Filtering Through Combination of Positive Filters

Luca Benvenuti, Lorenzo Farina, and Brian D. O. Anderson, *Fellow, IEEE*

***Abstract*—The linear filters characterized by a state-variable realization given by matrices with nonnegative entries (called positive filters) are heavily restricted in their achievable performance. Nevertheless, such filters are the only choice when dealing with the charged coupled device MOS technology of charge routing networks (CRN's), since nonnegativity is a consequence of the underlying physical mechanism. In order to exploit the advantages offered by this technology, the authors try to overcome the above-mentioned limitation by realizing an arbitrary transfer function as a difference of two positive filters.**

***Index Terms*—Charge routing network, discrete-time filtering, positive systems.**

I. Introduction

The class of filters having a discrete-time state-variable realization characterized by matrices with nonnegative entries and called positive filters [16], [20], has been analyzed in detail in [4]. Their main feature is nonnegativity of the impulse response. Such filters are heavily restricted in their performance, e.g., as low-pass filters, since the most widely used filters (Butterworth, Chebyshev, etc.) have no sign limitation on their impulse response. Moreover, approximating a given filter with a positive one may lead to unsatisfactory performance, as discussed in [5].

Nevertheless, positive filters are the only choice when dealing with a charged coupled device technology such as charge routing networks (CRN's). This class of filters was introduced by Gersho and Gopinath [12] and is based on a family of functional solid-state electronic devices using MOS technology. Under the application of a sequence of clock pulses, these devices move quantities of electrical charge in a controlled manner across a semiconductor substrate and, using this basic mechanism, they can perform a wide range of electronic functions, including image sensing, data storage, signal processing, or logic operations.

CRN's offer the possibility of achieving discrete-time signal processing on a MOS integrated circuit chip with the advantage of lighter weight, smaller size, lower power consumption, and improved reliability with respect to an equivalent digital implementation. Moreover, they can perform many sampled-data filtering functions directly in the analog domain. In fact, CRN's are inherently analog and, as such, they are ideally suited to a number of sampled data signal processing functions. In a sense, CRN's combine the features of digital and analog techniques. Like digital filters, CRN's are controlled by a master clock and their characteristic is as stable as the master oscillator, but the requirement for analog-to-digital conversion is eliminated and all functions are performed in the analog domain.

The scenario depicted above provides the motivation for trying to overcome the described limitations imposed by the positivity of the filter, by realizing an arbitrary filter using some straightforward combination of positive ones. However, for the bulk of this paper, we will focus on realizing an arbitrary filter as the difference of two positive filters. Note that the only compositions which make sense, in this context, are subtraction and feedback, since addition and multiplication of positive filters still give a positive filter.

The basic question is, is it always possible to realize an arbitrary filter as the difference of two appropriate positive filters?

If the answer is affirmative, then another question arises. Can one give some *a priori* bound for the orders of such positive filters?

This paper tackles the above questions and provides answers.

An outline of the paper is as follows. In Section II, we give some preliminary definitions and known results on positive filters. Section III develops some helpful propositions in order to state Theorem 8, which is the main result of the paper. This theorem describes how to realize a given filter as the difference of two positive filters of some a priori known order. Section IV is dedicated to an introductory description of a CRN and Section V describes how to implement a digital filter using CRN's. Concluding remarks are given in Section VI.

II. Preliminary Results

A. Definitions

A set $\mathcal{K}$ is said to be a cone, provided that $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$. If $\mathcal{K}$ contains an open ball of $\mathbb{R}^n$, then $\mathcal{K}$ is said to be solid and if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$, then $\mathcal{K}$ is said to be pointed. A cone which is closed, convex, solid, and pointed will be called a proper cone. A cone $\mathcal{K}$ is said to be polyhedral if it is expressible as the intersection of a finite family of closed half spaces and cone$(v_1, \cdots, v_M)$ denotes the polyhedral closed convex cone consisting of all nonnegative linear combinations of vectors $v_1, \cdots, v_M$. Finally, conv$(v_1, \cdots, v_M)$ denotes the

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L. Benvenuti is with the Dipartimento di Ingegneria Elettrica, Universitá degli Studi di L'Aquila, 67140, L'Aquila, Italy.

L. Farina is with the Dipartimento di Informatica e Sistemistica, Universitá degli Studi di Roma "La Sapienza", 00184, Rome, Italy.

B. D. O. Anderson is with the Department of Systems Engineering, Research School of Information Sciences and Engineering, Australian National University, ACT 0200, Australia.

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polytope consisting of all the convex combinations of vectors $v_1, \cdots, v_M$.

Given a matrix A, $\sigma(A)$ denotes its spectrum. Any eigenvalue λ of A such that $|\lambda| = \rho(A) := \max\{|\lambda|: \lambda \in \sigma(A)\}$ will be called a dominant eigenvalue of A and $\rho(A)$ will be called the spectral radius of A. A matrix A is said to be nonnegative, provided that $a_{ij} \geq 0$ and column-stochastic, provided that $a_{ij} \geq 0$, $\Sigma_i \; a_{ij} = 1$, where the a_{ij}'s are the entries of A [17], [22]. Given a set $\mathcal{S}$, $\text{int}(\mathcal{S})$ denotes its interior.

As stated in the introduction, positive filters have a nonnegative dynamic state matrix A and this implies that positive filters cannot have arbitrary pole patterns, as shown by Karpelevic [13]. In particular, one of the dominant poles is positive (Perron–Frobenius theorem [17]), i.e., $\rho(A) \in \sigma(A)$. However, the restrictions on nondominant poles become less restrictive as the dimension n of the dynamic state matrix increases but, for any n, some restrictions are always present.

We give below a definition which will be used in the next section.

Definition 1: $\mathcal{P}_n$ denotes the set of points in the complex plane that lie in the interior of the regular polygon with $n > 2$ edges having one vertex in point 1 and inscribed in the unit disk centered at the origin of the complex plane.

The region $\mathcal{P}_m$ can be defined in polar coordinates by the following inequalities:

$$\mathcal{P}_m = \left\{\rho, \theta: \rho \; \cos\left[\frac{(2k+1)\pi}{m} - \theta\right] < \cos \; \frac{\pi}{m}\right\} \tag{1}$$

with $k = 0, 1, \cdots, m-1$.

B. Positive Systems

We shall give next some definitions and known results on positive systems [16], [20]. The set of matrices $A_+ \in \mathbb{R}_+^{n \times n}$, $b_+ \in \mathbb{R}_+^n$, $c_+ \in \mathbb{R}_+^n$, and $d_+ \in \mathbb{R}_+$ for which

$$H(z) = c_+^T (zI - A_+)^{-1} b_+ + d_+$$

is called a positive realization of $H(z)$ and the system $\{A_+, b_+, c_+^T, d_+\}$ is called a positive system. A transfer function $H(z)$ is said to be positively realizable if it has a positive realization of some finite dimension. The poles of maximum modulus of $H(z)$ will be called dominant poles and their modulus will be denoted by $\rho(H(z))$.

The next two theorems provide conditions for positive realizability of a given transfer function. Without loss of generality, we will consider stricly proper transfer functions since the direct transmission term d_+ must be obviously nonnegative.

Theorem 2: [18] Let $H(z)$ be a strictly proper rational transfer function and let $\{A, b, c^T\}$ be a minimal (i.e., jointly reachable and observable) realization of $H(z)$. Then, $H(z)$ has a positive realization if and only if there exists a polyhedral proper cone $\mathcal{K}$ such that the following holds:

1) $A\mathcal{K} \subset \mathcal{K}$, i.e., $\mathcal{K}$ is A invariant;
2) $\mathcal{K} \subset \mathcal{O}$;
3) $b \in \mathcal{K}$

where

$$\mathcal{O} = \{x | c^T A^k x \geq 0, \qquad k = 0, 1, \cdots\}$$

is called the observability cone. Moreover, a positive realization $\{A_+, b_+, c_+^T\}$ is obtained by solving

$$AK = KA_+, \qquad b = Kb_+, \qquad c_+^T = c^T K \tag{2}$$

where K is such that $\mathcal{K} = \text{cone}(K)$.

The above theorem provides a geometrical interpretation of the positive realization problem. Given any minimal realization of a transfer function, then to any positive realization corresponds an invariant cone $\mathcal{K}$, satisfying Conditions (1-3) and vice versa. Moreover, the number of edges of the cone $\mathcal{K}$ equals the dimension of the positive realization.

The following theorem states sufficient conditions for positive realizability of a given transfer function.

Theorem 3: [2] Let $H(z)$ be a strictly proper rational transfer function of order n. If it has a unique simple dominant pole and its impulse response $h(k)$ is nonnegative for all $k \geq 0$, then $H(z)$ has a positive realization of some finite order $N \geq n$.

The general case has been treated in [9] and [14] where necessary and sufficient conditions for positive realizability are given. It is worth noting that the realization problem for positive linear systems is inherently different from that of ordinary linear systems. In fact, a positively realizable transfer function of order n may not have a positive realization of the same order, as shown in [1] and [3].

III. Main Result

The following theorem provides an affirmative answer to the question of whether an arbitrary transfer function can be realized as the difference of the transfer functions of two positive systems.

Theorem 4: Let $H(z)$ be a strictly proper asymptotically stable transfer function of order n . Then $H(z)$ can be realized as the difference of an N-dimensional positive system and a one-dimensional positive system for some $N > n$, dependent on $H(z)$.

Proof: Since $H(z)$ is asymptotically stable, the associated impulse response $h(k)$ is such that

$$|h(k)| < R \cdot \Lambda^k, \qquad \text{for every } k \geq 0$$

for appropriate positive values of R and $\rho(H(z)) < \Lambda < 1$.

It follows that

$$H_+^1(z) = H(z) + \frac{R}{z - \Lambda}$$

has a nonnegative impulse response and has a unique dominant pole Λ. Hence, by Theorem 3, $H_+^1(z)$ has a positive realization of some order $N > n$. Moreover, $H_+^2(z) = R/(z - \Lambda)$ obviously has a first-order positive realization, so that one can write

$$H(z) = H_+^1(z) - H_+^2(z)$$

from which the theorem remains proved. ∎

The previous theorem states that it is always possible to realize an arbitrary system with transfer function $H(z)$ as the difference of an N-dimensional positive system with transfer function $H_+^1(z)$ and a one-dimensional positive system with transfer function $H_+^2(z)$, but it does not provide the value N.

It is worth noting that it may well be the case that the order N of the positive realization of a system must be much larger than its transfer function order n, as shown in [3]. Moreover, a systematic way to pin down the value N is not known, so that it seems to be very hard to obtain an *a priori* upper bound for the dimension of the positive realization of $H_+^1(z)$.

To gain partial insight into this problem, we will initially consider the case of first- and second-order transfer functions and later explore what can be done by decomposing an arbitrary transfer function into a sum of first- and second-order transfer functions.

Proposition 5: Let $H(z)$ be a first-order asymptotically stable strictly proper transfer function. Then, there exist two positive real values R and λ with $\lambda < 1$ and nonnegative matrices $A_+ \in \mathbb{R}_+^{2\times 2}$, b_+, $c_+ \in \mathbb{R}_+^{2\times 1}$ such that

$$H_+^1(z) = H(z) + \frac{R}{z-\lambda} = c_+^T(zI - A_+)^{-1}b_+.$$

Proof: Let $H(z) = r/(z-p)$ with p, $|p|<1$, and $r \in \mathbb{R}$. Consider then the Jordan realization of $H_+^1(z)$, i.e.,

$$A = \begin{pmatrix} p & 0 \\ 0 & \lambda \end{pmatrix}, \quad b = \begin{pmatrix} r \\ R \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with λ, $R \in \mathbb{R}_+$. By choosing the change of coordinates defined by

$$T = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

one obtains

$$\begin{aligned} A_+ &= TAT^{-1} = \tfrac{1}{2}\begin{pmatrix} p+\lambda & -p+\lambda \\ -p+\lambda & p+\lambda \end{pmatrix} \\ b_+ &= Tb = \begin{pmatrix} -r+R \\ r+R \end{pmatrix}, \qquad c_+^T = cT^{-1} = (0 \quad 1) \end{aligned}$$

which are nonnegative for $R \geq |r|$ and $1 > \lambda \geq |p|$.

Remark 6: Note that when p and r are positive, one can directly realize $H(z)$ as a one-dimensional positive system, i.e.,

$$H(z) = c_+^T(zI - A_+)^{-1}b_+$$

with

$$A_+ = p \qquad b_+ = r \qquad c_+ = 1.$$

Proposition 7: Let $H(z)$ be a second-order asymptotically stable strictly proper transfer function with strictly complex poles. Then, there exist two positive real values R and λ with $\lambda < 1$, and nonnegative matrices $A_+ \in \mathbb{R}_+^{m\times m}$, b_+, $c_+ \in \mathbb{R}_+^{m\times 1}$ such that

$$H_+^1(z) = H(z) + \frac{R}{z-\lambda} = c_+^T(zI - A_+)^{-1}b_+$$

provided that the poles of $H(z)$ lie inside the region $\mathcal{P}_m$.

Proof: Let $\rho \quad \cos\theta \pm i\rho \quad \sin\theta \in \mathcal{P}_m$ be the poles of $H(z)$. Consider then the Jordan realization of $H_+^1(z) = H(z) + R/(z-\lambda)$, i.e.,

$$\begin{aligned} A(\lambda) &= \left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & \lambda \end{array}\right) \\ &= \left(\begin{array}{cc|c} \rho\ \cos(\theta) & \rho\ \sin(\theta) & 0 \\ -\rho\ \sin(\theta) & \rho\ \cos(\theta) & 0 \\ \hline 0 & 0 & \lambda \end{array}\right) \\ 6(R) &= \begin{pmatrix} r_i \\ r_a \\ R \end{pmatrix}, c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

where $\lambda, R \in \mathbb{R}_+$, $\lambda < 1$ and r_1, r_2 are appropriate real values. We will show next that there exists an $A(\lambda)$-invariant cone $\mathcal{K}$ with m edges satisfying the conditions of Theorem 2 for appropriate values of λ and R. Consequently, $H_+^1(z)$ admits a positive realization of order m.

To see this, consider the cone $\mathcal{K}_\alpha = \text{cone}(K(\alpha))$ with

$$K(\alpha) = D(\alpha)K$$

where

$$D(\alpha) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}_+$$

and the equation at the bottom of this page holds. Note that the $A(\lambda)$ invariance of $\mathcal{K}_\alpha$ is retained for any $\alpha \in \mathbb{R}_+$ if it holds for $\alpha = 1$. In fact, $A(\lambda)K = KA_+$, with A_+ nonnegative, implies

$$\begin{aligned} A(\lambda)K(\alpha) &= A(\lambda)D(\alpha)K = D(\alpha)A(\lambda)K \\ &= D(\alpha)KA_+ = K(\alpha)A_+. \end{aligned}$$

Then, once the $A(\lambda)$ invariance of $\mathcal{K}_1 = \text{cone}(K(1))$ is proved, one can arbitrarily choose the value of α without loss of such property. We shall prove later that $\mathcal{K}_1$ is $A(\lambda)$ invariant. However, first we prove that α can be chosen in such a way that $\mathcal{K}_\alpha \subset \mathcal{O}$ and R in such a way that $b(R) \in \mathcal{K}_\alpha$.

$$K = \begin{pmatrix} 1 & \cos\left(2\pi\dfrac{1}{m}\right) & \cos\left(2\pi\dfrac{2}{m}\right) & \cdots & \cos\left(2\pi\dfrac{m-1}{m}\right) \\ 0 & \sin\left(2\pi\dfrac{1}{m}\right) & \sin\left(2\pi\dfrac{2}{m}\right) & \cdots & \sin\left(2\pi\dfrac{m-1}{m}\right) \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

To prove the first, note that since $\lambda > 0$, then

$$c^T A(\lambda)^k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c^T A(\lambda)^k e_3 = \lambda^k > 0, \qquad k = 0, 1, \cdots.$$

Hence, in view of the definition of the observability cone $\mathcal{O}$, it follows that $e_3 \in \text{int}(\mathcal{O})$. Moreover, as α goes to zero, $\mathcal{K}_\alpha$ reduces to cone(e_3), i.e.,

$$\lim_{\alpha \to 0} \mathcal{K}_\alpha = \text{cone}(e_3)$$

so that there exists a sufficiently small value of α such that $\mathcal{K}_\alpha \subset \mathcal{O}$.

To prove the existence of a value R such that $b(R) \in \mathcal{K}_\alpha$, it suffices to observe that since

$$e_3 = \frac{1}{m} K(\alpha)(1 \quad \cdots \quad 1)^T$$

then $e_3 \in \text{int}(\mathcal{K}_\alpha)$ for any $\alpha \in \mathbb{R}_+$. Moreover

$$\lim_{R \to +\infty} \frac{b(R)}{R} = e_3$$

so that there exists a sufficiently large value of R such that $b(R) \in \mathcal{K}_\alpha$ for any $\alpha \in \mathbb{R}_+$.

It remains to prove the $A(\lambda)$ invariance of $\mathcal{K}_\alpha$, i.e., the $A(\lambda)$ invariance of $\mathcal{K}_1$ for some $0 < \lambda < 1$. To do this, we study first the case $\lambda = 1$. If this is the case, the $A(1)$-invariance of $\mathcal{K}_1$, i.e., $A(1)K = KA_+$, can be written as

$$\begin{pmatrix} A_{11}Q \\ \hline 1 \quad \cdots \quad 1 \end{pmatrix} = \begin{pmatrix} Q \\ \hline 1 \quad \cdots \quad 1 \end{pmatrix} A_+ \tag{3}$$

where the equation at the bottom of this page holds. The ith column of Q will be denoted by q_i. Equation (3) is nothing but the A_{11} invariance of the polytope $\mathcal{Q}_m = \text{conv}(Q)$.[1] Since the polytope $\mathcal{Q}_m$ can be equivalently described in polar coordinates by the following inequalities:

$$\mathcal{Q}_m = \left\{ r, \varphi \colon r \ \cos\left[\frac{(2k+1)\pi}{m} - \varphi\right] \le \cos \ \frac{\pi}{m} \right\}$$

with $k = 0, 1, \cdots, m-1$, then the A_{11} invariance of $\mathcal{Q}_m$ can be proved by showing that the polar coordinates of $A_{11}q_j$ still satisfy the above inequalities for $j = 1, \cdots, m$. To do this, we can write

$$A_{11}q_j = \rho e^{i\psi}, \qquad \text{with } \psi = 2\pi \frac{j-1}{m} - \theta \tag{4}$$

[1] It is well known that each point of a polytope can be expressed as a convex combination of its vertices. Hence, each transformed vector $A_{11}q_i$ which is contained in $\mathcal{Q}_m$ by invariance can be expressed as a convex combination of $q_1, \cdots, q_m$ from which

$$(1 \quad \cdots \quad 1) = (1 \quad \cdots \quad 1)A_+$$

follows.

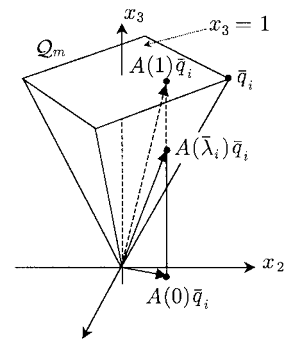

Fig. 1. Selection of the value λ in the case $m = 4$.

so that $A_{11}q_j \in \mathcal{Q}_m$ if and only if

$$\rho, \psi \colon \rho \cos\left[\frac{(2k+1)\pi}{m} - \psi\right] \le \cos \ \frac{\pi}{m}, \quad k = 0, \cdots, m-1.$$

Using (4) and setting $l = k - j + 1$, the above inequalities can be rewritten as

$$\rho \ \cos\left[\frac{(2l+1)\pi}{m} + \theta\right] \le \cos \ \frac{\pi}{m}, \qquad l = -m+1, \cdots, m-1. \tag{5}$$

Since the poles $\rho e^{\pm i\theta}$ of $H(z)$ lie inside the region $\mathcal{P}_m$, then, by (1), one has that

$$\rho, \theta \colon \rho \cos\left[\frac{(2k+1)\pi}{m} \mp \theta\right] < \cos \frac{\pi}{m}, \qquad k = 0, \cdots, m-1 \tag{6}$$

holds. As a consequence, (5) holds, i.e., $\mathcal{Q}_m$ is A_{11} invariant. Moreover, since (6) strictly holds, then $\mathcal{Q}_m$ is strictly A_{11} invariant, i.e., the image of each of the vertices q_i of $\mathcal{Q}_m$ belongs to int$(\mathcal{Q}_m)$. Then, as Fig. 1 makes clear, for each edge $\bar{q}_i = (q_i \quad 1)^T$ of $\mathcal{K}_1$ there exists a value $\bar{\lambda}_i < 1$ such that for all $\lambda \in [\bar{\lambda}_i, 1)$, $A(\lambda)\bar{q}_i \in \mathcal{K}_1$. Consequently, for each value $\lambda \in [\max_i \bar{\lambda}_i, 1)$, $\mathcal{K}_1$ is $A(\lambda)$ invariant.

In conclusion, we have shown that given $\mathcal{K}_\alpha$, there exists a sufficiently small value of α such that

$$\mathcal{K}_\alpha \subset \mathcal{O}$$

a sufficiently large value of R such that

$$b(R) \in \mathcal{K}_\alpha$$

and a positive value $\lambda < 1$ such that $\mathcal{K}_1$ (and consequently $\mathcal{K}_\alpha$) is $A(\lambda)$ invariant, i.e.,

$$A(\lambda)\mathcal{K}_\alpha \subset \mathcal{K}_\alpha$$

$$Q = \begin{pmatrix} 1 & \cos\left(2\pi\frac{1}{m}\right) & \cos\left(2\pi\frac{2}{m}\right) & \cdots & \cos\left(2\pi\frac{m-1}{m}\right) \\ 0 & \sin\left(2\pi\frac{1}{m}\right) & \sin\left(2\pi\frac{2}{m}\right) & \cdots & \sin\left(2\pi\frac{m-1}{m}\right) \end{pmatrix}$$

Then, by Theorem 2, $H_+^1(z)$ has a positive realization $\{A_+, b_+, c_+^T\}$ which can be found by solving

$$A(\lambda)K(\alpha) = K(\alpha)A_+, \quad b(R) = K(\alpha)b_+, \\ c_+^T = c^T K(\alpha). \tag{7}$$

Moreover, since $\mathcal{K}_\alpha$ has m edges, the dimension of the positive realization is m. ■

We can state now the main result of the paper. The following theorem gives an *a priori* upper bound for the dimension of the positive realization of $H_+^1(z)$.

Theorem 8: Let $H(z)$ be a strictly proper asymptotically stable transfer function of order n with simple poles. Then $H(z)$ can be realized as the difference of an N-dimensional positive system and a one-dimensional positive system with

$$N = n + N_2 + \sum_{i \geq 3} (i-2)N_i$$

where N_2 is the sum of the number of negative real poles and the number of nonnegative real poles with negative residues of $H(z)$, N_3 is the number of pairs of complex poles of $H(z)$ belonging to $\mathcal{P}_3$, and N_i $(i > 3)$ is the number of pairs of complex poles of $H(z)$ belonging to

$$\mathcal{P}_i \Big/ \bigcup_{j=3}^{i-1} \mathcal{P}_j.$$

Proof: Since the transfer function is assumed to be asymptotically stable and $\mathcal{P}_j$ covers the unit disk as j goes to infinity, we can assume, without loss of generality, that all the poles of $H(z)$ belong to $\mathcal{P}_q$ where q is the minimal value for which this occurs. Then, let us write $H(z)$ as

$$H(z) = \sum_{i=1}^{q} \sum_{j=1}^{N_i} H_i^j(z) + d$$

where $H_1^h(z)$ is the strictly proper transfer function corresponding to the hth nonnegative real pole with positive residue, $H_2^j(z)$ corresponds to the jth pole of the remaining real ones, $H_3^l(z)$ to the lth pair of complex poles belonging to $\mathcal{P}_3$, and $H_i^k(z)$ to the kth pair of complex poles belonging to

$$\mathcal{P}_i \Big/ \bigcup_{j=3}^{i-1} \mathcal{P}_j, \qquad i = 4, \cdots, q.$$

Then

$$H(z) = \sum_{j=1}^{N_1} H_1^j(z) + \sum_{i=2}^{q} \sum_{j=1}^{N_i} \left(H_i^j(z) + \frac{R_i^j}{z-\Lambda} \right) + d - \frac{\sum_{i=2}^{q} \sum_{j=1}^{N_i} R_i^j}{z - \Lambda}$$

where $\Lambda > 0$, $\Lambda < 1$, and $R_i^j > 0$ are such that

$$H_i^j(z) + \frac{R_i^j}{z-\Lambda}$$

has a positive realization $\{A_i^j, b_i^j, c_i^j\}$ of order i. To this end, one can resort to Propositions 5 and 7 so that one can choose Λ as the maximum among all the λ's chosen in such propositions.[2] Then one obtains

$$H(z) = H_+^1(z) - H_+^2(z)$$

where $H_+^i(z) := (c_+^i)^T(zI - A_+^i)^{-1}b_+^i + d_+^i$ and

$$A_+^1 = \mathrm{diag}(p^1, \cdots, p^{N_1}, A_2^1, \cdots, A_2^{N_2}, \cdots, A_q^1, \cdots, A_q^{N_q})$$
$$b_+^1 = (r^1, \cdots, r^{N_1}, (b_2^1)^T, \cdots, (b_2^{N_2})^T, \cdots, (b_q^1)^T, \cdots, (b_q^{N_q})^T)^T$$
$$c_+^1 = (1, \cdots, 1, (c_2^1)^T, \cdots, (c_2^{N_2})^T, \cdots, (c_q^1)^T, \cdots, (c_q^{N_q})^T)^T$$
$$A_+^2 = \Lambda \qquad b_+^2 = \sum_{i=2}^{q} \sum_{j=1}^{N_i} R_i^j \qquad c_+^2 = 1$$

and

$$d_+^1 = \begin{cases} d, & \text{if } d > 0 \\ 0, & \text{otherwise} \end{cases} \qquad d_+^2 = \begin{cases} d, & \text{if } d < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, A_+^1 has dimension N which is equal to

$$N = N_1 + 2N_2 + 3N_3 + \cdots + qN_q.$$

Finally, since $n = N_1 + N_2 + 2(N_3 + \cdots + N_q)$, one has

$$N = n + N_2 + \sum_{i=3}^{q} (i-2)N_i.$$

■

Remark 9: Note that since $\rho(A_+^2) = \Lambda < 1$ and $\rho(A_+^1) < 1$, then $H(z)$ can be viewed as the difference of two asymptotically stable positive systems.

IV. The CRN's Implementation

A CRN [12] consists of a collection of storage cells, locations where a packet of charge can be stored and maintained isolated from the others, and of a specific periodically repeating routing procedure operation, involving the packets of charge stored in the cells.

The basic operations consist of applying a charge packet to a storage cell such that the packet's size is proportional to a given positive voltage amplitude (injection), of splitting the charge packet of a cell into positive components and transferring these parts into distinct cells previously empty (splitting and transfer), of combining charge packets from different cells and transferring them simultaneously into the same cell (addition), and of emptying a cell by removing its charge packet from the network while generating a voltage amplitude proportional to the size of the extracted packet (extraction).

The cells of the network are divided into p disjoint groups or phases labeled $\{0, 1, \cdots, p-1\}$ and all the operations are controlled by a clock whose period is divided into p equal phases.

At each integer-valued time instant $t = i \pmod p$, all charge in the network is stored in cells of phase i and in the interval between two successive time instants $t = i \pmod p$ and $t+1$, all charge packets are removed from phase-i cells, and each

[2] This is possible since Propositions 5 and 7 continue to hold for all the values $\lambda < 1$ greater than the value chosen in the proposition.

packet is either extracted from the network (generating an output voltage) or transferred into phase $i+1$ cells while new charge packets (proportional to input voltage) may be injected in some of the phase $i+1$ cells (for convenience phase 0 may also be referred to as phase p). Hence, for each cell in phase i, either all of its charge packet exits from the network as an output (sink cell), or the entire packet is routed into one cell of phase $i+1$ or it is divided into two or more components with each component containing a fixed fraction of the original packet, routed into a particular cell of phase $i+1$. Furthermore, each phase $i+1$ cell either receives only a single new packet as an input to the network (source cell) or receives charge packets coming from one or more cells in phase i. All the cells which are neither source nor sink will be classified as internal cells.

From the above considerations it follows that every CRN can be completely described by a directed graph whose nodes represent cells and whose branches indicate routes of charge transfer between cells. Each directed branch has a positive weight value which indicates the fraction of the charge of the starting node transferred along that route. We assume, for the remainder of the paper, that the network has only one input and only one output and that the source and sink cells are located in phase zero. Let $u(t)$ be the size of the charge packet contained in the source cell at time t, $y(t)$ be the size of the charge packet contained in the sink cell at time t, and $x(t)$ the vector whose components are the size of the charge packets contained in the internal cells of phase zero at time t.

Defining a new time scale by setting $t = kp$ and setting $u(k) = u(kp)$, $y(k) = y(kp)$, and $x(k) = x(kp)$, one can derive the dynamic equations for a CRN, as described in [12]

$$\begin{cases} x(k+1) = Ax(k) + bu(k) \\ y(k+1) = c^T x(k) + du(k). \end{cases} \tag{8}$$

Equation (8) is called the reduced form representation of the CRN. Finally, the transfer function relating $u(z)$ and $y(z)$ is

$$H(z) = z^{-1} c^T (zI - A)^{-1} b + z^{-1} d. \tag{9}$$

V. CRN Graph Realization Procedure

The following theorem gives necessary and sufficient conditions for a discrete-time system to be realizable through a CRN in terms of its state-space description.

Theorem 10: [12] Given a discrete-time system of the form (8), there exists a p-phase CRN for any integer $p > 1$ having (8) as reduced form representation if and only if the system matrix

$$S = \begin{pmatrix} A & b \\ c^T & d \end{pmatrix}$$

is column stochastic.

If the conditions of Theorem 10 are met, then, as shown in [4], one can convert the Coates graph [7] obtained from (8) into a p-phase CRN using a simple algorithm, which will be described in Step 6 of the procedure described below.

In order to find a reduced form representation $\{A, b, c, d\}$ satisfying (9) such that the system matrix S is column-stochastic, one needs first to find a positive realization $\{A_+, b_+, c_+, d_+\}$ satisfying (9) (this point being explained in the sequel) and then make use of the two following Lemmas.

Lemma 11: Given a positive realization $\{A_+, b_+, c_+, d_+\}$ of $H(z)$ then there always exists a positive realization $\{\tilde{A}_+, \tilde{b}_+, \tilde{c}_+, \tilde{d}_+\}$ of $H(z)$ such that the following holds.

1) $\rho(\tilde{A}_+)$ is a pole of $H(z)$.
2) any internal node of the Coates graph associated to $\{\tilde{A}_+, \tilde{b}_+, \tilde{c}_+, \tilde{d}_+\}$ has a path to the output node.

Proof: If condition 2) does not hold, then, without loss of generality, we can reorder the entries of the state vector in such a way that $x^T = (x_1^T, x_2^T)$ and

$$A_+ = \begin{pmatrix} A_{+_{11}} & A_{+_{12}} \\ 0 & A_{+_{22}} \end{pmatrix} \quad b_+ = \begin{pmatrix} b_{+_1} \\ b_{+_2} \end{pmatrix} \quad c_+ = \begin{pmatrix} 0 \\ c_{+_2} \end{pmatrix}$$

where x_1 corresponds to state variables whose nodes (of the Coates graph associated to the realization) have no path to the output node. We can then remove the unobservable states x_1 without destroying nonnegativity of the realization. In fact, a lower dimension but still nonnegative realization is provided by

$$\{A_{+_{22}}, b_{+_2}, c_{+_2}, d_+\}.$$

Consider then, without loss of generality, a positive system $\{A_+, b_+, c_+, d_+\}$ satisfying condition 2). As shown in [17], A_+ has an eigenvalue at $\rho(A_+)$ and the associated right and left eigenvectors u_+ and v_+ are nonnegative. As in [2], we will show that either $c_+^T u_+ \neq 0$, i.e., the eigenvalue is observed, or unobservable states can be removed without destroying the nonnegativity of the realization.

Suppose then that $c_+^T u_+ = 0$. Without loss of generality, reorder the entries of the state vector so that

$$c_+^T = (c_{+_1}^T \quad 0 \quad 0), \qquad u_+^T = (0 \quad 0 \quad u_{+_3}^T)$$

with c_{+_1} and u_{+_3} positive. Let

$$A_+ = \begin{pmatrix} A_{+_{11}} & A_{+_{12}} & A_{+_{13}} \\ A_{+_{21}} & A_{+_{22}} & A_{+_{23}} \\ A_{+_{31}} & A_{+_{32}} & A_{+_{33}} \end{pmatrix}.$$

Because $A_+ u_+ = \rho(A_+) u_+$, the zeros in u_+ force $A_{+_{13}} = 0$, $A_{+_{23}} = 0$. But now the zero blocks in c_+ and A_+ mean that an unobservable part is displayed, and a lower dimension, but still nonnegative realization, is provided by

$$\left\{ \begin{pmatrix} A_{+_{11}} & A_{+_{12}} \\ A_{+_{21}} & A_{+_{22}} \end{pmatrix}, \begin{pmatrix} b_{+_1} \\ b_{+_2} \end{pmatrix}, \begin{pmatrix} c_{+_1} \\ 0 \end{pmatrix}, d_+ \right\}.$$

Similarly, if $v_+^T A_+ = \rho(A_+) v_+^T$ and $v_+^T b_+ = 0$, we can immediately eliminate certain uncontrollable blocks and still retain the nonnegativity of the realization.

Then, starting with an arbitrary nonnegative realization, we can reduce it by eliminating certain uncontrollable and/or unobservable blocks and retaining the nonnegativity until the real dominant eigenvalue is controllable and observable, i.e., condition 1). ∎

Lemma 12: Given a positive system $\{A_+, b_+, c_+, d_+\}$, if the following holds:

1) $\rho(A_+) < 1$;
2) $c_+^T(I - A_+)^{-1}b_+ + d_+ = 1$;
3) any internal node of the Coates graph associated to the given positive system has a path to the output node

then the system matrix

$$S = \begin{pmatrix} TA_+T^{-1} & Tb_+ \\ c_+^T T^{-1} & d_+ \end{pmatrix}$$

with $T = \mathrm{diag}(c_+^T(I - A_+)^{-1})$ is column stochastic.

Proof: It is known [19] that conditions 1) and 3) imply strict positivity of the diagonal entries of T. From this it follows that T is full rank and that S is a nonnegative matrix. By considering then the rescaling

$$\begin{aligned} z &= Tx = \mathrm{diag}(c_+^T(I - A_+)^{-1})x \\ &=: \mathrm{diag}(\alpha_1 \cdots \alpha_N)x \end{aligned} \tag{10}$$

the system matrix becomes

$$S = \begin{pmatrix} a_{11}^+ & a_{12}^+ \dfrac{\alpha_1}{\alpha_2} & \cdots & a_{1N}^+ \dfrac{\alpha_1}{\alpha_N} & \alpha_1 b_1^+ \\ a_{21}^+ \dfrac{\alpha_2}{\alpha_1} & a_{22}^+ & \cdots & a_{2N}^+ \dfrac{\alpha_2}{\alpha_N} & \alpha_2 b_2^+ \\ \vdots & & \ddots & \vdots & \vdots \\ a_{N1}^+ \dfrac{\alpha_N}{\alpha_1} & a_{N2}^+ \dfrac{\alpha_N}{\alpha_2} & \cdots & a_{NN}^+ & \alpha_N b_N^+ \\ \dfrac{c_1^+}{\alpha_1} & \dfrac{c_2^+}{\alpha_2} & \cdots & \dfrac{c_N^+}{\alpha_N} & d_+ \end{pmatrix}$$

where the a_{ij}^+'s are the entries of A_+, b_i^+ the entries of b_+, and c_i^+ the entries of c_+. This new matrix is column stochastic if

$$\alpha_k - \sum_{i=1}^{N} \alpha_i a_{ik}^+ = c_k^+, \qquad k = 1, \cdots, N \tag{11}$$

and

$$\sum_{i=1}^{N} \alpha_i b_i^+ + d_+ = 1. \tag{12}$$

To see that (11) hold, it suffices to rewrite them as follows:

$$(\alpha_1 \quad \cdots \quad \alpha_N) = c_+^T(I - A_+)^{-1}.$$

Moreover, from condition 2) it follows that

$$1 = c_+^T(I - A_+)^{-1}b_+ + d_+ = (\alpha_1 \quad \cdots \quad \alpha_N)b_+ + d_+$$

so that (12) also holds. ■

In view of Theorem 10 and of the results of this section and of Section III, we can now state a procedure to realize an arbitrary asymptotically stable transfer function using a CRN.

VI. CRN Design Procedure

Let

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

be an asymptotically stable transfer function of order n with simple poles.

Step 1: Consider the strictly proper transfer function $H'(z)$ obtained from $H(z)$ by eliminating the direct transmission term b_0

$$H'(z) = H(z) - b_0.$$

Step 2: Realize $zH'(z)$ as the difference of an N-dimensional positive system $H_+^1(z)$ and a one-dimensional positive system $H_+^2(z)$. By virtue of Theorem 4, this is always possible. Moreover, using Theorem 8 one has

$$\begin{aligned} H_+^1(z) &= \sum_{j=1}^{N_1} H_1^j(z) + \sum_{i\geq 2} \sum_{j=1}^{N_i} \left(H_i^j(z) + \frac{R_i^j}{z - \Lambda} \right) \\ &\quad + \delta(b_1 - b_0 a_1) \\ H_+^2(z) &= \frac{\sum_{i\geq 2} \sum_{j=1}^{N_i} R_i^j}{z - \Lambda} + (1 - \delta) \cdot (b_1 - b_0 a_1) \end{aligned}$$

$$\text{with } \delta = \begin{cases} 1, & \text{if } (b_1 - b_0 a_1) > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$N = n + N_2 + \sum_{i\geq 3} (i - 2)N_i$$

where N_2 is the sum of the number of negative real poles and the number of positive real poles with negative residues, N_3 is the number of pairs of complex poles belonging to $\mathcal{P}_3$, and N_i $(i > 3)$ is the number of pairs of complex poles belonging to

$$\mathcal{P}_i \Big/ \bigcup_{j=3}^{i-1} \mathcal{P}_j.$$

Note that one can choose a value Λ such that $\rho(zH'(z)) \leq \Lambda < 1$ and then select sufficiently large values R_i^j in order to make the impulse response of $H_i^j(z) + R_i^j/(z - \Lambda)$ nonnegative.

Step 3: Find a positive realization $\{A_+^i, b_+^i, c_+^i, d_+^i\}$ of $H_+^i(z)$ for $i = 1$, 2. To this end, one needs to find a positive realization $\{A_i^j, b_i^j, c_i^j\}$ of order i for each subsystem with transfer function

$$H_i^j(z) + \frac{R_i^j}{z - \Lambda}$$

and then apply the parallel composition described in the proof of Theorem 8. To find the positive realizations $\{A_i^j, b_i^j, c_i^j\}$, one can resort to the proof of Propositions 5 and 7.

Step 4: Following the proof of Lemma 11, reduce the positive realization $\{A_+^1, b_+^1, c_+^1, d_+^1\}$ of $H_+^1(z)$ in order to find a positive realization (still denoted as $\{A_+^1, b_+^1, c_+^1, d_+^1\}$) satisfying the conditions of Lemma 11.

Step 5: Consider the systems with transfer function $(1/H_+^i(1))H_+^i(z)$ $(i = 1, 2)$ and evaluate their positive realization $\{A_+^i, b_+^i/H_+^i(1), c_+^i, d_+^i/H_+^i(1)\}$.

Since by Lemma 11 $\rho(A_+^i)$ is a pole of $H_+^i(z)$ and $H_+^i(z)$ is asymptotically stable, then $\rho(A_+^i) < 1$. Hence, all

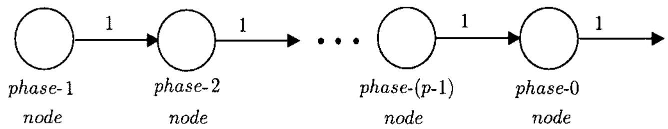

Fig. 2. Chain of p nodes in a CRN graph.

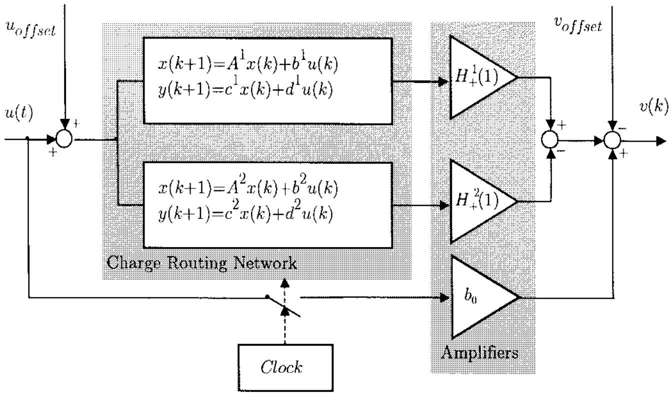

Fig. 3. The structure of the circuit implementing the Chebyshev filter considered in the example.

the conditions of Lemma 12 hold true so that a column-stochastic realization $\{A^i, b^i, c^i, d^i\}$ of $(1/H_+^i(1))H_+^i(z)$ can be obtained by considering the rescaling

$$T^i = \operatorname{diag}((c_+^i)^T(I - A_+^i)^{-1}).$$

Step 6: Convert the Coates graphs associated to $\{A^i, b^i, c^i, d^i\}$ $(i = 1, 2)$ into two p-phase CRN's. To this end, for each graph do the following.

(6.1) Replace every internal node and the output node of the graph by the chain of p nodes depicted in Fig. 2, considering the input node as a phase-0 node.

(6.2) Turn the incoming branches of every original node of the graph into the incoming branches of the phase-one node of the corresponding chain and the outgoing branches into the outgoing branches of the phase-zero node.

The structure of the circuit implementing the overall system is depicted in Fig. 3. The input to the filter is assumed to be an analog signal. As previously noted, the CRN is controlled by a master clock and the requirement for analog-to-digital conversion is eliminated. This conversion is needed, on the contrary, for the direct transmission term (if present). Last, it is worth noting that the nonnegative physical constraint on the input signal of a CRN needs appropriate input and output offsets, as explained in [12].

Let us see an example of how the above procedure works.

Example 13: Consider the fourth-order low-pass digital Chebyshev filter with 0.5 db of ripple in the passband and with cut-off frequency 0.5-times half the sample rate

$$H_{ch}(z) = \frac{0.067\,28(z+1)^4}{(z^2+0.0526z+0.7095)\,(z^2-0.5843z+0.2314).}$$

Step 1: In this step we compute the asymptotically strictly proper transfer function $H_{ch}^{'}(z)$ of order four obtained from $H_{ch}(z)$, by eliminating the direct transmission term $b_0 = 0.067\,28$

$$H_{ch}^{'}(z) = \frac{0.304\,89z^3+0.342\,44z^2+0.296\,19z+5.6234\times 10^{-2}}{(z^2+0.0526z+0.7095)\,(z^2-0.5843z+0.2314)}.$$

Step 2: We consider now the transfer function $zH_{ch}^{'}(z)$, whose poles are $\rho_1\cos\theta_1 \pm i\rho_1\sin\theta_1 = -0.0263 \pm 0.841\,91i$ and $\rho_2\cos\theta_2 \pm i\rho_2\sin\theta_2 = 0.292\,15 \pm 0.382\,16i$. All the poles lie inside $\mathcal{P}_4$, as one can easily verify. In particular, the poles $\rho_2\cos\theta_2 \pm i\rho_2\sin\theta_2$ lie inside $\mathcal{P}_3$. Since $N_2 = 0$, $N_3 = 1$, and $N_4 = 1$, then, in view of Theorem 8, we have

$$zH_{ch}^{'}(z) = H_+^1(z) - H_+^2(z)$$

where $H_+^1(z)$ is a fifth-order transfer function of a positive system of order seven and $H_+^2(z)$ is a first-order transfer function of a one-dimensional positive system. Following the proof of Theorem 8, one can write

$$zH_{ch}^{'}(z) = H_3^1(z) + H_4^1(z) + d$$

with $d = 0.304\,89$ so that

$$H_+^1(z) = \left(H_3^1(z) + \frac{R_3^1}{z-\Lambda}\right) + \left(H_4^1(z) + \frac{R_4^1}{z-\Lambda}\right) + d$$

and

$$H_+^2(z) = \frac{R_3^1 + R_4^1}{z-\Lambda}$$

with

$$H_3^1(z) := \frac{0.39048z - 0.13606}{z^2 - 0.5843z + 0.2314}$$
$$H_4^1(z) := \frac{0.114\,07z + 0.200\,86}{z^2 + 0.0526z + 0.7095}$$

and where Λ, $|\Lambda| < 1$,, and R_i^j are such that

$$H_i^j(z) + \frac{R_i^j}{z-\Lambda}$$

has a positive realization $\{A_i^j, b_i^j, c_i^j, d_i^j\}$ of order i.

Step 3: A positive realization of $H_3^1(z) + R_3^1/(z-\Lambda)$ can be easily found by solving (7) with $r_1 = 0.224$, $r_2 = 0.166\,48$, $\rho = \rho_2$, $\theta = \theta_2$ $\lambda = \Lambda$, and $\alpha = 2/(1+\sqrt{3})$. One has

$$A_3^1 = \frac{1}{3}\begin{pmatrix} 0.5843+\Lambda & 0.36977+\Lambda & -0.95407+\Lambda \\ -0.954\,07+\Lambda & 0.5843+\Lambda & 0.369\,77+\Lambda \\ 0.369\,77+\Lambda & -0.954\,07+\Lambda & 0.5843+\Lambda \end{pmatrix}$$

$$b_3^1 = \begin{pmatrix} 0.611\,98+R_3^1 \\ 8.7906\times 10^{-2}+R_3^1 \\ -0.699\,89+R_3^1 \end{pmatrix}, \quad c_3^1 = \begin{pmatrix} \dfrac{2+\sqrt{3}}{2\sqrt{3}+3} \\ \dfrac{1+\sqrt{3}}{2\sqrt{3}+3} \\ 0 \end{pmatrix}.$$

A positive realization of $H_4^1(z) + R_4^1/(z-\Lambda)$ can be easily found by solving (7) with $r_1 = -6.0472\times 10^{-2}$, $r_2 =$

$0.174\,54$, $\rho = \rho_1$, $\theta = \theta_1$, $\lambda = \Lambda$, and $\alpha = 1$. One has

$$A_4^1 = \frac{1}{2}\begin{pmatrix} 0 & 0.81561+\Lambda & 0.0526 & -0.868\,21+\Lambda \\ -0.868\,21+\Lambda & 0 & 0.815\,61+\Lambda & 0.0526 \\ 0.0526 & -0.868\,21+\Lambda & 0 & 0.815\,61+\Lambda \\ 0.815\,61+\Lambda & 0.0526 & -0.868\,21+\Lambda & 0 \end{pmatrix}$$

$$b_4^1 = \frac{1}{4}\begin{pmatrix} -0.120\,94+R_4^1 \\ 0.349\,08+R_4^1 \\ 0.120\,94+R_4^1 \\ -0.349\,08+R_4^1 \end{pmatrix}, \quad c_4^1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

Then, by choosing $\Lambda = 0.954\,07$, $R_3^1 = 0.699\,89$, $R_4^1 = 0.349\,08$ and resorting to the proof of Theorem 8, one has $H_+^i(z) = (c_+^1)^T(zI - A_+^i)^{-1}b_+^i + d_+^i$ with

$$A_+^1 = \left(\begin{array}{c|c} A_3^1 & 0 \\ \hline 0 & A_4^1 \end{array}\right), \quad b_+^1 = \left(\begin{array}{c} b_3^1 \\ \hline b_4^1 \end{array}\right)$$
$$c_+^1 = \left(\begin{array}{c} c_3^1 \\ \hline c_4^1 \end{array}\right), \quad d_+^1 = d$$

and

$$A_+^2 = \Lambda = 0.95407, \quad b_+^2 = 1.04897, \quad c_+^2 = 1, \quad d_+^2 = 0.$$

Step 4: As one can easily check, the positive realization $\{A_+^1, b_+^1, c_+^1, d_+^1\}$, obtained in the previous step, already satisfies the conditions of Lemma 11 so that no reduction is needed.

Step 5: Computing $H_+^1(1) = 23.716$, $H_+^2(1) = 22.838$ and using

$$T^1 = \mathrm{diag}(7.3802\ 7.552\ 6.84\ 21.877\ 22.832\ 21.668\ 20.712)$$
$$T^2 = 21.772$$

one finally obtains

$$S_1 = \left(\begin{array}{c|c} A^1 & b^1 \\ \hline (c^1)^T & d^1 \end{array}\right) \times 10^{-2}$$

where

$$A^1 = \begin{pmatrix} 51.279 & 43.124 & 0 & 0 & 0 & 0 & 0 \\ 0 & 51.279 & 48.721 & 0 & 0 & 0 & 0 \\ 40.898 & 0 & 51.279 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 84.783 & 2.6554 & 4.5345 \\ 0 & 0 & 0 & 4.4804 & 0 & 93.237 & 2.8992 \\ 0 & 0 & 0 & 2.6049 & 4.0741 & 0 & 92.568 \\ 0 & 0 & 0 & 83.772 & 2.3858 & 4.1036 & 0 \end{pmatrix}$$

$$(b^1)^T = (40.825 \quad 25.086 \quad 0 \quad 5.2612 \quad 16.803 \quad 10.736 \quad 0)$$

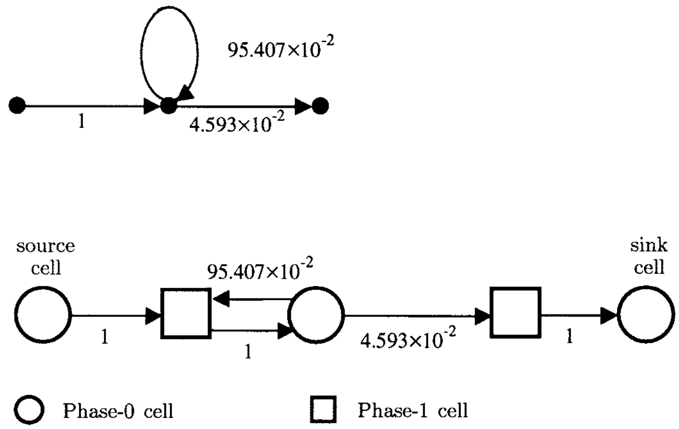

Fig. 4. (top) The Coates graph associated to S_2. (bottom) Its corresponding two-phase CRN.

$$(c^1)^T = (7.823 \quad 5.5965 \quad 0 \quad 9.142 \quad 8.7596 \quad 0 \quad 0)$$
$$d_1 = 1.2856$$
$$S_2 = \left(\begin{array}{c|c} A^2 & b^2 \\ \hline (c^2)^T & d^2 \end{array}\right) = \left(\begin{array}{c|c} 95.407 & 100 \\ \hline 4.593 & 0 \end{array}\right) \times 10^{-2}.$$

The matrices S_1 and S_2 completely describe the CRN's having transfer function $(1/H_+^1(1))H_+^1(z)$ and $(1/H_+^2(1))H_+^2(z)$, respectively.

Step 5: For illustrative purposes, we shall restrict ourselves to apply this step only to the system associated to S_2. This is shown in Fig. 4, where a two-phase CRN is considered.

VII. Concluding Remarks

The linear filters characterized by a state-variable realization given by matrices with nonnegative entries (called positive filters) are heavily restricted in their performance. Nevertheless, such filters are the only choice when dealing with the charged coupled device MOS technology of CRN's, since nonnegativity is a consequence of the underlying physical mechanism. In this paper, in order to exploit the advantages offered by such technology we have presented some theoretical results whose proof has led to a systematic design procedure for realizing an arbitrary transfer function as a difference of two positive filters. It remains open to investigate the case of other ways of interconnecting positive filters, e.g., feedback, to see if lower dimension realizations can be obtained. Moreover, we note that an open problem remains for the case of transfer functions with multiple poles.

As in [12], we have neglected topological constraints and nonideality of the charge transfer mechanism. However, the possibilities and limitations of charge transfer filtering are reviewed and examined in [11] and [23]. A more detailed analysis of transfer inefficiency, coefficients tolerance, insertion loss, and dynamic range is carried out, for example, on the multibridge implementation in [10].

Finally, a CRN can be completely described by a directed graph and, to date, various approaches for matching such a graph to a MOS technology circuit have been proposed in the literature (see, for example, [8], [10], [15], [21], and [24]).

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Luca Benvenuti was born in Rome, Italy, on February 8, 1966. He received the "Laurea" degree in electrical engineering (summa cum laude) and the Ph.D. degree in systems engineering from the University of Rome, "La Sapienza," Rome, Italy, in 1992 and 1995, respectively. He was a visiting graduate student at the University of California, Berkeley, in 1995.

In 1997 he became a Scientific Consultant for Magneti Marelli. He is currently a Research Fellow at the Department of Electrical Engineering, the University of L'Aquila, L'Aquila, Italy, and scientific consultant at PARADES, a European economic interest group funded by Cadence, Magneti-Marelli, SGS-Thomson and the Italian National Research Council (CNR). His research interests are in the areas of control of nonlinear systems, analysis and control of hybrid systems, and positive linear systems.

Lorenzo Farina was born in Rome, Italy, on October 3, 1963. He received the "Laurea" degree in electrical engineering (summa cum laude) and the Ph.D. degree in systems engineering from the University of Rome, "La Sapienza," Rome, Italy, in 1992 and 1997, respectively.

He was a Scientific Consultant at the Interdepartmental Research Centre for Environmental Systems and Information Analysis, the University of Milan, Milan, Italy, in 1993. He was the Project Coordinator at Tecnobiomedica S.p.A., in the field of remote monitoring of patients with heart diseases, in 1995, and held a visiting position at the Research School of Information Sciences and Engineering, the Australian National University in 1997. Since 1996 he has been with the Department of Computer and Systems Science, the University of Rome "La Sapienza," where he is currently an Assistant Professor. His research interests include constrained control and positive linear systems.

Brian D. O. Anderson (S'62–M'66–SM'74–F'75) was born in Sidney, Australia, and received the B.S. degree in pure mathematices and electrical engineering from the University of Sidney, Sidney, Australia, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA.

He has in industry in Silicon Valley and served as a faculty member in the Department of Electrical Engineering, Stanford University. From 1967 to 1981 he was was a Professor of Electrical Engineering at the University of Newcastle, Newcastle, Australia, and is now a Professor of Systems Engineering at the Australian National University and Director of the Research School of Information Sciences and Engineering. His research interests are in control and signal processing.

Dr. Anderson is a Fellow of the Royal Society, the Australian Academy of Science, Australian Academy of Technological Sciences and Engineering, and an Honorary Fellow of the Institution of Engineers, Australia. He holds doctorates (honoris causa) from the Université Catholique de Louvain, Belgium, Swiss Federal Institute of Technology, Zürich, University of Sydney and University of Melbourne. From 1990 to 1993 he served as President of the International Federation of Automatic Control and is currently President of the Australian Academy of Science. His awards include the 1997 IEEE Control Systems Award.