \quad (3.14)$$

This family contains all parameter vectors having misfit at most  $\varepsilon$ , with respect to the data matrix  $\tilde{\mathcal{M}}$ .

The *problem* of explicitly characterizing the family  $\mathcal{F}$  arises. This will provide an uncertain family of plants which is guaranteed to contain the plant that generated the data; subsequently, one can then search for a controller which achieves robust stability and performance across the entire set of plants. The following result which is proved in Antoulas (1997) and Zhang and Antoulas (1998) provides the characterization of  $\mathcal{F}$ .

**Lemma 3.1.** Consider a set of plants obeying Eq. (3.3), with noisy measurements of the plant input and output sequences given by Eq. (3.2). Define the data matrix associated with the measurements by Eq. (3.5), and assume that when decomposed as in Eq. (3.6), the bound (3.7) on the noise holds. Let  $\hat{M}$  have the SVD defined by Eqs. (3.8) and (3.9), and assume that the low noise condition (3.10) holds. Then,  $x \in \mathcal{F}$ , where  $\mathcal{F}$  is defined by Eq. (3.14), if, and only if,

$$x \in \text{span} \{ \hat{u}_1 + \delta_2 \hat{\gamma}_2 \hat{u}_2 + \cdots + \delta_r \hat{\gamma}_r \hat{u}_r \},$$

$$\hat{\gamma}_i := \sqrt{\frac{\varepsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_i^2 - \varepsilon^2}}, \quad i = 2, \dots, r$$

for some  $\delta_i, i = 2, \dots, r$ , satisfying

$$\delta_2^2 + \cdots + \delta_r^2 < 1.$$

If the family  $\mathcal{F}$  is viewed in a basis defined by the columns of  $\hat{U}$ , i.e.  $\hat{u}_1, \dots, \hat{u}_r$ , it is evidently a cone around  $\hat{u}_1$ , with half-axes

$$\hat{\gamma}_i, \quad i = 2, \dots, r.$$

Of course,  $\Delta := (\delta_r \delta_{r-1} \cdots \delta_2)^* \in \mathbb{R}^{r-1}$  is a contraction. Hence  $\mathcal{F}$  is a convex set.

With the center of the set  $\mathcal{F}$  defined by  $\hat{u}_1$ , one can view  $\hat{u}_1$  as providing the coefficients of the nominal model resulting from identification based on the measurements.

From the coefficient vectors  $\hat{u}_i$  we define the corresponding polynomials

$$\hat{\theta}_i(\xi)^* := \begin{pmatrix} 1 & \xi & \cdots & \xi^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \xi & \cdots & \xi^n \end{pmatrix}$$

$$\hat{u}_i := (\hat{\mathbf{p}}_i(\xi)^*, -\hat{\mathbf{q}}_i(\xi)^*)^*.$$

The family  $\mathcal{P}_\Delta$  of uncertain plants is defined by

$$\theta_\Delta(\xi) = \hat{\theta}_1(\xi) + \sum_{i=2}^r \delta_i \hat{\gamma}_i \hat{\theta}_i(\xi), \quad \sum_{i=2}^r \delta_i^2 \leq 1, \quad (3.15)$$

where  $\theta_\Delta(\xi) =: (\mathbf{p}_\Delta(\xi), -\mathbf{q}_\Delta(\xi))$ . If  $p_\Delta$  and  $q_\Delta$  are coprime for all contractions  $\Delta$ , we can form the transfer function of the corresponding uncertain models:

$$\frac{\mathbf{p}_\Delta(\xi)}{\mathbf{q}_\Delta(\xi)} = \frac{\hat{\mathbf{p}}_1(\xi) + \sum_{i=2}^r \delta_i \hat{\gamma}_i \hat{\mathbf{p}}_i(\xi)}{\hat{\mathbf{q}}_1(\xi) + \sum_{i=2}^r \delta_i \hat{\gamma}_i \hat{\mathbf{q}}_i(\xi)}. \quad (3.16)$$

**Remark 3.1.** (1) As already mentioned, the coefficient vector  $(\mathbf{p}^*, -\mathbf{q}^*)$  of the system which generated the data, can be obtained from the data matrix  $\mathcal{M}$  by computing its *left kernel*, which in turn, can be obtained by means of the SVD of  $\mathcal{M}$ . More precisely, if the orders  $m, n$  of the numerator and denominator are known, the (left) kernel of  $\mathcal{M}$  has dimension one; in this case the vector of numerator and denominator coefficients is determined up to a non-zero constant, and consequently, the transfer function is uniquely determined. If however only upper bounds of the degrees of the numerator and the denominator are available, the left kernel of  $\mathcal{M}$  has dimension larger than one, and the numerator and denominator polynomials of the transfer function of the system can be determined uniquely up to common factors.

(2) The results currently available in the literature about how  $u_1$  depends on the singular vectors  $\hat{u}_i$  of the perturbed data matrix, are asymptotic; this means that these results hold as the norm of the perturbation matrix goes to zero (see e.g. Stewart and Sun, 1990). In contrast, the result given in Lemma 3.1 is *not* asymptotic, i.e. is valid for any perturbation matrix satisfying Eq. (3.7).

(3) The largest angle  $\psi$  of the cone described in Lemma 3.1, with respect to  $\hat{u}_2$ , is given by  $\tan \psi = \hat{\gamma}_2$ . This angle gives a distance measure between the identified plant corresponding to  $\hat{u}_2$  and the true plant. The sine of this angle is actually the *gap* distance between the singular spaces spanned by the corresponding parameter vectors (for details see Stewart and Sun, 1990). It should be stressed that our motivation for using this particular distance measure is that a convex set  $\mathcal{F}$  of uncertain plants can be obtained.

(4) The uncertainty considered above is *norm bounded*. The characterization of uncertainty is *linear fractional*, with  $\Delta$  real. This allows in general the use of standard robust control tools. Notice however that  $\mathcal{P}_\Delta$  may contain models with  $\deg \mathbf{p}_\Delta > \deg \mathbf{q}_\Delta$  (non-proper models) and/or  $\mathbf{p}_\Delta, \mathbf{q}_\Delta$  *not* coprime, in which case the transfer function does not exist. In these cases behavioral robust control theory should be applied.

(5) For the case (3.10), the volume  $\mathcal{V}$  of the cone in the coordinate system defined by the columns of the matrix  $\hat{U}$ , is proportional to the product of the  $\hat{\gamma}_i$ :

$$\mathcal{V} \sim \hat{\gamma}_2 \cdots \hat{\gamma}_r.$$

This volume gives a measure of the magnitude of the uncertainty resulting from Eq. (3.7). Furthermore, the tangent of the largest angle between the central axis of the cone and the sides is equal to  $\hat{\gamma}_2$ . Clearly, the larger the uncertainty, the larger  $\varepsilon$ , the *larger*  $\mathcal{V}$ ; on the other hand, the larger  $\hat{\sigma}^2$  the *smaller*  $\mathcal{V}$  (we will elaborate on this in the next subsection).

(6) *Special case:* if the disturbance is white noise, affecting both the input and the output with equal intensity, as  $N \rightarrow \infty$ , there holds:  $\varepsilon \rightarrow \hat{\sigma}_1$ . This implies  $\hat{\gamma}_i \rightarrow 0$ . We thus recover the well known fact that if the perturbation is white noise and the number of measurements goes to infinity, the left singular vector of the data matrix corresponding to the smallest singular value provides the coefficients of the system which generated the data.

(7) The robust controller design problem for the family of plants  $\mathcal{F}$ , is discussed in Zhang and Antoulas (1998).

### 3.3. Formulation of the optimal input design problem

A result in the perturbation theory of singular values asserts that the singular values of  $\hat{\mathcal{M}}$  and  $\mathcal{M}$  related by Eq. (3.6), satisfy

$$\max_{i=1, \dots, r} |\hat{\sigma}_i - \sigma_i| \leq \|\hat{\mathcal{M}} - \mathcal{M}\| = \|\hat{\mathcal{M}}\| \leq \varepsilon.$$

The first inequality is standard (see Stewart and Sun, 1990); the second follows from assumption (3.7). The above inequality for  $i = 2$  yields

$$|\hat{\sigma}_2 - \sigma_2| \leq \varepsilon \Rightarrow \sigma_2 - \varepsilon \leq \hat{\sigma}_2 \leq \sigma_2 + \varepsilon.$$

Thus, assuming that  $\sigma_2 > 2\varepsilon$ , the largest semi-axis  $\hat{\gamma}_2$  of the cone described above is bounded as follows:

$$\hat{\gamma}_2^2 \leq \frac{\varepsilon^2 - \hat{\sigma}_1^2}{(\sigma_2 - \varepsilon)^2 - \varepsilon^2} \leq \frac{\varepsilon^2}{(\sigma_2 - \varepsilon)^2 - \varepsilon^2}. \quad (3.17)$$

In the above expressions,  $\varepsilon$  does *not* depend on the input  $\mathbf{u}$ ; it is the a priori given upper bound of the norm of the perturbation (3.7). The quantity  $\sigma_2$  depends on the perturbation-free input  $\mathbf{u}$ , which is chosen by the system operator. Thus, by maximizing  $\sigma_2$ , we minimize the upper bound given above. The idea behind this paper is to choose the input so as to maximize  $\sigma_2$ :

$$\mathbf{u}_* = \arg \max \sigma_2(\mathcal{M}(\mathbf{u}, \mathbf{y})), \quad \mathbf{q}(\sigma)\mathbf{y} = \mathbf{p}(\sigma)\mathbf{u}, \quad (3.18)$$

where  $\mathcal{M}(\mathbf{u}, \mathbf{y})$  is the data matrix defined by Eq. (3.5), with the dependence on the input and the output shown explicitly. Thus our *motivation* for studying problem (3.18), is to find inputs which will result in *small* uncertainty ellipsoids (see Section 7.3 for an illustration of this procedure).

As suggested by the discussion of Section 2, in an actual identification experiment, since the true plant which generated the data is not known, we propose the following *recursive* procedure for an intelligent choice of inputs:

$$\mathbf{u}^{[k]}, \mathbf{y}^{[k]} \stackrel{\text{id}}{\mapsto} \Sigma^{[k+1]} = (\mathbf{p}^{[k+1]}, \mathbf{q}^{[k+1]}), \quad k = 0, 1, 2, \dots$$

Initially, for systems with rational transfer functions and known numerator and denominator orders, use some input  $\mathbf{u}_0$  and the resulting output  $\mathbf{y}_0$  to identify the coefficients of the numerator and denominator of the transfer function of a system  $\Sigma_1$ . At the  $k$ th step, let the data matrix be  $\hat{\mathcal{M}}^{[k]}$ , the corresponding left singular vectors and singular values  $\hat{u}_i^{[k]}$  and  $\hat{\sigma}_i^{[k]}$ , respectively. At the  $(k + 1)$ th step, the input to the system is the one which maximizes the second-smallest singular value of the perturbation-free data matrix generated by the model obtained from the left singular vector  $\hat{u}_1^{[k]}$  associated with the smallest singular value  $\hat{\sigma}_1^{[k]}$ :

$$\mathbf{u}_*^{[k+1]} = \arg \max \sigma_2(\mathcal{M}(\mathbf{u}, \mathbf{y})), \quad \mathbf{q}_1^{[k]}(\sigma) \mathbf{y} = \mathbf{p}_1^{[k]}(\sigma) \mathbf{u}. \quad (3.19)$$

Thus at the  $(k + 1)$ th step we obtain a cone  $\mathcal{F}^{[k+1]}$  which contains the true (unknown) system generating the data and has the additional property that an upper bound for the largest semi-axis has been minimized, i.e. an upper bound for the uncertainty at the  $(k + 1)$ th step has been minimized.

As shown in Section 7.3, this recursive procedure yields better results than the identification procedure where a single input is used throughout. For details on recursive excitation function designs, we refer to Goodwin and Payne (1977, Chapter 7.7).

The remainder of this paper is devoted to the study of problem (3.18).

#### 4. Problem description in the frequency domain

In this section, we will derive an alternative to the noiseless data matrix  $\mathcal{M}$ , termed  $\mathcal{R}$ ; its singular values and the associated singular vectors are closely linked to those of the data matrix  $\mathcal{M}$  (actually, as the measurement interval tends to infinity, there is no difference in the singular values). Moreover the alternative is linked to a frequency domain description of inputs.  $\mathcal{R}$  will be useful in considering the optimal input problem using spectral ideas.

**Remark 4.1.** To actually perform an identification, we had proposed using the noisy data matrix  $\hat{\mathcal{M}}$ . However for the purposes of defining optimum inputs to use in the identification experiment, we worked with the noiseless

matrix  $\mathcal{M}$ , see Eq. (3.18). The frequency domain analog described in this section is an analog of  $\mathcal{M}$ , not  $\hat{\mathcal{M}}$ , and its use is like that of  $\mathcal{M}$  for experiment design.

We begin by transforming the data to the frequency domain. Let the Fourier transform of the sequence  $\mathbf{u}(t)$ ,  $t = 0, 1, \dots, N$  be  $\alpha_i$ ,  $i = 0, 1, \dots, N$ ; then

$$\mathbf{u}(t) = \sum_{k=0}^N \alpha_k \mu_k^t, \quad t = 0, 1, \dots, N, \quad (4.1)$$

where

$$\mu_k = e^{jk2\pi/(N+1)}, \quad j = \sqrt{-1}$$

are the  $(N + 1)$ th-roots of unity. For normalization purposes we will assume that  $\mathbf{u}$  has unit power, i.e.

$$\sum_{i=0}^N |\alpha_i|^2 = 1. \quad (4.2)$$

It follows that a possible response (the steady-state response) is

$$\mathbf{y}(t) = \sum_{k=0}^N \psi_k \alpha_k \mu_k^t$$

where  $\psi_k := \frac{\mathbf{p}(\mu_k)}{\mathbf{q}(\mu_k)}, \quad t, k = 0, 1, \dots, N, \quad (4.3)$

i.e.  $\psi_k$  is the value of the transfer function of the system at the frequency  $\mu_k$ . Next we introduce the truncated Vandermonde matrix

$$V_k := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mu_0 & \mu_1 & \dots & \mu_N \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0^k & \mu_1^k & \dots & \mu_N^k \end{pmatrix} \in \mathbb{C}^{(k+1) \times (N+1)}, \quad k \leq N$$

Assuming without loss of generality that  $n \geq m$ , we can write

$$\mathcal{M}_1 = V_m A \bar{V}_{N-n+1}^* \quad \text{and} \quad \mathcal{M}_2 = V_n \Psi A \bar{V}_{N-n+1}^*,$$

where  $(\cdot)^*$  denotes complex conjugation and transposition,  $(\bar{\cdot})$  denotes complex conjugation,  $A := \text{diag}\{\alpha_i\}$ , and  $\Psi := \text{diag}\{\psi_i\}$ . With the notation just introduced we can write the data matrix (3.5) as follows:

$$\mathcal{M} = \frac{1}{\sqrt{N+1}} \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} = \frac{1}{\sqrt{N+1}} \begin{bmatrix} V_m \\ V_n \Psi \end{bmatrix} A \bar{V}_{N-n+1}^*. \quad (4.4)$$

In the sequel, for technical reasons, we will consider the *modified data matrix*  $\mathcal{R}$ :

$$\mathcal{R} := \frac{1}{\sqrt{N+1}} \mathcal{V} A, \quad \mathcal{V} := \begin{bmatrix} V_m \\ V_n \Psi \end{bmatrix}, \quad \mathcal{R}, \mathcal{V} \in \mathbb{C}^{(m+n+2) \times (N+1)}. \quad (4.5)$$

Since the matrix  $(1/(\sqrt{N+1}))V_N$  is unitary, the singular values of  $\mathcal{R}$  and  $(1/(\sqrt{N+1}))\mathcal{R}\bar{V}_N^*$  are the same; furthermore, the equality

$$\mathcal{M} = \mathcal{R}\bar{V}_N^* \begin{bmatrix} I_{N-n+1} \\ 0 \end{bmatrix} = \mathcal{R}\bar{V}_{N-n+1}^*$$

implies that  $\mathcal{M}$  is a submatrix of  $\mathcal{R}\bar{V}_N^*$ ; consequently the singular values of  $\mathcal{R}$  and  $(1/(\sqrt{N+1}))\mathcal{M}$  satisfy the inequalities (see e.g. Stewart and Sun, 1990)

$$\sigma_i(\mathcal{R}) \geq \frac{1}{\sqrt{N+1}} \sigma_i(\mathcal{M}).$$

Finally, as  $N$  goes to infinity, the difference between the above quantities goes to zero:

$$\sigma_i(\mathcal{R}) \xrightarrow{N \rightarrow \infty} \frac{1}{\sqrt{N+1}} \sigma_i(\mathcal{M}). \quad (4.6)$$

In addition to Eq. (3.11) there also holds for finite or infinite  $N$

$$(\mathbf{p}^*, -\mathbf{q}^*)\mathcal{V}A = 0, \quad \forall A = \text{diag}\{\alpha_i\}. \quad (4.7)$$

For later use we also introduce the *data covariance matrix*

$$\begin{aligned} \mathcal{D} &:= \mathcal{R}\mathcal{R}^* = \frac{1}{N+1} \mathcal{V}\mathcal{B}\mathcal{V}^* \in \mathbb{R}^{(n+m+2) \times (n+m+2)}, \\ \mathcal{B} &:= AA^* =: \text{diag}(\beta_0, \dots, \beta_N) \end{aligned}, \quad (4.8)$$

where  $\beta_k := \alpha_k \alpha_k^* = |\alpha_k|^2$ . It should be noticed that  $\mathcal{D}$  is a square matrix with *real* entries (the justification for this assertion is given in Section 5.2). Furthermore, the kernel of  $\mathcal{D}$  is the same as the kernel of  $\mathcal{B}$ , which as already noted, is spanned by  $(\mathbf{p}^*, -\mathbf{q}^*)^*$ .

**Remark 4.2** (*The power of the input*). The input signal class we consider contains in effect finite power signals. However, it never makes sense to identify over an infinitely long interval (in principle one can learn the parameters perfectly, and so there is no question of optimal input design with virtually any design being optimal, but also in this case the parameters are only learnt after an infinitely long delay, which is unsatisfactory from a practical viewpoint); in practice then, the input signals will be truncated after some large time  $N$ . Given this truncation, they become finite energy signals — however the energy grows linearly with  $N$ .

## 5. Optimal inputs maximizing $\sigma_*$

In Eq. (3.18), we stated the optimal identification problem in terms of the data matrix  $\mathcal{M}$ . In the last section we

have suggested replacing this data matrix with the modified (frequency domain-based) data matrix  $\mathcal{R}$ , showing that asymptotically the singular values are the same. These considerations suggest the following.

**Problem 5.1.** *Given a linear, time-invariant, discrete-time system, design an OPTIMAL input  $\mathbf{u}_*(t)$ , defined for  $t \geq 0$ , which will maximize the second-smallest singular value  $\sigma_*$  of the data matrix  $\mathcal{R}$  defined by Eq. (4.5):*

$$\max_A \sigma_*(\mathcal{V}A) \quad \text{where } A = \text{diag}(\alpha_i) \quad \text{and } \sum_i |\alpha_i|^2 = 1. \quad (5.1)$$

This problem can be converted into an eigenvalue optimization problem by considering the second-smallest eigenvalue of the data covariance matrix  $\mathcal{D}$  defined by Eq. (4.8):

$$\max_{\mathcal{B}} \lambda_*(\mathcal{V}\mathcal{B}\mathcal{V}^*) \quad \text{where } \mathcal{B} = \text{diag}(\beta_i) \geq 0, \text{ trace } \mathcal{B} = 1. \quad (5.2)$$

Such eigenvalue optimization problems have been extensively studied. For a recent account, see Overton (1992). The difficulty in dealing with such problems arises from the fact that a fixed eigenvalue (the second smallest in our case) is piecewise continuous but it is *not* a smooth (i.e. differentiable) function of the parameters, namely  $\beta_i$ . For this purpose methods of non-smooth analysis have been developed.

In the remainder of this section, our aim is to study this eigenvalue optimization problem in more depth. We will show how it can be restated using linear matrix inequalities (LMIs), and we will conclude that optimal inputs are obtainable via a finite sum of sinusoids.

### 5.1. Linear matrix inequalities (LMI) formulation

In some instances eigenvalue optimization problems turn out to be *convex* or *concave*, in which case they can be cast into the framework of and solved using LINEAR MATRIX INEQUALITIES, abbreviated LMIs. Given a hermitian matrix  $M(x)$  depending linearly on the parameter vector  $x = (x_1 \cdots x_q)$ :

$$M(x) = \sum_{k=1}^q M_k x_k, \quad M_k = M_k^*,$$

it can be shown that the *largest* and *smallest eigenvalues*  $\lambda_{\max}(M)$  and  $\lambda_{\min}(M)$  are convex and concave functions of  $x$ , respectively. Therefore the minimization and/or maximization of these eigenvalues can be cast as an LMI problem.

In our case (5.2), we want to maximize the second-smallest eigenvalue  $\lambda_*(\mathcal{D}) = \sigma_*^2(\mathcal{D})$  of  $\mathcal{D}$  defined in Eq. (4.8). In general, the second-smallest eigenvalue is *not*



a convex (or concave) function of the parameters. However  $\mathcal{D}$  has a kernel which is constant, i.e. does not depend on the parameters  $\beta_i$ . This fact allows the reduction of the problem to that of maximizing the *smallest* eigenvalue of an associated matrix  $\mathcal{D}_1$ . Let  $\mathcal{W}$  be a constant unitary matrix whose last row is, up to scaling  $(\mathbf{p}^*, -\mathbf{q}^*)$ . Clearly  $\mathcal{W}\mathcal{D}\mathcal{W}^*$  has the form

$$\mathcal{D}_\# := \frac{1}{N+1} \mathcal{W}\mathcal{D}\mathcal{W}^* = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\mathcal{D}_1 := \frac{1}{N+1} \mathcal{V}_\# \mathcal{B} \mathcal{V}_\#^*$$

and

$$\mathcal{V}_\# := [I_{n+m+1}, 0] \mathcal{W} \mathcal{V}. \tag{5.3}$$

Furthermore, the non-zero eigenvalues of  $\mathcal{D}$  and  $\mathcal{D}_1$  are the same. Consequently

$$\lambda_*(\mathcal{D}) = \lambda_{\min}(\mathcal{D}_1)$$

and  $\mathcal{D}_1$  depends on the same parameters as  $\mathcal{D}$ . Thus the problem is reduced to

$$\max_{\mathcal{B}} \lambda_{\min}(\mathcal{V}_\# \mathcal{B} \mathcal{V}_\#^*) \quad \text{where } \mathcal{B} = \text{diag}(\beta_i) \geq 0, \text{ trace } \mathcal{B} = 1. \tag{5.4}$$

The maximization of the smallest eigenvalue of  $\mathcal{D}_\#$  is equivalent to the minimization of the largest eigenvalue of  $-\mathcal{D}_\#$ . Since the minimization leads to a convex problem, the maximization leads to a concave problem. The constraints (positivity of the  $\beta_i$  and their sum equal 1) are *convex*. Hence the resulting problem is concave and can be formulated as an *LMI problem* (see e.g. Boyd et al., 1994):

maximize  $\lambda > 0$  subject to

$$\begin{aligned} \mathcal{V}_\# \mathcal{B} \mathcal{V}_\#^* &\geq \lambda I \\ \mathcal{B} = \text{diag}(\beta_i) &\geq 0 \end{aligned} \tag{5.5}$$

trace  $\mathcal{B} \leq 1$ .

Alternatively, one seeks to maximize  $\lambda$  subject to the *LMI constraint*  $\Gamma(\mathcal{B}) \geq 0$ , where

$$\Gamma(\mathcal{B}) := \begin{pmatrix} \mathcal{V}_\# \mathcal{B} \mathcal{V}_\#^* - \lambda I & & \\ & \beta_0 & \\ & & \ddots & \\ & & & \beta_N & \\ & & & & 1 - \sum_i \beta_i \end{pmatrix}. \tag{5.6}$$

The problem can now be solved using the LMI routines which can be found in the MATLAB TOOLBOX: LMI-TOOL. Of course identifying the  $\beta_i$  which together with  $\lambda$  solve Eq. (5.5) is part of the problem.

### 5.2. Optimal inputs

We shall now look at the characterization of optimal input functions  $\mathbf{u}_*$ , of which there are in general infinitely many. In the light of Problem 5.1, it is evident that we can solve an eigenvalue maximization problem to obtain an optimal data covariance matrix  $\mathcal{D}_*$  (whether  $\mathcal{D}_*$  is unique, in the sense that maximizing the second-smallest eigenvalue specifies  $\mathcal{D}_*$  completely, is a question we leave open). We shall study below how  $\mathbf{u}_*$  may be chosen to produce a particular  $\mathcal{D}_*$ .

Let  $\alpha_i, i = 0, \dots, N$ , be the Fourier coefficients of an optimal input and  $\beta_i$  the corresponding magnitudes squared. The optimal data covariance matrix  $\mathcal{D}_*$  defined by Eq. (4.8) can be expressed as a sum of  $N+1$  rank one matrices

$$\begin{aligned} \mathcal{D}_* &= \sum_{i=0}^N \beta_i \mathcal{D}_i \in \mathbb{R}^{\kappa \times \kappa} \\ \text{where } \mathcal{D}_i &:= \frac{1}{N+1} \mathcal{V}_i \mathcal{V}_i^*, \quad \kappa := n+m+2 \end{aligned} \tag{5.7}$$

and  $\mathcal{V}_i$  denotes the  $i$ th column of  $\mathcal{V}$  defined in Eq. (4.5). Since  $\mathbf{u}_*$  is a real signal, its Fourier transform is symmetric;

$$\beta_i = \beta_{N+1-i}, \quad i = 1, \dots, \bar{N}, \quad \bar{N} := \begin{cases} \frac{N+1}{2}, & N \text{ is odd,} \\ \frac{N}{2} + 1, & N \text{ is even.} \end{cases}$$

Thus we can group the summands in Eq. (5.7) as follows:

$$\begin{aligned} \mathcal{D}_* &= \sum_{i=0}^{\bar{N}} \bar{\beta}_i \bar{\mathcal{D}}_i \quad \text{where } \bar{\mathcal{D}}_i := \mathcal{D}_i + \mathcal{D}_{N+1-i}, \\ \bar{\beta}_i &:= \begin{cases} \beta_i, & i = 0, \\ \beta_i, & 0 < i < \bar{N}, \\ \beta_i, & i = \bar{N}, N \text{ is even,} \\ \beta_i/2, & i = \bar{N}, N \text{ is odd.} \end{cases} \end{aligned} \tag{5.8}$$

A moment's reflection shows that the  $\bar{\mathcal{D}}_i$  matrices are real and symmetric. Hence  $\mathcal{D}_*$  is real, symmetric and parametrized in terms of at most  $\kappa(\kappa+1)/2$  parameters; however  $\mathcal{D}_*$  has a kernel of dimension 1 irrespective of the  $\beta_i$ . Hence the number of parameters drops to

$$\frac{\kappa(\kappa-1)}{2}.$$

As argued above the set of all optimal solutions of (5.5) is *concave*. Hence, it follows from a result due to

Carathéodory (see e.g. Goodwin and Payne, 1977) that  $\mathcal{D}_*$  can be expressed as a *convex combination* of at most

$$L := \frac{\kappa(\kappa - 1)}{2} + 1, \quad \kappa = n + m + 2 \quad (5.9)$$

matrices  $\bar{\mathcal{D}}_i$ . In practice the number of measurements is much larger than the order of the system. Thus

$$N, \bar{N} \gg L.$$

There exist therefore  $L$  indices

$$\{i_1, \dots, i_L\} \subset \{0, 1, \dots, \bar{N}\}$$

such that

$$\mathcal{D}_* = \sum_{j=1}^L \gamma_j \bar{\mathcal{D}}_{i_j}, \quad \sum_{j=1}^L \gamma_j = 1, \gamma_j \geq 0. \quad (5.10)$$

A consequence of this result is the existence of an optimal input function  $\mathbf{u}_*$  which is a *multi-sine*:

$$\mathbf{u}_*(t) = \sum_{i=1}^L \bar{\gamma}_i \cos \omega_j t,$$

where for  $N + 1$  even  $\bar{\gamma}_j = 2\gamma_j$ , for all  $j$  such that  $i_j \neq 0$  and  $i_j \neq \bar{N}$ .

**Remark 5.1.** A problem related to Eq. (5.2), i.e. the design of optimal inputs for the purpose of identification, can be found in Chapter 6 of Goodwin and Payne (1977). In this approach, as discussed in Section 2, the information matrix replaces the data covariance matrix and the objective is to find an input function which maximizes the logarithm of its determinant.

## 6. Computation of the stationary points of the eigenvalues of $\mathcal{D}$

The formulation of the non-smooth optimization Problem 5.1 in terms of LMIs provides a way of obtaining the optimal  $\sigma_*$  and a corresponding input  $\mathbf{u}_*$ . In order to obtain more insight into the problem, in particular in order to determine how the solutions might depend *more explicitly* on the system parameters, we compute in this section the stationary and extremal points of the *eigenvalues* of  $\mathcal{D}$  as a function of the  $\beta_i$ . This computation can be performed quite explicitly. There results an upper and a lower bound for the optimal  $\sigma_*$ , which coincides with  $\sigma_*$  in those cases where the maximum occurs at differentiable points, as opposed to an intersection of two eigenvalues.

Consider a system described by the transfer function  $\psi(z) = \mathbf{p}(z)/\mathbf{q}(z)$ ,  $\deg \mathbf{p} = m \leq n = \deg \mathbf{q}$ , and let  $\mu_k$ ,  $k = 0, 1, \dots, N$ , be the  $(N + 1)$ th roots of unity. The functions  $\mathbf{u}$ ,  $\mathbf{y}$  defined by Eq. (4.1) and (4.3), respectively, constitute

a possible (perturbation-free) input–output experiment conducted on the system in question. Let the eigenvalue decomposition of the corresponding data covariance matrix  $\mathcal{D}$  defined by Eq. (4.8), be given by

$$\mathcal{D}(\beta_i) = W(\beta_i)\Lambda(\beta_i)W^*(\beta_i), \quad W(\beta_i) := [w_1 \ w_2 \ \dots],$$

$$\Lambda(\beta_i) := \text{diag}[\lambda_1, \lambda_2, \dots], \quad (6.1)$$

where  $\lambda_1 = 0$ ,  $\lambda_i$ ,  $w_i$  are the eigenvalues, eigenvectors of  $\mathcal{D}$ ,<sup>3</sup> and the dependence on  $\beta_i$  is shown explicitly. It should be noted that the  $w_i$  and of course the  $\lambda_i$  are real with the  $w_i$  orthonormal. We will also assume that the norm of the input signal is 1 (cf. Eq. (4.2)),

$$\sum_{i=0}^N \beta_i = 1. \quad (6.2)$$

The following result holds true.

**Lemma 6.1.** *Let the eigenvalue decomposition of the data covariance matrix  $\mathcal{D}(\beta_i) = 1/(N + 1)\mathcal{V}\mathcal{B}\mathcal{V}^*$ ,  $\mathcal{B} = \text{diag}(\beta_0, \dots, \beta_N)$ , with  $\beta_i \geq 0$ , trace  $\mathcal{B} = 1$ , and  $\mathcal{V}$  as in Eq. (4.5), be given by Eq. (6.1), and let Eq. (6.2) hold. The partial derivative of the eigenvalue  $\lambda_j$  of  $\mathcal{D}$  is given by*

$$\frac{\partial \lambda_j}{\partial \beta_i} = |w_j^* \mathcal{V} e_{i+1}|^2 - |w_j^* \mathcal{V} e_{N+1}|^2, \quad i = 0, 1, \dots, N - 1, \quad (6.3)$$

where  $e_i$  denotes the  $i$ th unit vector in  $\mathbb{R}^{(N+1)}$ .

**Proof.** The eigenvalue decomposition of  $\mathcal{D}$  given by Eq. (6.1) implies  $w_j^* \mathcal{D} = \lambda_j w_j^*$ . Recall that  $\mathcal{D}$  can be expressed as in Eq. (4.8) where due to the normalization (6.2),

$$\mathcal{B} = \text{diag}[\beta_0, \beta_1, \dots, \beta_{N-1}, 1 - \beta_0 - \beta_1 - \dots - \beta_{N-1}].$$

Taking partial derivatives of  $\mathcal{D}$  with respect to the  $\beta_i$ , where  $\mathcal{V}$  is independent of the  $\beta_i$ , we obtain

$$\begin{aligned} \frac{\partial w_j^*}{\partial \beta_i} \mathcal{V} \mathcal{B} \mathcal{V}^* + w_j^* \mathcal{V} [e_{i+1} e_{i+1}^* - e_{N+1} e_{N+1}^*] \mathcal{V}^* \\ = \lambda_j \frac{\partial w_j^*}{\partial \beta_i} + \frac{\partial \lambda_j}{\partial \beta_i} w_j^*. \end{aligned}$$

Upon multiplying the above expression by  $w_j$  on the right, the first terms on both sides become equal, and since  $w_j^* w_j = 1$ , Eq. (6.3) follows.  $\square$

This result leads to the following implicit characterization of stationary points of the singular vectors of  $\mathcal{D}$  as a function of the  $\beta_i$ .

<sup>3</sup> In the sequel, the normalization factor  $1/(N + 1)$  which appears in Eq. (4.8) will be omitted, for simplicity.

**Corollary 6.1.** Under the hypotheses of Lemma 6.1, the stationary points of the eigenvalues  $\lambda_j$  of  $\mathcal{D}$  as functions of the  $\beta_i$ , satisfy

$$|w_j^* \mathcal{V} e_{i+1}|^2 = |w_j^* \mathcal{V} e_{N+1}|^2, \quad i = 0, 1, \dots, N - 1.$$

Furthermore, the corresponding eigenvalue is

$$\lambda_j = |w_j^* \mathcal{V} e_{N+1}|^2.$$

**Proof.** The first part of the corollary is immediate from Eq. (6.3). For the second part, we have  $w_j^* \mathcal{D} = \lambda_j w_j^*$ , and so

$$\begin{aligned} \lambda_j &= w_j^* \mathcal{D} w_j = \sum_{k=1}^{N+1} w_j^* \mathcal{V} e_k \beta_{k-1} e_k^* \mathcal{V}^* w_j \\ &= \sum_{k=1}^{N+1} |w_j^* \mathcal{V} e_k|^2 \beta_{k-1} \\ &= |w_j^* \mathcal{V} e_k|^2 \sum_{k=1}^{N+1} \beta_{k-1} \\ &= |w_j^* \mathcal{V} e_k|^2 = |w_j^* \mathcal{V} e_{N+1}|^2. \end{aligned}$$

The proof of the corollary is thus complete.  $\square$

Below in Section 6.2, we provide an algorithm for finding a solution to the equation in the statement of Corollary 6.1. Evidently, stationary points are characterized by the fact that all the entries of the row vector

$$w_j^* \mathcal{V} \in \mathbb{C}^{1 \times (N+1)}$$

have the same magnitude. This, in turn, leads to a characterization of the eigenvectors corresponding to stationary points.

**Lemma 6.2.** Let  $\mathcal{D}(\beta_i) = (1/(N+1)) \mathcal{V} \mathcal{B} \mathcal{V}^*$  be a data covariance matrix, where  $\mathcal{B} = \text{diag}(\beta_0, \dots, \beta_N)$ , with  $\beta_i \geq 0$ , trace  $\mathcal{B} = 1$ , and  $\mathcal{V}$  as in Eq. (4.5). Let the transfer function of the model leading to  $\mathcal{D}$  be  $\mathbf{p}(z)/\mathbf{q}(z)$ , where the number of measurements  $N$  is larger than  $n+m+2$ . Consider real polynomials  $\mathbf{x}(z)$ ,  $\mathbf{y}(z)$ ,  $\mathbf{r}(z)$  of degree  $m$ ,  $n$ ,  $n+m$ , respectively. The vector  $(\mathbf{x}^*, \mathbf{y}^*)$  composed of the coefficients of  $\mathbf{x}(z)$  and  $\mathbf{y}(z)$  is an eigenvector of  $\mathcal{D}$  corresponding to a stationary point of some eigenvalue of  $\mathcal{D}$  as a function of the  $\beta_i$  if, and only if,

$$\mathbf{x}(z) + \mathbf{y}(z) \frac{\mathbf{p}(z)}{\mathbf{q}(z)} =: \kappa \frac{\mathbf{r}(z)}{\mathbf{q}(z)}, \quad (6.4)$$

where  $\mathbf{r}(z)/\mathbf{q}(z)$  is all-pass with respect to the unit circle and  $\kappa$  is a real constant.

**Proof.** From the definition (4.5) of  $\mathcal{V}$  follows that the  $(i+1)$ th entry of the row vector  $(\mathbf{x}^*, \mathbf{y}^*) \mathcal{V}$  is

$$(\mathbf{x}^* \ \mathbf{y}^*) \mathcal{V} e_{i+1} = \mathbf{x}(\mu_j) + \mathbf{y}(\mu_j) \frac{\mathbf{p}(\mu_j)}{\mathbf{q}(\mu_j)}.$$

Obviously, if the rational function

$$\mathbf{x}(z) + \mathbf{y}(z) \frac{\mathbf{p}(z)}{\mathbf{q}(z)}$$

is a scaled all pass, the entries of  $(\mathbf{x}^*, \mathbf{y}^*) \mathcal{V}$  will have the same magnitude. Conversely, if the rational function  $\mathbf{x}(z) + \mathbf{y}(z)(\mathbf{p}(z)/\mathbf{q}(z))$  has the same magnitude at a sufficiently large but finite number of values  $N$  on the unit circle, it must be an all pass. Taking degrees into account  $N \geq n+m+2$ .  $\square$

**Remark 6.1.** (1) Given the (real) polynomial  $\mathbf{q}(z)$  of degree  $n$  there are *finitely many* (real) polynomials  $\mathbf{r}(z)$  of degree  $n+m$  such that  $\mathbf{r}(z)/\mathbf{q}(z)$  is all-pass. They can be written down explicitly (see Example following Corollary 6.2).

(2) For each of the possible selections of  $\mathbf{r}(z)$ , Eq. (6.4) is equivalent to the *Bezout equation*

$$\mathbf{q}(z)\mathbf{x}(z) + \mathbf{p}(z)\mathbf{y}(z) = \kappa \mathbf{r}(z), \quad (6.5)$$

where the polynomials  $\mathbf{q}(z)$ ,  $\mathbf{p}(z)$ ,  $\mathbf{r}(z)$  of degrees  $n$ ,  $m$ ,  $n+m$  respectively, are given and the polynomials  $\mathbf{x}(z)$ ,  $\mathbf{y}(z)$  of degrees  $m$ ,  $n$  respectively, are to be determined. In addition, the normalization condition

$$\mathbf{x}^* \mathbf{x} + \mathbf{y}^* \mathbf{y} = 1 \quad (6.6)$$

has to be satisfied. Consequently, we need to solve a *normalized Bezout equation*.

### 6.1. The solution of the normalized Bezout equation

Let  $\mathbf{y}_0(z)$ ,  $\mathbf{x}_0(z)$  be a particular solution pair of the Bezout equation (6.5) with  $\kappa = 1$  and satisfying the degree constraints. The set of all solutions of Eq. (6.5) satisfying the same degree constraints is then given as follows:

$$\mathbf{x}(z) = \kappa \mathbf{x}_0(z) + k \mathbf{p}(z), \quad \mathbf{y}(z) = \kappa \mathbf{y}_0(z) - k \mathbf{q}(z), \quad k \in \mathbb{R}.$$

We have to identify the constants  $\kappa$  and  $k$ . This can be done by using two facts:

- (a) because  $(\mathbf{x}^*, \mathbf{y}^*)$  is an eigenvector of  $\mathcal{D}$  not corresponding to a zero eigenvalue, while  $(\mathbf{p}^*, -\mathbf{q}^*)$ ,<sup>4</sup> is an eigenvector corresponding to the zero eigenvalue,  $(\mathbf{x}^*, -\mathbf{y}^*)$  must be orthogonal to  $(\mathbf{p}^*, -\mathbf{q}^*)$ .
- (b) the normalization constraint (6.6) is satisfied.

<sup>4</sup> Recall notation (3.4): the vector of coefficients of the polynomial  $\mathbf{a}(z)$  is denoted by  $\mathbf{a} = (a_0 \ a_1 \ \dots)^*$ .

The first constraint immediately implies

$$k = \frac{\mathbf{y}_0^* \mathbf{q} - \mathbf{x}_0^* \mathbf{p}}{\mathbf{p}^* \mathbf{p} + \mathbf{q}^* \mathbf{q}} \kappa.$$

The normalization constraint gives

$$\begin{aligned} \mathbf{x}^* \mathbf{x} + \mathbf{y}^* \mathbf{y} &= k^2 (\mathbf{p}^* \mathbf{p} + \mathbf{q}^* \mathbf{q}) + 2(\mathbf{x}_0^* \mathbf{p} - \mathbf{y}_0^* \mathbf{q}) k \kappa \\ &+ (\mathbf{x}_0^* \mathbf{x}_0 + \mathbf{y}_0^* \mathbf{y}_0) \kappa^2 = 1. \end{aligned}$$

Substituting the expression for  $k$  in the above equation we obtain the following value for  $\kappa$ :

$$\kappa_*^2 = \frac{\mathbf{p}^* \mathbf{p} + \mathbf{q}^* \mathbf{q}}{(\mathbf{p}^* \mathbf{p} + \mathbf{q}^* \mathbf{q})(\mathbf{x}_0^* \mathbf{x}_0 + \mathbf{y}_0^* \mathbf{y}_0) - (\mathbf{x}_0^* \mathbf{p} - \mathbf{y}_0^* \mathbf{q})^2}. \quad (6.7)$$

The above considerations imply that (once  $\mathbf{r}$  is selected) there is a *unique* solution of the normalized Bezout equations (6.5) and (6.6). We have thus proved the following.

**Theorem 6.1.** *Given the system described by the transfer function  $\psi(z) = \mathbf{p}(z)/\mathbf{q}(z)$ , consider the data covariance matrix  $\mathcal{D}$  defined by Eq. (4.8); for a fixed  $\psi(z)$  regard  $\mathcal{D}$  as a function of the input power spectrum, i.e. a function of the  $\beta_j \geq 0$ ,  $\sum \beta_j = 1$ . A non-zero eigenvalue  $\lambda_j$  of  $\mathcal{D}$  is stationary with respect to the variables  $\beta_j$  if, and only if, there exists a polynomial  $\mathbf{r}(z)$  of degree at most the sum of the degrees of  $\mathbf{p}(z)$  and  $\mathbf{q}(z)$ , such that  $\mathbf{r}(z)/\mathbf{q}(z)$  is all-pass with respect to the unit circle and  $\lambda_j = \kappa_*^2$  given by Eq. (6.7). The vectors  $\mathbf{x}_0$ ,  $\mathbf{y}_0$  in Eq. (6.7) are formed from the coefficients of polynomials  $\mathbf{x}_0(z)$ ,  $\mathbf{y}_0(z)$  of degree  $m$ ,  $n$  respectively, where  $\mathbf{q}(z)\mathbf{x}_0(z) + \mathbf{p}(z)\mathbf{y}_0(z) = \mathbf{r}(z)$ .*

Thus, the stationary eigenvalues given by formula (6.7) are parametrized by means of the elements of the finite set of restricted degree polynomials  $\mathbf{r}(z)$  which have the property that  $\mathbf{r}(z)/\mathbf{q}(z)$  is all-pass (see Example following Corollary 6.2).

## 6.2. An algorithm for computing optimal input spectra

As promised, we now take up the task of solving the equation of corollary 6.1. From Lemma 6.1 follows that the change  $\delta\lambda_2$  of  $\lambda_2$ , as a function of the change  $\delta\beta_i$  of  $\beta_i$ , satisfies

$$\delta\lambda_2 = \sum_{i=1}^N \frac{\partial\lambda_2}{\partial\beta_i} \delta\beta_i = \sum_{i=0}^{N-1} (|w_2^* \mathcal{V} e_{i+1}|^2 - |w_2^* \mathcal{V} e_{N+1}|^2) \delta\beta_i.$$

The above result suggests the optimization *algorithm* described below. Define

$$I := \max_i \{ |w_2^* \mathcal{V} e_i|^2 : \beta_i < 1 \},$$

$$K := \min_i \{ |w_2^* \mathcal{V} e_i|^2 : \beta_i > 0 \}. \quad (6.8)$$

The rule for changing the  $\beta_i$  in order to maximize  $\lambda_2$  is

$$\left. \begin{aligned} \beta_I &\rightarrow \beta_I + \varepsilon, \quad \varepsilon > 0 \\ \beta_K &\rightarrow \beta_K - \varepsilon \\ \beta_i &\rightarrow \beta_i, \quad i \neq I, K \end{aligned} \right\} \Rightarrow \delta\lambda_2 = \varepsilon (|w_2^* \mathcal{V} e_I|^2 - |w_2^* \mathcal{V} e_K|^2). \quad (6.9)$$

**Remark.** Let the SVD of  $\mathcal{R}$  be  $\mathcal{R} = \mathcal{V}A = W\Sigma V^*$ , whereby the normalizing factor  $1/(N+1)$  has been omitted for simplicity; hence  $W^* \mathcal{V}A = \Sigma V^*$  and consequently  $w_2^* \mathcal{V}A e_i = \sigma_2 v_2^* e_i$ . Thus assuming that  $A e_i = \alpha_i e_i \neq 0$  we have

$$w_2^* \mathcal{V} e_i = \sigma_2 \frac{v_{i,2}}{\alpha_i},$$

where  $v_{i,2}$  denotes the  $i$ th entry of  $v_2$  (the right singular vector of  $\mathcal{R}$  corresponding to the singular value  $\sigma_2$ ). Hence substituting in Eq. (6.9) we obtain the percentage change of  $\lambda_2$ :

$$\frac{\delta\lambda_2}{\lambda_2} = \varepsilon \left( \left| \frac{v_{I,2}}{\alpha_I} \right|^2 - \left| \frac{v_{K,2}}{\alpha_K} \right|^2 \right).$$

## Algorithm

- Preliminaries: pick system.
- Initial choice of  $\beta_i$ ,  $i = 1, \dots, N$ .
- Compute svd of  $\mathcal{D}$ .
- Compute  $\partial\lambda_2/\partial\beta_i$  according to Eq. (6.3).
- Determine  $I, K$  according to Eq. (6.8).
- Update  $\beta_i$  according to Eq. (6.9) with small  $\varepsilon$ .
- Compute change and percentage change of  $\lambda_2$  using Eq. (6.9).
- Repeat procedure with new values of  $\beta_i$ .
- Stop when  $\beta_i$  do not change significantly.

The *algorithm* described above, makes sure that for an arbitrary  $\mathcal{B}$  the first  $m$  diagonal entries of  $\mathcal{D}$  remain equal to 1 (normalization of the input spectra); the following  $n$  entries are equal to, say  $a$ , which depends on the choice of  $\mathcal{B}$ . Furthermore (in contrast to the LMI method) this algorithm may converge to a local extremum of  $\lambda_2$ .

## 6.3. AR systems

For the special case of AR systems (i.e. systems with no finite zeros)  $\mathbf{p}(z) = 1$  and  $\mathbf{x}(z)$ ,  $\mathbf{x}_0(z)$  are simply constants and formula (6.7) is simplified. With some manipulation, one obtains

**Corollary 6.2** (AR systems). *Consider a system given by the transfer function  $\psi(z) = 1/\mathbf{q}(z)$  where  $\mathbf{q}(z) = q_0 + \dots + q_n z^n$ ,  $q_n \neq 0$ . The extrema of the eigenvalues of the data covariance matrix  $\mathcal{D}$  with respect to the (normalized) spectrum of the input function, are given by*

$$\kappa_*^2 = \frac{1 + \mathbf{q}^* \mathbf{q}}{\mathbf{q}^* \mathbf{q} (1 + \mathbf{q}^* \mathbf{q}) - (\mathbf{q}^* \mathbf{r})^2}, \quad (6.10)$$

where  $\mathbf{q} := [q_0 \ \dots \ q_n]^*$ , and  $\mathbf{r} := [r_0 \ \dots \ r_n]^*$  is such that the associated rational function  $\mathbf{r}(z)/\mathbf{q}(z)$  is all-pass.

**Proof.** The equation  $\mathbf{q}(z)\mathbf{x}_0(z) + \mathbf{p}(z)\mathbf{y}_0(z) = \mathbf{r}(z)$  is satisfied by  $\mathbf{x}_0 = 0$  and  $\mathbf{y}_0 = \mathbf{r}$ . Further, the all-pass condition in this case implies  $\mathbf{r}(z)\mathbf{r}(z^{-1}) = \mathbf{q}(z)\mathbf{q}(z^{-1})$ , whence  $\mathbf{r}^*\mathbf{r} = \mathbf{q}^*\mathbf{q}$ , and Eq. (6.10) follows.  $\square$

**Example.** Consider the AR system given by  $\psi(z) = 1/\mathbf{q}(z)$ , where  $\mathbf{q}(z) := (z - 1/2)(z + 1/4)$ ; the resulting vector of coefficients is:  $\mathbf{q} = (-1/8, -1/4, 1)^*$ . There are exactly four choices of the second degree polynomial  $\mathbf{r}(z)$  which result in an all-pass  $\mathbf{r}(z)/\mathbf{q}(z)$ :

$$\mathbf{r}_1(z) = \mathbf{q}(z) \Rightarrow \mathbf{r}_1 = \begin{pmatrix} -\frac{1}{8} & -\frac{1}{4} & 1 \end{pmatrix}^* \Rightarrow \frac{\mathbf{r}_1(z)}{\mathbf{q}(z)} = 1$$

$$\Rightarrow \mathbf{q}^*\mathbf{r}_1 = \mathbf{q}^*\mathbf{q} = \frac{69}{64},$$

$$\mathbf{r}_2(z) = \left(\frac{z}{2} - 1\right)\left(z + \frac{1}{4}\right) \Rightarrow$$

$$\mathbf{r}_2 = \begin{pmatrix} -\frac{1}{4} & -\frac{7}{8} & 1 \end{pmatrix}^* \Rightarrow \frac{\mathbf{r}_2(z)}{\mathbf{q}(z)} = \frac{\frac{z}{2} - 1}{z - \frac{1}{2}} \Rightarrow \mathbf{q}^*\mathbf{r}_2 = \frac{3}{4},$$

$$\mathbf{r}_3(z) = \left(\frac{z}{2} - 1\right)\left(\frac{z}{4} + 1\right) \Rightarrow$$

$$\mathbf{r}_3 = \begin{pmatrix} -1 & \frac{1}{4} & 1 \end{pmatrix}^* \Rightarrow \frac{\mathbf{r}_3(z)}{\mathbf{q}(z)} = \frac{(\frac{z}{4} + 1)(\frac{z}{2} - 1)}{(z + \frac{1}{4})(z - \frac{1}{2})}$$

$$\Rightarrow \mathbf{q}^*\mathbf{r}_3 = \frac{3}{16},$$

$$\mathbf{r}_4(z) = \left(z - \frac{1}{2}\right)\left(\frac{z}{4} + 1\right) \Rightarrow$$

$$\mathbf{r}_4 = \begin{pmatrix} -\frac{1}{2} & \frac{7}{8} & 1 \end{pmatrix}^* \Rightarrow \frac{\mathbf{r}_4(z)}{\mathbf{q}(z)} = \frac{\frac{z}{4} + 1}{z + \frac{1}{4}} \Rightarrow \mathbf{q}^*\mathbf{r}_4 = \frac{3}{32}.$$

It follows that the corresponding extremal values of  $\kappa$  are

$$\kappa_*^2(1) = \frac{1}{\frac{69}{64} - \frac{64}{133}(\frac{69}{64})^2} = \frac{8512}{4416} = 1 + \frac{64}{69},$$

$$\kappa_*^2(2) = \frac{1}{\frac{69}{64} - \frac{64}{133}(\frac{3}{4})^2} = \frac{8512}{6873},$$

$$\kappa_*^3(3) = \frac{1}{\frac{69}{64} - \frac{64}{133}(\frac{3}{16})^2} = \frac{8512}{9033},$$

$$\kappa_*^2(4) = \frac{1}{\frac{69}{64} - \frac{64}{133}(\frac{3}{32})^2} = \frac{8512}{9141}.$$

Recall that the data covariance matrix  $\mathcal{D}$  in this case has size  $4 \times 4$  and one zero eigenvalue. The first value of  $\kappa_*$  is the minimum of the biggest eigenvalue of  $\mathcal{D}$ . For the particular system considered, the maximum  $\sigma_*^2$  of the second-smallest eigenvalue of  $\mathcal{D}$  (or the smallest non-zero eigenvalue) satisfies

$$\sigma_*^2 = \kappa_*^2(4) < \kappa_*^2(3)$$

that is  $\sigma_*^2$  is equal to  $\kappa_*^2(4)$ .

### 6.3.1. Numerical experiments

For second-order AR systems the values of the optimal  $\sigma_*$  were computed using the LMI formulation (LMI-Tool); They were compared with the values of  $\kappa_*(i)$ ,  $i = 1, \dots, 4$ , as a function of the pole locations. These experiments show that

$$\sigma_*^2 \leq \min_{i=2,3,4} \kappa_*^2(i) \leq \kappa_*^2(1).$$

The inequalities hold for all second-order systems with stable poles. The question which is still open, is whether such inequalities hold for stable AR systems of arbitrary order, and more generally for ARMA systems.

In the four plots of Fig. 1 the  $x$ - and  $y$ -axes show the values of the two poles; the  $z$ -axis shows the difference between the  $\lambda_*$  and  $\kappa_*(i)$ ,  $i = 2, 3, 4$ . The fourth is a binary plot of the difference between the minimum among  $\kappa_*(i)$ ,  $i = 2, 3, 4$  and  $\lambda_*$ ; the maximum of this plot (which is 0.0052) is equal to the actual maximum of this difference. Thus the optimal value of  $\lambda_*$  for second-order AR systems can be obtained exactly for the great majority of (stable) pole locations, by computing the value of  $\kappa_*(i)$ ,  $i = 2, 3, 4$ . The importance of the fourth plot of Fig. 1 (lower right) is that the regions where the maxima of  $\sigma_*$  are differentiable from the regions where they are not clearly shown. It has actually been possible to explicitly compute in terms of the system parameters, the equations defining the boundary of the two regions; results along these lines are reported in Antoulas and Astolfi (1998).

## 7. The case of first-order systems

The purpose of this section is to explicitly determine the value of the frequencies of multi-sine inputs which define the best inputs for identification purposes, according to the criterion defined in Section 5.1, i.e. the maximization of the second-smallest singular value  $\sigma_*$  of the data covariance matrix  $\mathcal{D}$ .

Given the first-order system described by the transfer function

$$\psi(z) = \frac{b}{z + a} \tag{7.1}$$

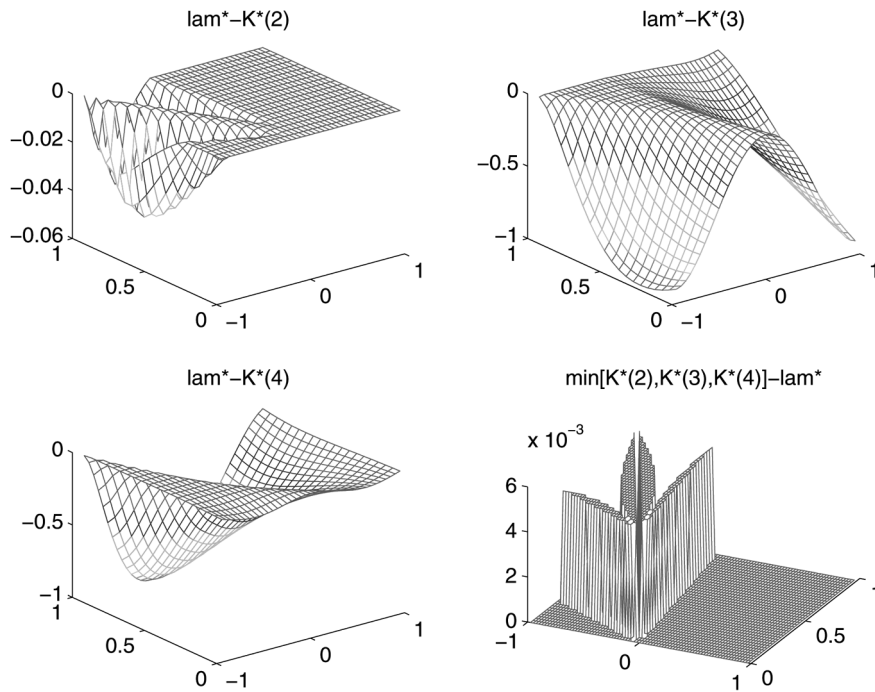


Fig. 1. The difference  $\lambda_*$  minus  $\kappa_*(i)$ ,  $i = 2, 3, 4$ , (first 3 plots), and  $\lambda_*$  minus the minimum among  $\kappa_*(i)$ ,  $i = 2, 3, 4$ , (last plot).

according to Eq. (6.7)  $\kappa$  has two different values, namely

$$\kappa_1^2 = \frac{b^2(1 + a^2 + b^2)}{(1 - a^2)^2 + b^2(1 + a^2)}, \quad \kappa_2^2 = 1 + \frac{b^2}{1 + a^2}. \quad (7.2)$$

Since for any  $b$  and  $|a| < 1$ ,

$$\kappa_1 < \kappa_2$$

it follows that the second-smallest eigenvalue of the data covariance matrix is a differentiable function of the input, and its maximum is precisely  $\kappa_1$ . Furthermore, the minimum of the largest eigenvalue is equal to  $\kappa_2$  (though we make no use of this fact).

### 7.1. One-sine input

In this subsection we consider the input function

$$\mathbf{u}(t) = \sqrt{2} \cos \omega_0 t \quad (7.3)$$

having unit power. The question we wish to answer in this section is: *does there exist a frequency  $\omega_0$  so that the second-smallest eigenvalue of the data covariance matrix  $\mathcal{D}$  is equal to  $\kappa_1$  given above?* As we show below the answer is, yes.

Let  $\phi(t)$  be the regression vector

$$\phi(t) := \begin{pmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) \\ \sigma \mathbf{y}(t) \end{pmatrix}.$$

The data covariance matrix for an infinite number of measurements can be expressed as follows:

$$\mathcal{D} := \lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \sum_{t=0}^{T-1} \phi(t) \phi(t)^* \right\} \in \mathbb{R}^{3 \times 3}. \quad (7.4)$$

The entries of  $\mathcal{D}$  are computed in Appendix A. Recall that  $\mathcal{D}$  has a zero eigenvalue; let  $\mathcal{D}_1$  be the  $2 \times 2$  matrix obtained from  $\mathcal{D}$  by means of a unitary transformation as in Eq. (5.3); hence  $\mathcal{D}$  and  $\mathcal{D}_1$  have the same non-zero eigenvalues. The following two quantities will be optimized:

$$\begin{aligned} \lambda_*(\mathcal{D}) = \lambda_{\min}(\mathcal{D}_1) & \quad \text{maximized,} \\ \Pi(\mathcal{D}) := \lambda_{\max}(\mathcal{D}) \lambda_*(\mathcal{D}) = \det(\mathcal{D}_1) & \quad \text{minimized.} \end{aligned} \quad (7.5)$$

Besides the first quantity which is of primary interest, for comparison purposes we also examine the optimization of the second quantity; this latter quantity was treated in Goodwin and Payne (1977) and Kosut et al. (1985). The optimization results, which were obtained in part using Mathematica, are tabulated in Table 1.

Fig. 2 shows numerical examples. Every sub-figure displays 10 curves, each one corresponding to a specific value of  $a$  and  $b$ ; in particular:  $b = 1$  and  $a$  varies from 0.9 to 0.0 in steps of 0.1. The horizontal axis represents frequency  $\omega$  over the interval  $[0, \pi/2]$  in both figures; thus 50 corresponds to  $\omega = \pi/4$ .

**Remark 7.1.** (1) The maximum of  $\sigma_*$  attained using a single sinusoid is equal to the overall maximum

Table 1

| Quantity optimized                    | Optimum $\cos(\omega_0)$ | $\lambda_*(\mathcal{D})$   | $\Pi(\mathcal{D})$       |
|---------------------------------------|--------------------------|--|--------------------------|
| $\lambda_*(\mathcal{D})$<br>maximized | $\frac{-2a}{1+a^2+b^2}$  | $\frac{b^2(1+a^2+b^2)}{(1-a^2)^2+b^2(1+a^2)}$  | $\lambda_*(1+\lambda_*)$ |
| $\Pi(\mathcal{D})$<br>minimized       | $\frac{-2a}{1+a^2}$      | $\frac{1}{2} + \kappa - \sqrt{\frac{1}{4} + \frac{4a^2}{(1+a^2)^2} \kappa^2}$ , where $\kappa := \frac{b^2(1+a^2)}{(1+a^2)^2}$ | $b^2(1+a^2)(1+a^2+b^2)$  |

(cf. Eq. (7.2) with the value in the  $\lambda_*$  column of Table 1). The largest number of sinusoids  $L$  predicted by Carathéodory's convexity theorem, and given by Eq. (5.9), is 2. Here it is shown that **one** sinusoid is enough to optimize  $\lambda_*(\mathcal{D})$ .

(2) Optimizing  $\lambda_*$  and  $\Pi$  yields different optimal signals. The sensitivity of the maximal  $\lambda_*$  design is better than the sensitivity of the minimal  $\Pi$  design. The flatness of the curve justifies the contention in Section 2 that one can intelligently choose inputs (even if they are improperly described as optimal) when there is only rough knowledge of the system.

(3) The  $\cos(\omega_0)$  column of Table 1 reveals that the optimization of the determinant (corresponding to the optimization of the Fisher information matrix in Goodwin and Payne (1977) yields an optimal frequency which does *not* depend on the gain  $b$ , in contrast to the design based on the optimization of  $\lambda_*$  (see also Mareels et al., 1987). In particular, as the pole  $a$  tends to 0, the optimal inputs in both cases coincide. However, as  $a$  tends to  $\pm 1$ , the optimal  $\Pi$  design gives  $\lambda_*^2 = b^2/2$ , while for the optimal  $\lambda_*$  design we have  $\lambda_*^2 = 1 + b^2/2$ ; it follows that the difference between these two designs for a pole close to the stability boundary, is most pronounced for small gains.

(4) Assume that  $a = -1 + \varepsilon$  where  $\varepsilon$  is small. The appropriate 3 db frequency is then approximately  $\varepsilon$  radians per second. The determinant minimization gives  $\omega_{\text{opt}} \simeq \varepsilon$  while maximization of  $\lambda_*$  leads to a somewhat larger value, depending on  $b^2$ .

## 7.2. Multi-sine inputs

In this subsection we will consider the same first-order system as in the previous subsection, namely the one described by the transfer function  $\psi(z) = b/(z+a)$ . Here however we will consider multi-sine inputs. First, let the input be a convex combination of two sinusoids of frequencies  $\omega_1$  and  $\omega_2$ :

$$\mathbf{u}(t) = \alpha\sqrt{2}\cos\omega_1 t + (1-\alpha)\sqrt{2}\cos\omega_2 t, \quad 0 \leq \alpha \leq 1.$$

It follows that the corresponding data covariance matrix  $\mathcal{D}$  can be written as a convex combination of the data

covariance matrices  $\mathcal{D}_1, \mathcal{D}_2$  which correspond to the two sinusoids:

$$\mathcal{D} = \alpha\mathcal{D}_1 + (1-\alpha)\mathcal{D}_2, \quad (7.6)$$

where  $\mathcal{D}_i, i = 1, 2$ , is defined as in Appendix A (Eq. (A.1)), with  $\omega_0$  replaced by  $\omega_i, i = 1, 2$ . From the calculation in Appendix A, it follows that for one sinusoid ( $\alpha = 1$ )  $\mathcal{D}$  can be written for  $a \neq 0$  as a sum

$$\mathcal{D} = M_1 + yM_2, \quad y := \frac{1}{a^2 + 2ax + 1}$$

and  $M_1, M_2$  are  $3 \times 3$  matrices which *do not* depend on the variable  $x$ :

$$M_1 := \begin{pmatrix} 1 & \frac{b}{2a} & \frac{b}{2} \\ \frac{b}{2a} & 0 & \frac{b^2}{2a} \\ \frac{b}{2} & \frac{b^2}{2a} & 0 \end{pmatrix},$$

$$M_2 := \begin{pmatrix} 0 & \frac{b(a^2-1)}{2a} & \frac{b(1-a^2)}{2} \\ \frac{b(a^2-1)}{2a} & b^2 & -\frac{b^2(a^2+1)}{2a} \\ \frac{b(1-a^2)}{2} & -\frac{b^2(a^2+1)}{2a} & b^2 \end{pmatrix}, \quad a \neq 0.$$

Thus  $\mathcal{D}$  corresponding to one sinusoid is *affine* in the variable  $y$ .<sup>5</sup> It follows from Eq. (7.6) that  $\mathcal{D}$  corresponding to two sinusoids is convex in the new variable  $\bar{y}$ , which is a convex combination of  $y_1$  and  $y_2$ :

$$\mathcal{D} = M_1 + \underbrace{(\alpha y_1 + (1-\alpha)y_2)}_{:= \bar{y}} M_2.$$

Hence  $\mathcal{D}$  is affine in the new variable  $\bar{y}$ . By the results of the previous subsection we can compute the optimal  $\bar{y}_*$ :

$$\bar{y}_* = \frac{1}{a^2 + 2ax_* + 1} \quad \text{where } x_* = \frac{-2a}{a^2 + b^2 + 1}.$$

Since  $x$  is a cosine, its values are between  $\pm 1$ ; hence  $y$  satisfies

$$y_{\min} := \frac{1}{(1+|a|)^2} \leq y \leq \frac{1}{(1-|a|)^2} =: y_{\max}.$$

<sup>5</sup> For  $a = 0$ ,  $\mathcal{D}_i$  is convex in the variable  $x$ .

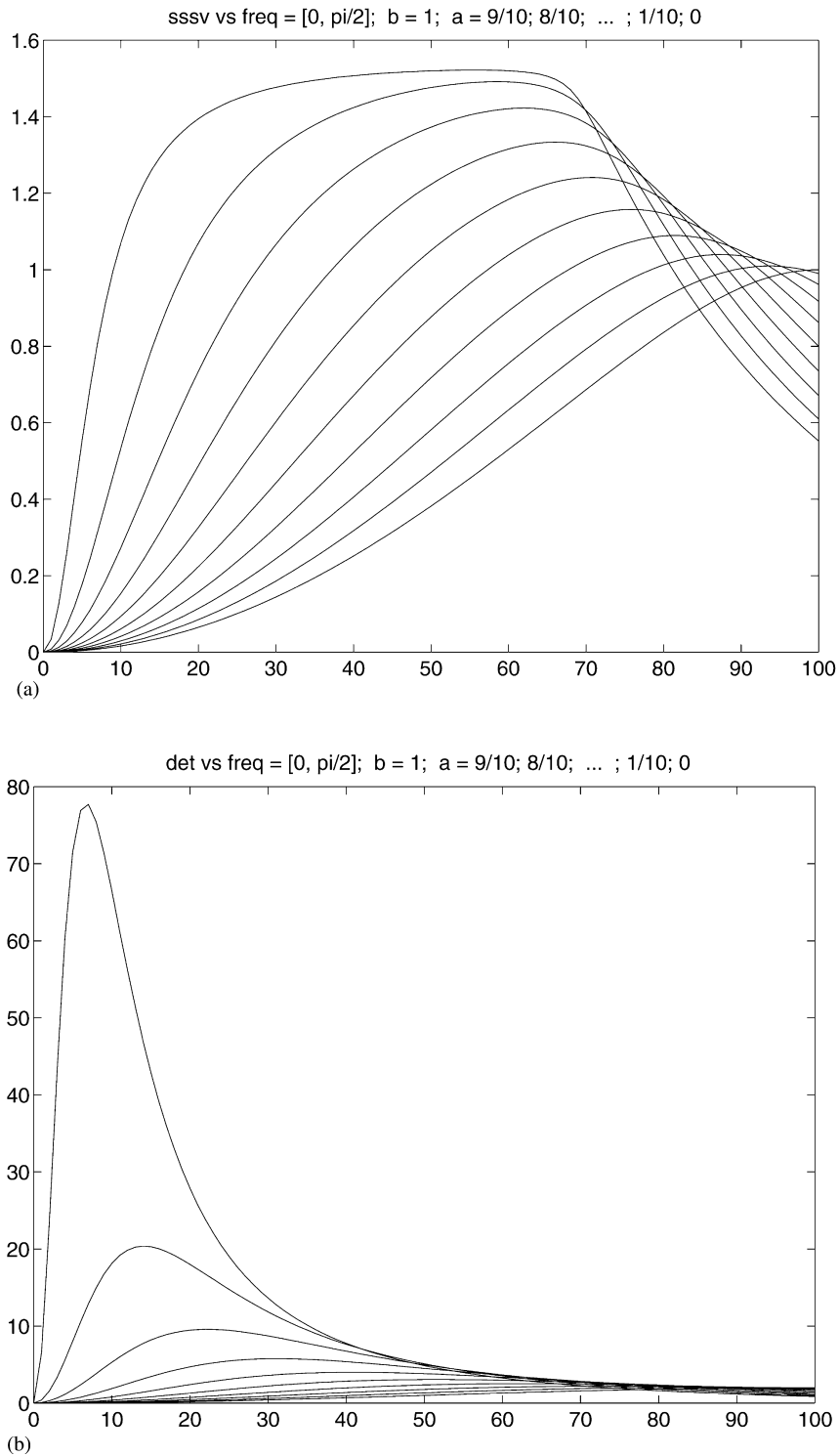


Fig. 2. Second-smallest eigenvalue  $\sigma_* = \lambda_*(\mathcal{D})$  versus frequency  $\in [0, \pi/2]$  (left) — Product of non-zero singular values versus frequency  $\in [0, \pi/2]$  (right).

The following thus holds true: either

$$y_{\min} \leq y_1 \leq \bar{y}_* \Leftrightarrow \bar{y}_* \leq y_2 \leq y_{\max}$$

or  $y_{\min} \leq y_2 \leq \bar{y}_* \Leftrightarrow \bar{y}_* \leq y_1 \leq y_{\max}$ .

This result is depicted in Fig. 3. The shaded regions show the allowable pairs of  $y_1, y_2$ . For any point in the interior of the shaded regions and for some points on the boundary, there is an  $\alpha \in [0, 1]$  such that  $\alpha y_1 + (1 - \alpha)y_2 = \bar{y}_*$ . Then an optimal input is defined



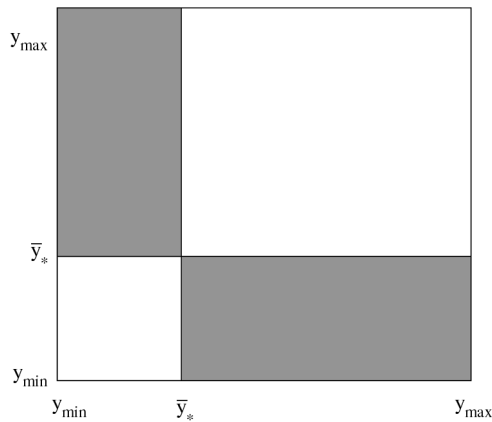


Fig. 3. Two sinusoids.

by the frequencies  $\omega_i$ , where  $y_i = 1/(a^2 + 2a \cos \omega_i + 1)$ , and powers  $\alpha$  and  $1 - \alpha$ . This result can be generalized: if the input signal is a convex combination of more than 2 sinusoids, at least one will have frequency with cosine below  $x_*$  and at least one will have frequency with cosine above  $x_*$ .

### 7.3. A simulation

To illustrate the identification method proposed above we have simulated a first-order discrete-time, linear system with transfer function  $\psi(z) = b/(z + a)$ . The various parameters were chosen as follows:

- *Nominal parameters:* pole  $a = -9/10$ , gain  $b = 1/10$ .

- *Number of experiments:* 600, length of each experiment: 5000 samples.
- *Input:* two input functions were used, namely a random input  $\mathbf{u}_1$ , and a cosine  $\mathbf{u}_2$  as given by Eq. (7.3). 600 experiments using  $\mathbf{u}_1$  were performed, followed by 600 experiments using  $\mathbf{u}_2$ . The frequency  $\omega_0$  of  $\mathbf{u}_2$  is chosen in a different way for each of the 600 experiments, namely, so that  $\lambda_*(\mathcal{D})$  given in Eq. (7.5) is maximized at each step.
- *Noise:* the output of the system was corrupted by additive white noise of standard deviation  $1/5$ .
- *Initial experiment.* The initial  $\mathbf{u}_2$  is chosen as in Eq. (7.3), where we arbitrarily choose  $\omega_0 = 1$  rad ( $57.29^\circ$ ).
- *Identification procedure.* The procedure used is the recursive one outlined in Section 3.3. The value of  $\varepsilon$  is the estimate of the standard deviation of the noise sequence, which can be found in any classical text on system identification (see Zhang and Antoulas, 1998, and references therein). After each experiment, a nominal model and an uncertainty set (ellipse) were identified. The coefficients of the nominal model were obtained from the left singular vector of the data matrix, which corresponds to the smallest singular value. The ellipse was computed as indicated in Section 3.2; for details see Zhang and Antoulas (1998).

The results of this simulation are summarized in Figs. 4 and 5. The former figure shows some of the ellipses obtained from the 600 experiments (every 50th); the dashed ellipses are obtained using  $\mathbf{u}_1$ , while the

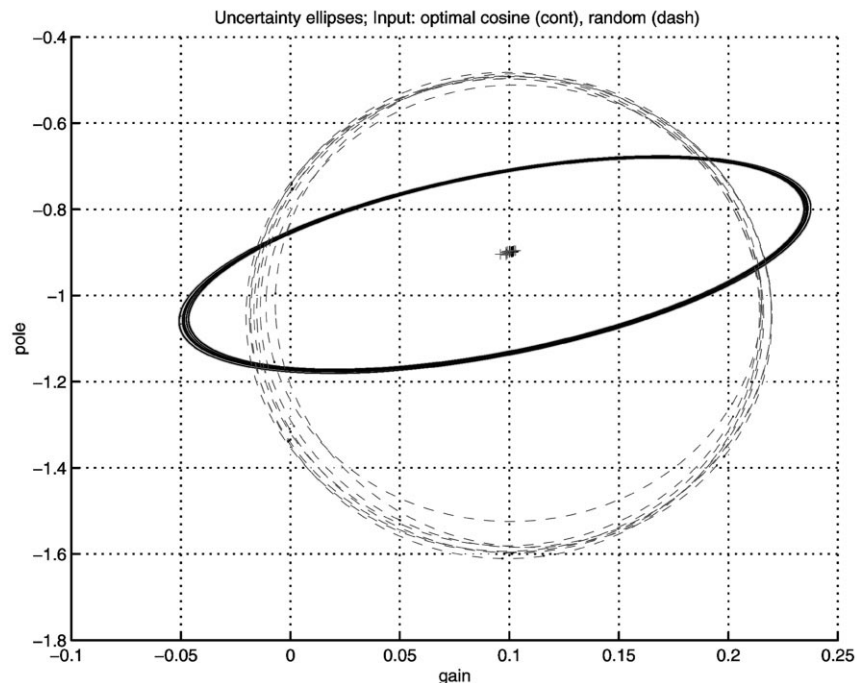


Fig. 4. Uncertainty ellipses.

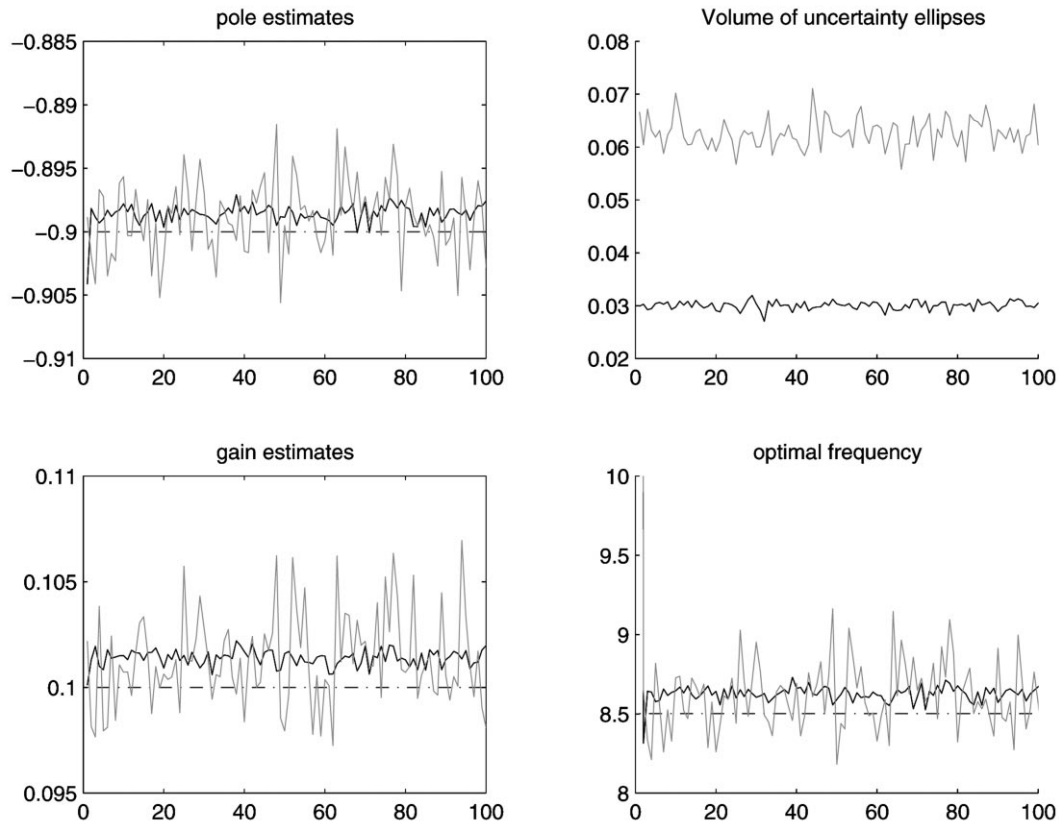


Fig. 5. Poles and gains of identified nominal models; volume of ellipses; estimated optimal frequency.

continuous ones are obtained by means of  $\mathbf{u}_2$ . The crosses indicate the identified nominal models at the corresponding steps.

Fig. 5 shows the pole estimates, that is the poles of the identified nominal models (upper left-hand plot); the gain estimates, that is the gains of the identified nominal models (lower left-hand plot); the area of the ellipses (upper right-hand plot), and the identified optimal frequency  $\omega_0$  (lower right-hand plot). The dash-dot lines in these figures show the true value of the parameters. For clarity, only the result of the first 100 experiments are shown.

The *conclusions* which can be drawn are:

- Fig. 4. The ellipses obtained using random inputs  $\mathbf{u}_1$  have larger area than those obtained using  $\mathbf{u}_2$ ; consequently the uncertain sets obtained using random inputs contain more plants. Furthermore the variance of the ellipses obtained using random inputs is larger than that of the ellipses obtained using the recursively optimized inputs  $\mathbf{u}_2$ . Thus, it seems reasonable to measure *convergence* in this framework, in terms of the variance of the uncertainty ellipses.
- Fig. 5. The two plots on the left-hand side show the pole and gain estimates (i.e. the poles and gains of the identified nominal models — the crosses in Fig. 4),

together with the true values. Again as mentioned earlier, the variance of the estimates using random inputs (dashed curves) is larger than that of the estimates obtained using the optimal inputs. We also notice that the estimates for both sets of inputs have a bias, which is due to the fact that only the output is corrupted with noise (the input is constructed at each step, and hence is known exactly).

On the right-hand side, the upper plot depicts the volume of the uncertainty ellipses; this plot provides a quantification of the observations made in connection with Fig. 4. In particular, it shows that the ellipses obtained using  $\mathbf{u}_1$  (dashed curve) are about double the size of the ones obtained using  $\mathbf{u}_2$  (continuous curve).

Finally, the lower plot on the right-hand side depicts the estimated optimal frequency (in deg) using both  $\mathbf{u}_1$  (dashed curve) and  $\mathbf{u}_2$  (continuous curve); it also shows the optimal frequency based on the true parameters (dash-dot line). The same observations as before, concerning the variance and the bias of the two curves can be made. Furthermore, it is worthwhile noting that the frequency whose initial value was chosen  $57.29^\circ$ , *converges rapidly* — after a few steps — to the true value ( $8.5^\circ$ ) and thereafter stays in its neighborhood.

The above simulation shows that the recursive identification scheme for robust control with successively optimized inputs proposed in this paper, has desirable features, and in the example chosen above compares favorably with the same recursive identification scheme where instead of optimizing the choice of inputs at each step, a (different at each step) classical persistently exciting random input signal is used throughout.

**Remark 7.2** (*The colored noise case*). The above experiment can also be conducted using colored measurement noise. The optimal inputs in this case depend on the characteristics of the noise; however, it is important to keep in mind that the main feature of our approach is the computation of uncertainty ellipses which are guaranteed to contain the true plant. This is accomplished in the colored noise case, provided that the parameter  $\varepsilon$  is chosen so that it satisfies Eq. (3.10). Finally, we mention that experiments we performed with the above nominal plant using colored measurement noise, exhibit features which are similar to the white noise case, namely the variance of the uncertainty ellipses and of the parameter estimates, is larger for the random input case.

**8. The inverse optimal-input problem**

Suppose that the input is fixed:  $\mathbf{u} = \delta$ , i.e. the unit impulse. The measurements in this case are impulse response measurements and the problem of maximizing  $\sigma_*$  over an appropriately defined class of systems can be posed, under the assumption  $N = \infty$ , i.e. the whole impulse response is measured. Thus we will consider what sort of systems will offer the least problems in identification, if we know in advance that the input is a delta function. This is the *inverse problem* to the optimal-input problem. The solution of this problem will yield an upper bound on the second-smallest singular value  $\sigma_*$  of  $\mathcal{D}$  which depends on the  $h_2$  norm of the system.

Notice that in this case ( $N = \infty$ ) the data covariance matrices  $\mathcal{R}\mathcal{R}^*$  and  $\mathcal{M}\mathcal{M}^*$  are the same, and consequently their singular values coincide (cf. Eq. (4.6)). The search for a system maximizing  $\sigma_*$  will be carried out over the class of systems which are linear, stable, discrete-time, finite-dimensional, and have *fixed  $h_2$ -norm*, say  $a$ . This problem can be solved explicitly. Its importance lies in the fact that it provides an upper bound for the optimal  $\sigma_*$  achievable by optimizing over a system class, rather than input class.

Given the (strictly proper) system described by

$$\mathcal{S} := (A, b, c),$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $|\lambda_i(A)| < 1$ ,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}^{1 \times n}$ , let  $\mathbf{h}$  denote its impulse response and

$$\psi(z) := \frac{\mathbf{p}(z)}{\mathbf{q}(z)} = c(zI - A)^{-1}b, \tag{8.1}$$

where  $\mathbf{p}(z) := p_0 + p_1z + \dots + p_mz^m$ ,  $\mathbf{q}(z) := q_0 + q_1z + \dots + q_{n-1}z^{n-1} + z^n$ ,  $m < n$  are coprime polynomials with real coefficients. The first step is to compute the data covariance matrix. Notice that  $\mathbf{u} = \delta$  implies that  $\mathcal{B}$  defined by Eq. (4.8) is the identity.

**Lemma 8.1.** *The data covariance matrix (4.8) for  $\mathbf{u} = \delta$  and  $N \rightarrow \infty$  is*

$$\mathcal{D} = \begin{pmatrix} I_{m+1} & \Delta_1 \\ \Delta_1^* & \Delta_2 \end{pmatrix} \in \mathbb{R}^{(m+n+2) \times (m+n+2)},$$

where  $I_{m+1}$  denotes the identity matrix of size  $m + 1$ ,  $\Delta_1$ , and  $\Delta_2$  are Töplitz matrices of size  $(m + 1) \times (n + 1)$ ,  $(n + 1) \times (n + 1)$ , respectively,

$$\Delta_1 = \begin{pmatrix} \beta_0 & \beta_1 & \dots & \beta_m & \dots & \beta_n \\ 0 & \beta_0 & \dots & \beta_{m-1} & \dots & \beta_{n+1} \\ & & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & \beta_0 & \dots & \beta_{n-m} \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_1 & \alpha_0 & \dots & \alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_{n-1} & \dots & \alpha_0 \end{pmatrix}$$

where  $\beta_0 := d$ ,  $\beta_i := cA^{i-1}b$ ,  $i > 0$ ;  $\alpha_i := cA^iPc^*$ ,  $i > 0$ , and  $P$  is the reachability Grammian satisfying the Lyapunov equation  $APA^* + bb^* = P$ .

We will use the following notation:

$$\mathcal{S}_{\alpha,n} := \{ \mathcal{S} = (A, b, c) : \text{McMillan degree}(\mathcal{S}) = n, \|\mathcal{S}\|_2 = \sqrt{\alpha} \}, \tag{8.2}$$

where  $\|\mathcal{S}\|_2$  denotes the  $h_2$ -norm of  $\mathcal{S}$ . We are now ready to formulate the following.

**Problem 8.1.** *Find the supremum of  $\lambda_*(\mathcal{D})$  for  $\mathcal{S} \in \mathcal{S}_{\alpha,n}$ , where  $\mathcal{D}$  is defined in Lemma 8.1 and  $\mathcal{S}_{\alpha,n}$  is defined by Eq. (8.2).*

The solution is given in the following result.

**Theorem 8.1.** *With the notation and set-up introduced above, the supremum of  $\lambda_*$  is*

$$\lambda_*^{\text{sup}}(\mathcal{D}) = \begin{cases} \frac{n+1}{n} \alpha & \text{for } \alpha \leq \frac{n}{n+1}, \\ 1 & \text{for } \frac{n}{n+1} \leq \alpha \leq 1, \\ \alpha & \text{for } \alpha \geq 1. \end{cases} \tag{8.3}$$

For  $\alpha \geq 1$  the supremum is actually a maximum and is attained by systems with poles at 0 and zeros at infinity, i.e. with poles and zeros as far apart as possible. For  $\alpha < 1$  the supremum is approached arbitrarily closely by systems having poles approaching the unit circle, and zeros at infinity, and consequently gain approaching zero.

This result says that the best systems from the point of view of identification under a unit impulse input and with  $h_2$  norm  $\alpha \geq 1$ , are those of the form  $kz^{-n}$  for some constant  $k$ . The proof of this result can be found in the appendix.

## 9. Conclusions

An approach to the identification problem for robust control has been proposed in Antoulas (1997) and Zhang and Antoulas (1998) under the assumption that an upper bound of the norm of the uncertainty affecting the data is a priori known. Following this approach it is possible to obtain a convex parametrization of an uncertain family of systems, guaranteed to contain the system which generated the data. It turns out that the size of this uncertain family is upper-bounded by a quantity which involves the second-smallest singular value of the noise-free data covariance matrix. In this paper we examined the problem of optimizing this particular singular value using a recursive procedure, thus minimizing the upper bound of the size of the uncertain family. We have also shown by means of a simple simulation experiment that in the framework of robust control, the proposed input generation scheme compares favorably with classical ones.

We have shown that the problem of optimizing this singular value can be reduced to an LMI problem. Furthermore, additional insight can be gained by computing the stationary points of this quantity explicitly. We also examined two special cases: that of first-order systems and that of second-order AR systems. Finally an inverse optimization problem was discussed.

Some open problems and future directions are worth mentioning at this point:

- Extension of the results to multi-input and/or multi-output systems. In this regard it should be mentioned that the eigenvalue optimization of the smallest non-zero eigenvalue of a positive semi-definite data matrix stemming from a MIMO system presents no difficulty. The effort at present is concentrated in obtaining an appropriate generalization of the cone  $\mathcal{F}$ .
- Explicit description of the regions of non-differentiable extrema of  $\sigma_*$  in terms of the system parameters. As a preliminary result we were able to obtain the equations of the curves shown in Fig. 1 (in particular the plot on the lower right) which separate the areas of differentiable from the areas of non-differentiable

maxima of  $\sigma_*$ . The importance of this result lies in the fact that in a non-differentiable optimization problem, one can describe explicitly the region of non-differentiability without using LMIs or more general non-smooth optimization methods.

- Influence of the location of the poles and zeros on  $\sigma_*$ . Some results concerning this issue, for second-order AR systems, are shown in Fig. 1.
- Study the connections with the area of subspace-based system identification. In particular, is it possible to extend the convex parametrization of the uncertainty of Lemma 3.1 to a set of plants obtained in state space by means of 4SID algorithms?

## Acknowledgements

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## Appendix A: The computation of $\mathcal{D}$ for first-order systems

The entries of  $\mathcal{D}$  defined by Eq. (7.4) can be computed by means of the power spectrum

$$S(\omega) := \frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0).$$

We have  $\mathcal{D} = \mathcal{D}^* \in \mathbb{R}^{3 \times 3}$ ,  $\mathcal{D}_{22} = \mathcal{D}_{33}$ , and

$$\mathcal{D}_{11} = \int_{-\infty}^{\infty} S(\omega) d\omega = 1,$$

$$\mathcal{D}_{12} = \int_{-\infty}^{\infty} S(\omega) \Re\{\psi(e^{j\omega})\} d\omega$$

$$= \frac{b(a + \cos \omega_0)}{a^2 + 2a \cos \omega_0 + 1},$$

$$\mathcal{D}_{13} = \int_{-\infty}^{\infty} S(\omega) \Re\{e^{j\omega} \psi(e^{j\omega})\} d\omega$$

$$= \frac{b(a \cos \omega_0 + 1)}{a^2 + 2a \cos \omega_0 + 1},$$

$$\mathcal{D}_{22} = \int_{-\infty}^{\infty} S(\omega) |\psi(e^{j\omega})|^2 d\omega = \frac{b^2}{a^2 + 2a \cos \omega_0 + 1},$$

$$\mathcal{D}_{23} = \int_{-\infty}^{\infty} S(\omega) \Re\{e^{j\omega}\} |\psi(e^{j\omega})|^2 d\omega = \mathcal{D}_{22} \cos \omega_0.$$

Thus  $\mathcal{D}$  can be written as follows:

$$\mathcal{D} = VV^*, \quad V := \frac{1}{\sqrt{a^2 + 2a \cos \omega_0 + 1}} \cdot \begin{pmatrix} a + \cos \omega_0 & \sin \omega_0 \\ b & 0 \\ b \cos \omega_0 & b \sin \omega_0 \end{pmatrix} \in \mathbb{R}^{3 \times 2}. \quad (\text{A.1})$$

The problem in this context is: maximize the second-smallest eigenvalue of  $\mathcal{D}$  as a function of the variable  $x := \cos \omega_0$ . The result is given in Section 7.1.

### Appendix B: The proof of Theorem 7.1

Recall the computation of the data covariance matrix given in Lemma 7.1. It follows that  $\sqrt{\alpha_0}$  is the  $h_2$  norm of the system. Moreover  $\mathcal{D} \geq 0$ ,  $\Delta_2 \geq 0$ , and  $\Delta_2 - \Delta_1^* \Delta_1 \geq 0$ . Recall the notation (3.4). We have

$$\mathcal{D} \begin{pmatrix} \mathbf{p} \\ -\mathbf{q} \end{pmatrix} = 0 \Rightarrow \begin{cases} \mathbf{p} = \Delta_1 \mathbf{q}, \\ (\Delta_2 - \Delta_1^* \Delta_1) \mathbf{q} = 0. \end{cases} \quad (\text{B.1})$$

Furthermore,

$$\Delta_2 - \Delta_1^* \Delta_1 = \mathcal{H}_n \mathcal{H}_n^* = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_{n+1} & \cdots & \alpha_{2n} \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$$

and  $\mathcal{H}_n$  is the Hankel matrix of Markov parameters  $\beta_i$  which has  $n + 1$  rows and infinitely many columns. Thus  $\mathcal{D}$  is positive semi-definite with one eigenvalue equal to 0 and corresponding eigenvector  $(\mathbf{p}^* - \mathbf{q}^*)^*$ . Moreover the second-smallest eigenvalue  $\sigma_*$  is zero if, and only if, the polynomials  $\mathbf{p}(z)$ ,  $\mathbf{q}(z)$  are not coprime. This is the case because from Eq. (B.1),  $\mathcal{D}$  will have a second eigenvalue equal to zero if and only if  $\mathcal{H}_n \mathcal{H}_n^*$  has two zero eigenvalues. This implies that  $\mathcal{H}_n$  has 2 zero eigenvalues, and by standard results in the theory of partial realizations it follows that the set of polynomials  $\mathbf{q}$  in the kernel of  $\mathcal{H}_n$  are multiples of a unique minimal degree polynomial.

#### B.1. Preliminary results

(A) The first preliminary result is a theorem due to Jacobi on the inertia of symmetric matrices. For a proof, see e.g. Gantmacher (1952).

**Jacobi's Theorem.** Given  $M = M^* \in \mathbb{R}^{r \times r}$ , let  $M_i$ ,  $i = 1, 2, \dots, r$ , denote the  $i$ th principal minor thereof. Consider the sequence

$$1, \det M_1, \det M_2, \dots, \det M_r,$$

where it is assumed that there are no more than two consecutive zeros. The number of positive eigenvalues of  $M$  is equal to the number of times the sign remains constant in the above sequence, while the number of negative eigenvalues is equal to the number of sign changes of this sequence.

(B) Let  $A \in \mathbb{R}^{n \times n}$  be stable and in companion form with ones on the super-diagonal and the negative of the coefficients of the characteristic polynomial on the last row. With  $b := (0 \cdots 0 \ 1)^* \in \mathbb{R}^n$ , let  $P > 0$  denote the reachability Grammian of  $A$ ,  $b$ . The following results hold.

**Proposition B.1.** With the assumptions and notation above, (i)  $P$  is a Töplitz matrix. (ii) If the characteristic polynomial of  $A$  is  $s^n + \gamma s^{n-1} + \cdots + \gamma s + \gamma$ , i.e. all coefficients but the leading one are equal, all off-diagonal entries of  $P$  are equal. Furthermore, the diagonal elements and the off-diagonal elements are, respectively,

$$\frac{1 + (n-1)\gamma}{(1-\gamma)(1+n\gamma)} \quad \text{and} \quad \frac{-\gamma}{(1-\gamma)(1+n\gamma)}.$$

In particular, if all eigenvalues of  $A$  are 0,  $P$  is the identity matrix  $I_r$ .

**Corollary B.1.** In case  $\gamma = 1 - \varepsilon$ ,  $\varepsilon > 0$ , as  $\varepsilon$  goes to zero, the Grammian becomes

$$P_\varepsilon = \frac{1}{(n+1)\varepsilon} \begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

The special class of systems where  $A$ ,  $b$  are as above and  $c := (\kappa \ 0 \cdots 0 \ 0) \in \mathbb{R}^n$ , will play an important role in the proof of Theorem 7.1 given below.

Case (i). Let  $\gamma = 0$  and  $\kappa = \sqrt{\alpha}$ . Then  $\Delta_1 = (0 \ 0 \cdots 0 \ \kappa) \in \mathbb{R}^{1 \times (n+1)}$ , and  $\Delta_2 = \alpha I_{n+1}$ . It readily follows that

$$\sigma_{n+2} = \alpha + 1, \sigma_i = \alpha, i = 2, 3, \dots, n + 1, \sigma_1 = 0. \quad (\text{B.2})$$

Case (ii). Let  $\gamma = 1 - \varepsilon$ ,  $1 > \varepsilon > 0$ . As  $\varepsilon$  goes to 0,  $\kappa = \sqrt{\alpha(1-\gamma)(n+1)/n}$ , and  $\Delta_1$  tends to the  $1 \times (n+1)$  zero matrix and the diagonal entries of  $\Delta_2$  are equal to  $\alpha$  while all the others are equal to  $-(1/n)\alpha$ . The (not necessarily ordered) eigenvalues of  $\mathcal{D}$  are readily com-

puted to be

$$\lambda_{n+2} = 1, \quad \lambda_i = \left(1 + \frac{1}{n}\right)\alpha, \quad i = 2, 3, \dots, n+1, \\ \lambda_1 = 0. \quad (\text{B.3})$$

### B.2. Proof of the theorem

*Case A.* If  $\alpha \geq 1$  and  $m > 0$ , it follows that  $\sigma_* \leq 1$ .

**Proof.** Given an arbitrary but fixed  $\varepsilon > 0$ , by applying the inertia theorem to the matrix

$$\mathcal{D}_\# := \mathcal{D} - (1 + \varepsilon)I_{m+n+2}, \quad \varepsilon > 0$$

the resulting first three terms of the sequence of determinants turns out to be

$$1, \quad -\varepsilon, \quad \varepsilon^2.$$

Since there are already 2 sign changes, at least 2 eigenvalues of  $\mathcal{D}_\#$  are negative, which in turn implies that at least 2 eigenvalues of  $\mathcal{D}$  are less than  $1 + \varepsilon$ . Furthermore, since  $\varepsilon$  is arbitrary, the claimed upper-bound follows.  $\square$

*Case B.* If  $\alpha \geq 1$  and  $m = 0$ , it follows that  $\sigma_* \leq \alpha$ .

**Proof.** By applying the inertia theorem to the matrix

$$\mathcal{D}_\# := \mathcal{D} - \alpha I_{n+2}$$

the resulting first four terms of the sequence of determinants turn out to be

$$1, \quad (1 - \alpha), \quad 0, \quad -(1 - \alpha)\alpha_1^2.$$

Since there are at least 2 sign changes in this sequence, at least 2 eigenvalues of  $\mathcal{D}_\#$  are negative, which implies that at least 2 eigenvalues of  $\mathcal{D}$  are less than  $\alpha$ . The claim follows. The maximum is actually attained for systems defined in the previous subsection with  $\gamma = 0$ .  $\square$

*Case C.* If  $1 \geq \alpha \geq n/(n+1)$  it follows that  $\sigma_* \leq 1$ .

**Proof.** Again for a given  $\varepsilon > 0$ , by applying the inertia theorem to the matrix

$$\mathcal{D}_\# := \mathcal{D} - (1 + \varepsilon)I_{m+n+2}, \quad \varepsilon > 0,$$

the resulting first three terms of the sequence of determinants turn out to be

$$1, \quad -\varepsilon, \quad (1 + \varepsilon - \alpha)\varepsilon.$$

This gives 2 sign changes and by the same argument as above the result follows. In this case the supremum can

be approached arbitrarily closely for systems defined in case (ii) above.  $\square$

*Case D.* If  $n/(n+1) \geq \alpha$  it follows that  $\sigma_* \leq ((n+1)/n)\alpha$ .

**Proof.** Systems defined in case (ii) above show that  $((n+1)/n)\alpha$  is a lower bound for  $\sigma_*$ . However, there always holds  $\sigma_{n+2} \geq 1$ , since the 2 norm of  $\mathcal{D}$  is greater than the 2 norm of any of its sub-matrices. Therefore, the value claimed for  $\sigma_*$  is indeed the supremum in this case, because otherwise if  $\sigma_* > ((n+1)/n)\alpha$ , since the sum of all  $\sigma_i$  is constant and equal to  $m+1+(n+1)\alpha$ , the largest singular value would have to increase, which is a contradiction.  $\square$

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