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Dual Form of a Positive Real Lemma

Abstract—**Systems theory descriptions alternative to those already known are presented for transfer function matrices which are positive real. They are derived using the fact that the transpose of a positive real matrix is itself positive real.**

In Anderson [1] a system theory criterion for rational positive real matrices was given. [A matrix $Z(s)$ of rational functions of a complex variable s is positive real if 1) the poles of $Z(s)$ lie in Re $s<0$, or are simple on Re $s=0$, and at any pole on Re $s=0$, the associated residue matrix is non-negative definite hermitian; 2) elements of $Z(s)$ are real for real s; 3) $Z(j\omega)+Z'(-j\omega)$ is non-negative definite hermitian for all ω, other than those values for which $j\omega$ is a pole of an element of Z.] The criterion is as follows.

Positive Real Lemma

Let $Z(s)$ be an $n \times n$ matrix of rational functions of s, with $Z(\infty)<\infty$, and with (F, G, H, J) a minimal realization for $Z(s)$ in the sense that

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$$Z(s) = J + H'(sI - F)^{-1}G \quad (1)$$

with F square and of minimal dimension. Then $Z(s)$ is positive real if and only if there exist matrices P, L, and W_0, with P positive definite symmetric, such that

$$PF + F'P = -LL'; \quad PG = H - LW_0; \quad W_0'W_0 = J + J'. \quad (2)$$

There are computational difficulties in finding P, L, and W_0, which in essence define a matrix $W(s)$ such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \quad (3)$$

via the formula

$$W(s) = W_0 + L'(sI - F)^{-1}G. \quad (4)$$

If (3) can be solved for $W(s)$ see, e.g., Youla [2], Davis [3], and Brockett [4], then P, L, and W_0 can be found. Another way of finding P is discussed in Anderson [5], which reduces, in the case where $J+J'$ is nonsingular, to solving the quadratic equation

$$P[F - G(J + J')^{-1}H'] + [F' - H(J + J')^{-1}G']P + PG(J + J')^{-1}G'P + H(J + J')^{-1}H' = 0. \quad (5)$$

In Anderson [5], a technique for solving this equation is discussed. Once a P has been found, L follows from the second of equations (2) (note that W_0 is nonsingular since $J+J'$ is); W_0 also follows from (2), at least to within multiplication by an arbitrary orthogonal matrix. Then $W(s)$ in (4) may be constructed.

It is often of interest to solve (3), i.e., to find $W(s)$ given $Z(s)$, with an additional constraint: not only should $W(s)$ be analytic in the right half-plane Re $s>0$, but also $W^{-1}(s)$ should be analytic there. [Note that the analyticity of $W(s)$ in the right half-plane follows from the fact that $W(s)$ and $Z(s)$ have the same poles, viz., the eigenvalues of F, and $Z(s)$ is analytic in Re $s>0$, being positive real.]

It is possible to ensure the constraint on W^{-1} in the case where there exists a $W^{-1}(s)$ analytic in Re $s\geq 0$ as distinct from Re $s>0$. One way is to solve (5) with the additional constraint,

$$\text{Re}\,\lambda_i[F + G(J + J')^{-1}(PG - H)'] < 0. \quad (6)$$

It is shown in Anderson [5] that there is a unique P satisfying (5) and (6) and that it generates a $W(s)$ having the desired property.

A second method is to use the following lemma, a trivial variant on a result in Anderson [6].

Lemma for Determination of P

With the same hypothesis as for the positive real lemma, let $Z(s)$ be positive real, $J+J'$ nonsingular, $Z(s)$ possess no $j\omega$-axis poles, and $Z(j\omega)+Z'(-j\omega)$ be positive definite. Then the equation

$$-\dot{\Pi} = \Pi[F - G(J + J')^{-1}H'] + [F' - H(J + J')^{-1}G']\Pi + \Pi G(J + J')^{-1}G'\Pi + H(J + J')^{-1}H' \quad (7)$$

with initial condition $\Pi(t_1)=0$ has a well-defined solution $\Psi(t, t_1)$ for all $t<t_1$. Moreover, $\lim_{t\to-\infty} \Psi(t, t_1)$ exists and equals P, the unique solution of (5) and (6).

Dual results follow by noting that if $Z(s)$ is positive real, so is $Y(s) = Z'(s)$; and if $Z(s)$ is given by (1), $Y(s)$ is given by

$$Y(s) = J' + G'(sI - F')^{-1}H. \quad (8)$$

Applying the positive real lemma to $Y(s)$ leads to:

Dual Positive Real Lemma

With the same hypothesis as for the positive real lemma, $Z(s)$ is positive real if and only if there exist matrices Q, M, and V_0 such that

$$QF' + FQ = -MM'; \quad QH = G - MV_0; \quad V_0'V_0 = J + J'. \quad (9)$$

Again, M allows determination of a matrix $V(s)$ such that

$$Y(s) + Y'(-s) = V'(-s)V(s) \quad (10)$$

through

$$V(s) = V_0 + M'(sI - F')^{-1}H. \quad (11)$$

Substituting $Z(s)$ into (10) and transposing leads to

$$Z(s) + Z'(-s) = V'(s)V(-s) \quad (12)$$

and thus $V(s)$ in (11) gives a spectral factorization of $Z(s)+Z'(-s)$, where the first term on the right of (12) is analytic in Re $s>0$, rather than the second, as in (3).

If $(J+J')$ is nonsingular, Q satisfies

$$Q[F' - H(J + J')^{-1}G'] + [F - G(J + J')^{-1}H']Q + QH(J + J')^{-1}H'Q + G(J + J')^{-1}G' = 0. \quad (13)$$

The particular solution of (12) leading to a $V(s)$ with stable inverse also satisfies the dual condition

$$\text{Re}\,\lambda_i[F' + H(J + J')^{-1}(QH - G)'] < 0 \quad (14)$$

and, with appropriate conditions on $Z(s)$, is the limiting solution of the Riccati equation

$$-\dot{\Pi} = \Pi[F' - H(J + J')^{-1}G'] + [F - G(J + J')^{-1}H']\Pi + \Pi H(J + J')^{-1}H'\Pi + G(J + J')^{-1}G' \quad (15)$$

with zero initial condition.

Finally, let us note that if P, L, and W_0 are matrices satisfying (2), when F, G, H, and J are known, one set of matrices satisfying (9) is provided by solving the invertible equations

$$QP = I; \quad PM = -L; \quad W_0 = V_0. \quad (16)$$

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