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Brief Paper

Robust Stabilization of Nonlinear Systems via Normalized Coprime Factor Representations*

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Key Words—Robust control; nonlinear systems; Hankel norm; robust stabilization; Isaac's equation; information state; normalized coprime factorization.

Abstract—A reasonably complete theory for the synthesis of robust controllers for a broad class of nonlinear systems is now available. We use this theory to generalize the linear theory of normalized coprime factor robustness optimization to the case of affine input nonlinear systems. In particular, we show that the equilibrium controller may be characterized in terms of the stabilizing and destabilizing solutions of the Hamilton-Jacobi equation used to calculate the normalized (right) coprime factors of the plant. We also show that the optimal robustness margin of

$$\sqrt{\left(1-\left\|\begin{bmatrix} M \\ N \end{bmatrix}\right\|_H^2\right)}$$

generalizes to the nonlinear case. In preparation for the nonlinear analysis, we review the linear case in a way which motivates our approach to the nonlinear case and highlights the parallels with it.

1 *Introduction*

At least in the case of finite dimensional linear systems, the use of normalized coprime factor representations as a basis for the study of robust stability is well established. The synthesis theory for controllers which optimise the robust stability of linear systems with respect to normalized coprime factor perturbations is remarkably simple (Glover and MacFarlane, 1989). At an algorithmic level, this simplicity stems from the fact that the controller synthesis only requires the stabilizing and destabilizing solution of a single linear quadratic (LQ) optimal control type Riccati equation. It turns out that this Riccati equation is also the one used to calculate normalized coprime factors of the plant. Once the stabilizing solution is known, the destabilizing solution may be computed via the solution of a (linear) Lyapunov equation and need not be computed explicitly. In addition, there is a simple formula, in terms of the Hankel norm of the coprime factors, for the highest achievable robustness margin. This formula obviates the need for an iterative solution to the synthesis problem.

The aim of this paper is to examine the extent to which these ideas generalize to the case of affine input nonlinear systems of the form

$$\dot{x} = A(x) + B(x)u, \tag{1.1}$$

$$y = C(x). \tag{1.2}$$

In the interests of a clear exposition we have not included a direct feedthrough term in the output equation. This term simply introduces algebraic clutter and does not lead to any new insights.

In the linear case one could argue that a state-space based solution to the normalized coprime factor robustness problem is unnecessarily clumsy. One treatment of the equivalent gap metric robustness optimization problem has no state-space computations (Georgiou and Smith, 1990). In our study of the restricted class of nonlinear systems described above, a state-space approach seems appropriate, because of the state-space characterisation of the class of nonlinear plants under examination. We will show that an equilibrium controller may be characterized in terms of the stabilizing and destabilizing solutions of a Hamilton-Jacobi equation. We will also show that this Hamilton-Jacobi equation may be used to compute normalized right coprime factorizations of the plant. As with the linear case, there is no need to solve any game theory equations of the Hamilton-Jacobi-Isaacs variety. We show that the highest achievable robustness margin for the equilibrium plant is

$$\sqrt{\left(1-\left\|\begin{bmatrix} M \\ N \end{bmatrix}\right\|_H^2\right)},$$

where $\|\cdot\|_H$ denotes an appropriate generalization of the Hankel norm to nonlinear systems.

Many of the ideas we require can be found in the literature. Normalized coprime factorizations and the closely related stable kernel representations of nonlinear systems can be found in Paice and Van der Schaft (1994) and Scherpen and Van der Schaft (1994). Normalized coprime factors can be computed using a Hamilton-Jacobi equation via the Bounded Real Lemma (Hill and Moylan, 1976; Scherpen and Van der Schaft, 1994). The general synthesis theory for H_∞ controllers for nonlinear systems can be found in Helton and James (1996a), Isidori and Astolfi (1992), James (1995), James *et al.* (1994), James and Baras (1995), James and Baras (1996), Krener (1994) and Van der Schaft (1991, 1992). Some results on normalized coprime factor robustness have already been obtained in Van der Schaft (1995). In this paper the plant was described in terms of a stable kernel representation of a left coprime factorization. The nature of this problem is such that if a normalized left coprime factorization exists, the state estimation problem is easy to solve. This is not true in the case of normalized right coprime factorizations.

Section 2 contains a review of the linear case. This is done in a manner which most closely resembles the approach we will take in the nonlinear case. Section 3 contains a review of the general synthesis theory for the class of nonlinear systems of interest to us here. The certainty equivalence controller requires the solution of an unforced Hamilton-Jacobi-Isaacs equation

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which solves the state-feedback problem, while a second forced partial differential equation, which must be solved in real time, generates a conditional storage function which is known as the information state. The "best" state estimate corresponds to the supremal value of a third conditional storage function. In the case that a certainty equivalence controller does not exist, the solutions of the forced partial differential equation for the information state and an unforced infinite dimensional partial differential equation are required. Section 4 contains the main result of the paper which is Theorem 4.3. An illustrative example is presented in Section 5 and some concluding comments are given in Section 6.

The following notation will be used. Transfer function matrices and operators will be represented by bold face symbols. State-space realizations are denoted by

$$\mathbf{G}(s) = D + C(sI - A)^{-1}B \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and $\mathbf{G}^{\sim}(s) = \mathbf{G}'(-\bar{s})$. RH_{∞}^{+} denotes the space of proper rational matrix valued functions of $s \in C$, analytic in the closed right half plane. $\mathbf{G} = \mathbf{N}\mathbf{M}^{-1}$ is a right coprime factorization of $\mathbf{G}$ if

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}(s)$$

has full column rank for all $Re(s) \geq 0$. Left coprimeness may be defined in a parallel manner via the right coprimeness of $\mathbf{G}'(s)$.

2. *Review of the linear case*

The aim of this section is to review normalized coprime factor robustness and controller synthesis for robustness optimization in the linear case. Our development of this topic has been carried out in such a way as to motivate the nonlinear analysis, and to highlight analogies between the linear and nonlinear cases which will be presented in a later section. Because we will be studying nonlinear systems with a certain state-space structure, our analysis will necessarily be state-space oriented. In the linear case we show that all admissible controllers are parametrized in terms of the stable and anti-stable solutions of a single LQ type Riccati equation which is used in the computation of the normalized (right) coprime factors of the plant. Once the stable solution to the algebraic Riccati equation has been found, the anti-stable solution is easily computed via the solution of a linear equation. In the nonlinear case we will require the stable and anti-stable solutions of a Hamilton–Jacobi equation. Again, this equation arises in the computation of the normalized (right) coprime factors of the plant and a completely analogous development goes through.

If $\mathbf{G}$ is a given plant model, then

$$\mathbf{G} = \mathbf{N}\mathbf{M}^{-1} \tag{2.1}$$

is a normalized right coprime factorization of $\mathbf{G}$ if $\mathbf{M}, \mathbf{N} \in RH_{\infty}^{+}$ are coprime and satisfy

$$\mathbf{M}^{\sim}\mathbf{M} + \mathbf{N}^{\sim}\mathbf{N} = I. \tag{2.2}$$

Given such a normalized right coprime factorization, we define the model set

$$G_{R\gamma} = \left\{(\mathbf{N} + \Delta_N)(\mathbf{M} + \Delta_M)^{-1} : \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in RH_{\infty}^{+}, \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_{\infty} < \gamma^{-1} \right\} \tag{2.3}$$

in which Δ_N and Δ_M are perturbations to the numerator and denominator respectively. The aim is to find a controller which stabilizes $G_{R\gamma}$ with γ minimized. To do this, we seek a controller which (internally) stabilizes the loop illustrated in Fig. 1 and which minimizes the infinity norm of the operator which maps

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \phi.$$

This operator is obtained by breaking the connection in Fig. 1 at a, b and ϕ and treating a, b as inputs and ϕ as the output of the lower subsystem resulting from the break. Because

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}\phi, \tag{2.4}$$

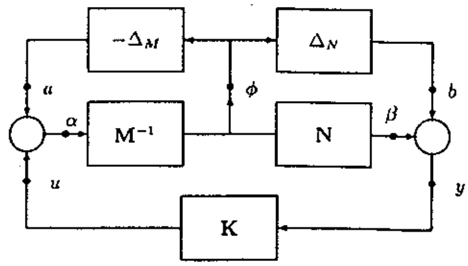

Fig. 1. Normalized right coprime factor robustness optimization.

it follows from $\phi^{\sim}\phi = \alpha^{\sim}\alpha + \beta^{\sim}\beta$ that an equivalent problem results from studying the operator which maps

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

We can see from Fig. 1 that the generalized plant associated with this problem is given by

$$\begin{bmatrix} \alpha \\ \beta \\ y \end{bmatrix} = \begin{bmatrix} I & 0 & I \\ \mathbf{G} & 0 & \mathbf{G} \\ \mathbf{G} & I & \mathbf{G} \end{bmatrix} \begin{bmatrix} a \\ b \\ u \end{bmatrix}. \tag{2.5}$$

If

$$\mathbf{G} = C(sI - A)^{-1}B \tag{2.6}$$

is a minimal realization of $\mathbf{G}$, then a minimal realization of the generalized plant given in equation (2.5) is given by

$$\begin{bmatrix} I & 0 & I \\ \mathbf{G} & 0 & \mathbf{G} \\ \mathbf{G} & I & \mathbf{G} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|ccc} A & B & 0 & B \\ \hline 0 & I & 0 & I \\ C & 0 & 0 & 0 \\ C & 0 & I & 0 \end{array}\right]. \tag{2.7}$$

We can now transform this realization into standard form by invoking known loop shifting and scaling transformations (Green and Limebeer, 1995a) to yield:

$$\mathbf{P} \overset{s}{=} \left[\begin{array}{c|cc} A & [0 \;\; (1-\gamma^{-2})^{-1/2}B] & (1-\gamma^{-2})^{-1/2}B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ C & [I \;\; 0] & 0 \end{array}\right]. \tag{2.8}$$

The idea is that any controller which solves the original problem described by equation (2.7), in the sense of yielding an internally stable closed-loop with closed-loop gain less than γ, will solve the transformed problem described by equation (2.8) and vice versa. A detailed account of these arguments may be found in Green and Limebeer (1995a), refer in particular to Problem 8.12 and the solution which is given on p. 110 of Green and Limebeer (1995b).

2.1. *The computation of normalized right coprime factors.*

It is well known that any state feedback gain matrix F which stabilizes $A - BF$ will generate right coprime factors via

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|c} A - BF & B \\ \hline -F & I \\ C & 0 \end{array}\right]. \tag{2.9}$$

If Q_{+} is the stabilizing solution (meaning $A - BB'Q_{+}$ is asymptotically stable) to

$$A'Q + QA - QBB'Q + C'C = 0, \tag{2.10}$$

then

$$F = B'Q_+ \tag{2.11}$$

generates normalized right coprime factors (Glover and MacFarlane, 1989; Green and Limebeer, 1995a).

2.2. *The Hankel norm of the normalized coprime factor pair.* It follows from equations (2.10) and (2.11) that $Q_+ > 0$ is the observability gramian of the normalized right coprime factor realization given in equation (2.9). The controllability gramian comes from

$$(A - BB'Q_+)P + P(A - BB'Q_+)' + BB' = 0. \tag{2.12}$$

If $Q_- < 0$ is the anti-stable solution to equation (2.10) (meaning $-(A - BB'Q_-)$ is asymptotically stable), we can show that

$$(A - BB'Q_+)(Q_+ - Q_-)^{-1} + (Q_+ - Q_-)^{-1}$$

$$(A - BB'Q_+)' + BB' = 0. \tag{2.13}$$

Thus by uniqueness

$$P = (Q_+ - Q_-)^{-1} \tag{2.14}$$

which shows that Q_- can be computed from P given Q_+. It is now clear that

$$\left\| \begin{bmatrix} \mathrm{M} \\ \mathrm{N} \end{bmatrix} \right\|_H^2 = \lambda_{\max}(Q_+(Q_+ - Q_-)^{-1}) \tag{2.15}$$

in which $\|\cdot\|_H$ denotes the Hankel norm. See Glover and MacFarlane (1989) and Green and Limebeer (1995a) for more details.

2.3. *The two game theory Riccati equations of robust control.* We will now show that the two game theory Riccati equations of $\mathscr{H}_\infty$ control, for the plant (2.8), can be solved in terms of Q_+ and Q_-. Direct substitution from equation (2.8) of equation (8.3.2) into Green and Limebeer (1995a) (for the stabilizing solution X_∞ of the control Riccati equation) gives

$$0 = A'X_\infty + X_\infty A - X_\infty BB'X_\infty + C'C. \tag{2.16}$$

Thus

$$X_\infty = Q_+ \tag{2.17}$$

since this solution is nonnegative and has the correct stability properties. Direct substitution into the filtering Riccati equation (8.3.12) given on p. 298 of Green and Limebeer (1995a) yields

$$0 = AY_\infty + Y_\infty A' - (1 - \gamma^{-2})Y_\infty C'CY_\infty + (1 - \gamma^{-2})^{-1}BB'. \tag{2.18}$$

If Y_+ is the stabilizing solution to the Kalman filter equation

$$0 = AY + YA' - YC'CY + BB', \tag{2.19}$$

then it is clear that

$$Y_\infty = (1 - \gamma^{-2})^{-1}Y_+ \tag{2.20}$$

has the correct properties for $\gamma > 1$. Since

$$0 = A'Y^{-1} + Y^{-1}A + Y^{-1}BB'Y^{-1} - C'C \tag{2.21}$$

we see that $Y = -Q_-^{-1}$ solves the Kalman filter equation and has the correct sign and stability properties. This means that

$$Y_\infty = (1 - \gamma^{-2})^{-1}(P^{-1} - Q_+)^{-1} \tag{2.22}$$

so that

$$Y_\infty^{-1} = (1 - \gamma^{-2})(P^{-1} - Q_+)$$

$$= -(1 - \gamma^{-2})Q_-. \tag{2.23}$$

To help us form one last link in the linear theory, we observe from Green and Limebeer (1995a) Theorem 8.3.2 that

$$Z_\infty^{-1} = Y_\infty^{-1} - \gamma^{-2}X_\infty$$

$$= -(1 - \gamma^{-2})Q_- - \gamma^{-2}Q_+. \tag{2.24}$$

In order that $Z_\infty > 0$, we observe that

$$Z_\infty^{-1} > 0 \Leftrightarrow -Q_- > \gamma^{-2}(Q_+ - Q_-)$$

$$\Leftrightarrow I - Q_+(Q_+ - Q_-)^{-1} > \gamma^{-2}I$$

$$\Leftrightarrow \gamma > \left(1 - \left\| \begin{bmatrix} \mathrm{M} \\ \mathrm{N} \end{bmatrix} \right\|_H^2 \right)^{-1/2}. \tag{2.25}$$

For the rest of the paper, our aim is to repeat these calculations in the nonlinear setting.

3. *The generalized regulator for a class of nonlinear systems*

We will now review the synthesis theory developed in Helton and James (1996a), James (1995), (James *et al.* (1994), James and Baras (1995, 1996) for a class of nonlinear systems. The basic formulae in this section come from Helton and James (1996a). Consider a generalized plant of the form

$$\dot{x} = A(x) + B_1(x)w + B_2(x)u, \quad x(0) = x_0, \tag{3.1}$$

$$z = C_1(x) + D_{12}(x)u, \qquad D'_{12}(x)D_{12}(x) = I_m, \tag{3.2}$$

$$y = C_2(x) + D_{21}(x)w, \qquad D_{21}(x)D'_{21}(x) = I_q. \tag{3.3}$$

In these equations $x(t) \in \mathbf{R}^n$ denotes the state of the system and $y(t) \in \mathbf{R}^q$ is a vector of observations. The output to be regulated (as closely as possible to zero) is $z(t) \in \mathbf{R}^p$. The control input is $u(t) \in \mathbf{R}^m$, while $w(t) \in \mathbf{R}^l$ is an exogenous disturbance input. We will also assume that $p \geq m$ and that $l \geq q$. We assume that all the problem data are smooth functions of x with bounded first derivatives. We assume that the origin is an equilibrium state: $A(0) = 0$, $C_1(0) = 0$ and $C_2(0) = 0$.

The (output feedback) controller is assumed to be a causal mapping $y \in L_{2,e} \mapsto u \in L_{2,e}$. Such a controller is called admissible if the closed-loop equations associated with $\mathbf{K}$ and (3.1)–(3.3) are well defined in the sense that they have unique solutions in $L_{2,e}$.

A controller $\mathbf{K}$ is said to solve the H^∞ control problem provided the closed-loop system is γ-dissipative and internally stable. The closed-loop system is γ-dissipative if

$$\int_0^T z'(s)\, z(s)\, ds \leq \gamma^2 \int_0^T w'(s) w(s)\, ds + \beta(x(0)) \tag{3.4}$$

for some non-negative function β with $\beta(0) = 0$ and every $w \in L_2[0, T]$, for all $T \geq 0$. The closed-loop system will be called weakly internally stable if for all $w \in L_2[0, \infty)$, all the signals $u(\cdot)$, $y(\cdot)$, $z(\cdot)$ as well as $x(\cdot)$ converge to 0 as $t \to \infty$, for any x_0.

3.1. *The state feedback problem.* Let us consider static state feedback controllers of the form

$$u = K(x).$$

Combining this with equations (3.1) and (3.2) gives the following closed-loop system:

$$\dot{x} = A(x) + B_2(x)K(x) + B_1(x)w, \tag{3.5}$$

$$z = C_1(x) + D_{12}(x)K(x). \tag{3.6}$$

If there exists a static state feedback controller such that this closed-loop system is γ-dissipative, we know that there exists a storage function $V(x) \geq 0$ which satisfies the Hamilton–Jacobi–Isaacs equation

$$\nabla_x V(A - B_2D'_{12}C_1) - \tfrac{1}{2}\nabla_x V(B_2B'_2 - \gamma^{-2}B_1B'_1)\nabla'_x V$$

$$+ \tfrac{1}{2}C'_1(I - D_{12}D'_{12})C_1 = 0. \tag{3.7}$$

The function $V(x)$ need not necessarily be smooth, in which case the PDE (3.7) can be interpreted in the viscosity sense (James, 1993; Soravia, 1996). The existence of an output feedback controller which leads to a γ-dissipative closed-loop will also ensure the existence of a solution $V(\cdot) \geq 0$, $V(0) = 0$, to equation (3.7), see Helton and James (1996a).

Conversely, if equation (3.7) has a smooth solution $V(\cdot) \geq 0$, $V(0) = 0$, then the state feedback controller

$$u^s(x) = -(D'_{12}C_1 + B'_2\nabla'_x V) \tag{3.8}$$

renders the closed loop γ-dissipative. Since the control law depends on $\nabla_x V$, we assume that $V(\cdot)$ is differentiable in some sense (C^1 with globally Lipschitz derivative is enough). The stability of the closed loop follows from $V(\cdot) \geq 0$ and the zero state detectability of $((A + B_2K)(x), (C_1 + D_{12}K)(x))$ (the zero state detectability implies positive definiteness of V). The details are covered in Helton and James (1996a), James (1995), James *et al.* (1994), James and Baras (1995, 1996), Krener (1994), Van der Schaft (1991, 1992) and Isidori and Astolfi (1992).

3.2. *The information state.* We will now consider the case of (causal) output feedback controllers of the form K: $y \mapsto u$. In the case of output measurements, we transform the output feedback problem to a new state feedback problem using the information state $P(x, t)$ which is a (conditional) storage function defined by the equation

$$0 = \frac{\partial P}{\partial t} + \nabla_x P(A + B_1 D'_{21}(y - C_2) + B_2 u) - \tfrac{1}{2}\gamma^{-2} \nabla_x P B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x P + \tfrac{1}{2}\gamma^2 (y - C_2)'(y - C_2) - \tfrac{1}{2}(C_1 + D_{12}u)'(C_1 + D_{12}u). \quad (3.9)$$

In shorthand notation, we write $P_t(\cdot) = P(\cdot, t)$ and regard P_t as the state of a new dynamical system, with state equations

$$\dot{P} = F(P, u, y), \quad (3.10)$$

where $F(P, u, y)$ is the nonlinear differential operator defined in equation (3.9). The state space is an appropriate function space. It is known (see Helton and James, 1996a) that if a causal controller K : $y \mapsto u$ exists such that the closed-loop is γ-dissipative, then equation (3.9) has a solution, which we will assume to be smooth.

In the equilibrium (or steady-state) case that $u(t) = 0$ and $y(t) = 0$, this equation reduces to

$$\nabla_x P(A - B_1 D'_{21} C_2) - \tfrac{1}{2}\gamma^{-2} \nabla_x P B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x P + \tfrac{1}{2}\gamma^2 C'_2 C_2 - \tfrac{1}{2} C'_1 C_1 = 0 \quad (3.11)$$

or in shorthand, $F(P, 0, 0) = 0$, which is reminiscent of the equation defining $-\tfrac{1}{2} Y_\infty^{-1}$ (see Green and Limebeer, 1995a, equation (8.2.8)). Convergence of P_t to a solution P_∞ of (3.11) as $t \to \infty$ is discussed in Helton and James (1996a) and Helton and James (1996b). We will assume the existence of P_∞, smooth, non-positive, satisfying $P_\infty(0) = 0$, and $-(A - B_1 D_{21}{}'C_2 - \gamma^2 B_1 [I - D_{21}{}'D_{21}] B_1{}'\nabla_x P_\infty)$ globally exponentially stable, in the sense of Helton and James (1996a) (it is the unique such function).

3.3. *Central controller.* In the output feedback case, the analog of the PDE (3.7) is a nonlinear PDE on an infinite dimensional (function) space. In shorthand notation, it reads

$$\inf_u \sup_y \{\nabla_P W(P)[F(P, u, y)]\} = 0, \quad (3.12)$$

where $\nabla_P W(P)$ is a linear operator. For full details, see Helton and James (1996a) and James and Baras (1996). For us, it is sufficient to note that a smooth solution of equation (3.12) defines an ouput feedback controller (by evaluating the infimum in equation (3.12), called the central controller,

$$u^o(P) = \nabla_P W(P)[-D'_{12} C_1 + B'_2 \nabla'_x P]. \quad (3.13)$$

This controller feeds back the information state, and produces a γ-dissipative closed loop. For precise statements and stability results, see Helton and James (1996a) and James and Baras (1995).

One remaining fact concerning the central controller is needed in the sequel. The function $W(P)$ is related to the state feedback function $V(x)$ by

$$\max_x \{P(x) + V(x)\} \leq W(P), \quad (3.14)$$

and in particular

$$\max_x \{P_\infty(x) + V(x)\} \leq W(P_\infty) = 0 < +\infty. \quad (3.15)$$

Equation (3.15) is a necessary condition which holds irrespective of the smoothness of $W(P)$, and in fact does not depend on the existence of the central controller $u^o(P)$ (Helton and James, 1996a).

3.4. *Certainty equivalence.* Under certain conditions, the central controller simplifies to a controller which is equivalent to the certainty equivalence controller of Basar and Bernhard (1991). Suppose a causal controller of the form $K : y \mapsto u$ exists such that the corresponding closed-loop is γ-dissipative. This means that equations (3.7) and (3.9) have solutions, assumed smooth. Let $u^s(x)$ denote the optimal state feedback control, given by equation (3.8). Then the minimum stress state estimate defined by

$$\bar{x}(t) = \arg\max_x (P(x, t) + V(x)) \quad (3.16)$$

and the certainty equivalence controller

$$u^{ce}(t) = u^s(\bar{x}(t)) \quad (3.17)$$

leads to a γ-dissipative closed-loop provided $\bar{x}(t)$ is unique (Basar and Bernhard, 1991; James and Baras, 1996; Krener, 1994). If a certainty equivalence controller does not exist, it is necessary to solve equation (3.12) for the central controller (3.13), which is much more complex. When the certainty equivalence assumptions are valid, essentially one can take

$$W(P) = \max_x \{P(x) + V(x)\}. \quad (3.18)$$

Further details can be found in Helton and James (1996a), and James and Baras (1996).

If

$$S(x, t) = P(x, t) + V(x), \quad (3.19)$$

we can show that $S(x, t)$ satisfies another "information state" type of equation (Helton and James, 1996a, equation (8.13)) which is reminiscent of that which defines $-\tfrac{1}{2}\gamma^2 Z_\infty^{-1}$. Direct computation gives

$$\begin{aligned} 0 = \frac{\partial S}{\partial t} &+ \nabla_x S(A + B_1 D'_{21}(y - C_2) + B_2 u \\ &+ \gamma^{-2} B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x V) \\ &+ \nabla_x V(B_2 u - B_1 D'_{21} y + \tfrac{1}{2} B_2 B'_2 \nabla'_x V \\ &- \tfrac{1}{2}\gamma^{-2} B_1 D'_{21} D_{21} B'_1 \nabla'_x V) \\ &- \tfrac{1}{2} C'_1 (I - D_{12} D'_{12}) C_1 + \tfrac{1}{2}(C_1 + D_{12}u)'(C_1 + D_{12}u) \\ &- \tfrac{1}{2}\gamma^2 (y - C_2)'(y - C_2) - \tfrac{1}{2}\gamma^{-2} \nabla_x S B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x S. \end{aligned} \quad (3.20)$$

In the case equilibrium case that $u(t) = 0$ and $y(t) = 0$ we get

$$\begin{aligned} 0 = &\nabla_x S(A - B_1 D'_{21} C_2 + \gamma^{-2} B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x V) \\ &- \tfrac{1}{2}\gamma^2 \nabla_x S B_1 (I - D'_{21} D_{21}) B'_1 \nabla'_x S \\ &+ \tfrac{1}{2}\gamma^2 (C_2 + \gamma^{-2} D_{21} B'_1 \nabla'_x V)'(C_2 + \gamma^{-2} D_{21} B'_1 \nabla'_x V) \\ &- \tfrac{1}{2}(D'_{21} C_2 + B'_2 \nabla'_x V)'(D'_{21} C_2 + B'_2 \nabla'_x V). \end{aligned} \quad (3.21)$$

The interested reader may like to note the striking resemblance between this equation and its linear counterpart, see Green anmd Limebeer (1995a, equation (8.2.15)). This is not surprising, as equation (3.21) can be solved as a quadratic form involving Z_∞^{-1} in the linear case.

4. *The nonlinear coprime factor robustness problem*

We will now generalize the normalized right coprime factor results of Section 2 to the nonlinear case. To do this, we show that a nonlinear version of the generalized regulator problem given in equation (2.8) may be used. In the case that the certainty equivalence principle applies, we show that the lowest achievable value of γ can be computed in terms of the Hankel norm of a normalized right coprime factorization of the plant. In other words, we show that equation (2.25) carries over to the nonlinear case. In the general case that a certainty equivalence controller does not exist, equation (2.25) provides an upper bound on the achievable robust stability margin. This follows from James and Baras (1996); see Section 6 and equation (6.14) in particular. We begin by reviewing known results on normalized (right) coprime factorizations of nonlinear systems and generalizations of the idea of a Hankel norm.

4.1. *Normalized right coprime factorizations.* The computation of a right coprime factorization of the nonlinear system $\mathbf{G}$ is not difficult and requires little more than a nonlinear version of the bounded real lemma (Hill and Moylan (1976)). We assume that the system given in equations (1.1) and (1.2) is smooth, that $A(0) = 0$ and that $C(0) = 0$, that is, the system has an equilibrium at the origin. For standard technical reasons we also assume that this realization is zero-state detectable ($y(t) = 0$, $u(t) = 0$ for all $t \geq 0$ implies that $\lim_{t\to\infty} x(t) = 0$).

If we set

$$u(t) = -k(x) + \phi(t) \tag{4.1}$$

it follows that

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A(x) - B(x)k(x) & B(x) \\ \hline -k(x) & I \\ C(x) & 0 \end{array}\right] \tag{4.2}$$

is a right coprime factorization of the plant described by equations (1.1) and (1.2) provided $A(x) - B(x)k(x)$ is asymptotically stable (Scherpen, 1993; Van der Schaft, 1995).

A direct application of the bounded real lemma (Hill and Moylan, 1976) shows that we can choose

$$k(x) = B'(x)\nabla_x' V_+(x), \tag{4.3}$$

where $V_+(x)$ is the smooth positive definite solution to the Hamilton–Jacobi equation

$$\nabla_x V(x)A(x) - \tfrac{1}{2}\nabla_x V(x)B(x)B'(x)\nabla_x' V(x) + \tfrac{1}{2}C'(x)C(x) = 0, \tag{4.4}$$

$V_+(0) = 0$, to normalizes the factorization given in equation (4.2).

Lemma 4.1. Suppose there exists a smooth solution $V_+(x) > 0$ if $x \neq 0$, $V_+(0) = 0$, to equation (4.4). Then

$$k(x) = B'(x)\nabla_x' V_+(x), \tag{4.5}$$

ensures that $A(x) - B(x)k(x)$ is locally asymptotically stable (and globally asymptotically stable if V_+ is proper). For the choice of state feedback law given in equations (4.3) and (4.2) is a right coprime factorization of the plant $\mathbf{G}$. In addition, this factorization is normalized.

Proof. This follows from the bounded real lemma (Hill and Moylan, 1976). Further details can be found in Scherpen and Van der Schaft (1994) and Van der Schaft (1995). □

4.2. *The Hankel norm of nonlinear systems.* Suppose the nonlinear system $\mathbf{G}$ is described by equations (1.1) and (1.2). We also assume that the origin is an asymptotically stable equilibrium of $A(x)$ on a neighbourhood W of the origin (i.e. $A(0) = 0$) and that $C(0) = 0$.

By analogy with the linear case, we define the Hankel norm of $\mathbf{G}$ by

$$\|\mathbf{G}\|_H^2 = \sup_{u \in L_2(-\infty,0]} \left\{\frac{\int_0^\infty y'y\,dt}{\int_{-\infty}^0 u'u\,dt}\right\}. \tag{4.6}$$

As in the linear case, we define the observability gramian by

$$Q(x) = \frac{1}{2}\int_0^\infty y'y\,dt, \quad x(0) = x, \quad u(t) = 0,\ t \geq 0 \tag{4.7}$$

in which x is fixed but unspecified. We also assume that $Q(x)$ is finite. It follows by direct calculation that for all $x \in W$, $Q(x)$ is the unique solution of (Scherpen, 1993)

$$\nabla_x Q(x)A(x) + \tfrac{1}{2}C'(x)C(x) = 0. \tag{4.8}$$

If $x(-\infty) = 0$ and $x(0) = x$ is fixed, then we define the controllability gramian by

$$P(x) = \min_{u \in L_2(-\infty,0]} \frac{1}{2}\int_{-x}^0 u'u\,dt \tag{4.9}$$

which we assume to be finite. It follows from standard optimal control theory (Scherpen, 1993) that $P(x)$ satisfies

$$\nabla_x P(x)A(x) + \tfrac{1}{2}\nabla_x P(x)B(x)B'(x)\nabla_x' P(x) = 0 \tag{4.10}$$

provided it has a unique smooth solution $P_-(x)$ such that the origin is an asymptotically stable equilibrium of $-(A(x) + B(x)B'(x)\nabla_x' P(x))$. Under these assumptions $P(x) > 0$ and for all $x \in W$, $x \neq 0$ and

$$\|\mathbf{G}\|_H^2 = \sup_{x \neq 0}\left\{\frac{Q(x)}{P(x)}\right\}. \tag{4.11}$$

4.3. *The Hankel norm of normalized coprime factors.* The aim of this section is to derive a formula for the Hankel norm of the normalized right coprime factors given in equation (4.2). The result is given in terms of two solutions V_+, V_- to the Hamilton–Jacobi equation (4.4). A similar result is given in (Scherpen and Van der Schaft, 1994, Theorem 19).

We assume there exists a solution V_+ to (4.4) which is stabilizing in the sense that $A - BB'\nabla_x' V_+$ is globally exponentially stable and $V_+(x) > 0$ if $x \neq 0$, $V_+(0) = 0$, and we assume there exists a solution V_- to (4.4) which is antistabilizing in the sense that $-(A - BB'\nabla_x' V_-)$ is globally exponentially stable and $V_-(x) < 0$ if $x \neq 0$, $V_-(0) = 0$.

Lemma 4.2. For the normalized coprime factors given in equation (4.2) we have

$$\left\|\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}\right\|_H^2 = \sup_{x \neq 0}\left\{\frac{V_+(x)}{V_+(x) - V_-(x)}\right\} \tag{4.12}$$

where $V_+(x)$ and $V_-(x)$ are the stabilizing and antistabilizing solutions of the Hamilton–Jacobi equation (4.4). We assume $V_+(x) - V_-(x) > 0$ if $x \neq 0$.

Proof. The observability gramian $L_o(x)$ of the realization (4.2) with $k(x)$ given by equation (4.5) is the unique smooth solution of the equation

$$0 = \nabla_x L_o(A - BB'\nabla_x' V_+) + \tfrac{1}{2}C'C + \tfrac{1}{2}\nabla_x V_+ BB'\nabla_x' V_+ \tag{4.13}$$

satisfying $L_o(0) = 0$. Now V_+ is also a solution of this equation, and satisfies $V_+(0) = 0$. Therefore by uniqueness, we conclude that $L_o(x) = V_+(x)$.

The controllability gramian must satisfy

$$\nabla_x L_c(A - BB'\nabla_x' V_+) + \tfrac{1}{2}\nabla_x L_c BB'\nabla_x' L_c = 0 \tag{4.14}$$

with

$$-(A - BB'\nabla_x'(V_+ - L_c)) \tag{4.15}$$

stable.

If V_+ and V_- are the stabilizing and destabilizing solutions of equation (4.4), it follows that

$$0 = \nabla_x(V_+ - V_-)A - \tfrac{1}{2}\nabla_x V_+ BB'\nabla_x' V_+ + \tfrac{1}{2}\nabla_x V_- BB'\nabla_x' V_- \tag{4.16}$$

$$\Leftrightarrow 0 = \nabla_x(V_+ - V_-)(A - BB'\nabla_x' V_+) + \tfrac{1}{2}\nabla_x(V_+ - V_-)BB'\nabla_x'(V_+ - V_-). \tag{4.17}$$

Comparing this with equation (4.14) we see that

$$L_c = V_+ - V_- \tag{4.18}$$

is one solution. We must now check that this is the correct solution and therefore that

$$-(A - BB'\nabla_x'(V_+ - L_c)) \tag{4.19}$$

is stable. This follows immediately from

$$-(A - BB'\nabla_x'(V_+ - L_c)) = -(A - BB'\nabla_x' V_-) \tag{4.20}$$

which is stable by definition. The result now follows from equation (4.11). □

4.4. *The nonlinear generalized plant.* In Section 4.1 we showed that nonlinear systems of the form equatons (1.1) and (1.2) can be factorised. In addition, we showed that the factors can be normalized using a state feedback law which is expressible in terms of the solution to an Hamilton–Jacobi equation (4.4). With these results in place we turn our attention to Fig. 1 and pose the problem of synthesizing a stabilizing control law which minimizes the infinity norm of the mapping

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \phi.$$

Since

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} \phi$$

and since

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$$

is normalized, an equivalent problem is to synthesize a stabilizing control law which minimizes the infinity norm of the mapping

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

The associated generalized plant may be derived as

$$\left[\begin{array}{cc|c} I & 0 & I \\ \mathbf{G} & 0 & \mathbf{G} \\ \hline \mathbf{G} & I & \mathbf{G} \end{array}\right] = \left[\begin{array}{c|cc} A(x) & [B(x)\ 0] & B(x) \\ \hline \begin{bmatrix} 0 \\ C(x) \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C(x) & [0\ \ I] & 0 \end{array}\right] \tag{4.21}$$

which has a nonzero D_{11} entry. In a manner tightly analogous to the linear case (Green and Limebeer, 1995a), it is possible to remove the nonzero D_{11} term and replace the generalized plant in equation (4.21) with the equivalent system

$$\mathbf{P} = \left[\begin{array}{c|cc} A(x) & [0\ (1-\gamma^{-2})^{-1/2}B(x)] & (1-\gamma^{-2})^{-1/2}B(x) \\ \hline \begin{bmatrix} 0 \\ C(x) \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ C(x) & [I\ \ 0] & 0 \end{array}\right] \tag{4.22}$$

which is an exact counterpart to the linear generalized plant given in equation (2.8). Thus any controller which produces an internally stable closed-loop, with closed-loop gain less than γ, when combined with equation (4.21) does the same with equation (4.22).

4.5. *The partial differential equations of controller synthesis.* We will now substitute the problem data given in equation (4.22) into the general theory described in Section 3. Substitution into the state feedback H_∞ Hamilton–Jacobi equation (3.7) gives

$$\nabla_x V A - \tfrac{1}{2}\nabla_x V BB'\nabla_x' V + \tfrac{1}{2}C'C = 0. \tag{4.23}$$

This is exactly the same Hamilton–Jacobi equation as the one used in the evaluation of the normalized coprime factors described in Section 4.1. The stabilizing solution V_+ is the one we require since it has the correct properties. In the same way, the information state is given by

$$0 = \frac{\partial P}{\partial t} + \nabla_x P(A + (1-\gamma^{-2})^{-1/2}Bu) - \frac{1}{2(\gamma^2-1)}\nabla_x PBB'\nabla_x' P - \tfrac{1}{2}u'u - \tfrac{1}{2}C'C + \tfrac{1}{2}(y-C)'(y-C) \tag{4.24}$$

which becomes

$$0 = \nabla_x PA - \frac{1}{2(\gamma^2-1)}\nabla_x PBB'\nabla_x' P - \frac{1-\gamma^2}{2}C'C \tag{4.25}$$

under equilibrium conditions.

The following result links the solutions of equation (4.4) to that of equation (4.25) and finds conditions on γ such that $S(x) \le 0$.

Theorem 4.3. Assume the existence of stabilizing V_+ and anti-stabilizing V_- solutions to the Hamilton–Jacobi equation (4.4) as described in Section 4.3. Then a necessary condition for the existence of a controller which robustly stabilizes the system (1.1) and (1.2) against normalized right coprime factor perturbations is

$$\gamma \ge \left(1 - \left\|\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}\right\|_H^2\right)^{-1/2}$$

where

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$$

is given by equations (4.2) and (4.5). If the certainty equivalence assumption is valid, this condition is also sufficient.

Proof. Since

$$\nabla_x V_- A - \tfrac{1}{2}\nabla_x V_- BB'\nabla_x' V_- + \tfrac{1}{2}C'C = 0$$

it follows that

$$\nabla_x(\gamma^2-1)V_- A - \frac{1}{2(\gamma^2-1)}\nabla_x(\gamma^2-1)V_- BB'\nabla_x'(\gamma^2-1)V_- - \frac{1-\gamma^2}{2}C'C = 0.$$

Comparing this with equation (4.25) we conclude that

$$P_\infty(x) = (\gamma^2-1)V_- \tag{4.26}$$

since $P_\infty(x) \le 0$ for $\gamma \ge 1$ which is a required property of the information state.

In other words, the existence of a smooth antistabilizing solution P_∞ to the equilibrium information state equation (3.11) (i.e. equation (45.25) follows from the assumed existence of V_- and $\gamma \ge 1$. As discussed above, the function V_+ is the required stabilizing solution to the state feedback H_∞ Hamilton–Jacobi equation (3.7).

It now follows that

$$S(x) = (\gamma^2-1)V_-(x) + V_+(x). \tag{4.27}$$

We complete the proof by invoking equation (3.15). From this we to conclude that a necessary condition for the existence of a stabilizing equilibrium controller is $S(x) \le 0$. That is

$$(\gamma^2-1)V_-(x) + V_+(x) \le 0 \tag{4.28}$$

$$\Leftrightarrow \gamma^2 \ge \frac{V_+(x) - V_-(x)}{-V_-(x)} \tag{4.29}$$

$$\Leftrightarrow \gamma^2 \ge \frac{1}{1 - \dfrac{V_+(x)}{V_+(x) - V_-(x)}} \tag{4.30}$$

$$\Leftrightarrow \gamma \ge \left(1 - \left\|\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}\right\|_H^2\right)^{-1/2} \tag{4.31}$$

using Lemma 4.2. In the case that the certainty equivalence assumption is true, this condition is also sufficient, thanks to (3.18). Here we refer the reader to Section 6 of (James and Baras (1996) and equation (6.14) in particular. □

Remark 4.1. In the linear theory it is well known that

$$\gamma \ge \sqrt{1 + \lambda_{max}(Q_+ Y_+)} \tag{4.32}$$

for both left and right normalized coprime factor robustness; Q_+ and Y_+ are the stabilizing solutions of equations (2.10) and (2.19), respectively. It is also true that $Y_+ = -Q_-^{-1}$ where Q_- is the anti-stable solution to equation (2.10). Hence

$$\gamma \geq \sqrt{1 - \lambda_{\max}(Q_+Q_-^{-1})}. \qquad (4.33)$$

The nonlinear counterpart to equation (4.33) can be found from equation (4.29) and is

$$\gamma \geq \sqrt{1 - \frac{V_+(x)}{V_-(x)}}.$$

A nonlinear counterpart to equation (4.32) is given in Remark 3.2 of Van der Schaft (1995) in the case of normalized left coprime factorizations.

5. *Example*

For the sake of illustration, we will now apply our results to the special case of lossless systems. This example was also studied in Van der Schaft (1995).

Suppose that equations (1.1) and (1.2) is a lossless system, that is there exists an $H(x) > 0$ such that

$$\nabla_x H(x)A(x) = 0; \qquad \nabla_x H(x)B(x) = C'(x).$$

It follows from these equations that the stabilizing and anti-stabilizing solutions to equations (4.4) are given by

$$V_+(x) = H(x), \qquad V_-(x) = -H(x).$$

Using Lemma 4.2 we see that

$$\left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\|_H^2 = \frac{1}{2},$$

and hence that

$$\gamma > \sqrt{2}$$

from Theorem 4.3.

There are two approaches to the controller synthesis problem. In the first we set $K = -1$ in Fig. 1 and notice that

$$\alpha + \beta = a - b$$

hence

$$\|\alpha + \beta\|_2^2 = \|a - b\|_2^2.$$

Since $\langle \alpha, \beta \rangle = 0$, due to the lossless nature of the plant, we get

$$\|\alpha\|_2^2 + \|\beta\|_2^2 = \|a - b\|_2^2 \qquad (5.1)$$

$$\leq 2(\|a\|_2^2 + \|b\|_2^2). \qquad (5.2)$$

One admissible controller is therefore $K = -1$ since it is also stabilising.

If we set $\gamma = \sqrt{2}$, it follows from equation (4.26) that $P(x) = -H(x)$. Substituting this into equation (4.24) gives

$$0 = -\nabla_x H(1 - \tfrac{1}{2})^{-1/2} Bu - \tfrac{1}{2}u'u + y'y - 2y'u \qquad (5.3)$$

$$= -\sqrt{2}C'u - \tfrac{1}{2}u'u + y'y - 2y'u \qquad (5.4)$$

and hence that

$$u = -\sqrt{2}y$$

which is the required control for equation (4.22). It follows from the scaling arguments given in Green and Limebeer (1995a) that

$$u = -y$$

is the control required for equation (4.21). Since $y^2 = 2$, it follows from equation (4.27) that $S(x) = 0$ which is the required dissipation property.

6. *Conclusions*

This paper has studied the synthesis of robust stabilizing controllers for normalized right coprime factorizations of certain nonlinear systems. We have shown that many of the attractive properties of the linear problem solution generalize to the nonlinear case in a straight forward way. It is possible to synthesize an (infinite dimensional) controller in terms of two solutions to a single Hamilton–Jacobi equation; the equation used to generate the normalized right coprime factors. We also show that the optimal stability margin for the equilibrium system is no greater than

$$\sqrt{\left(1 - \left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\|_H^2\right)}.$$

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