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Survey Paper

From Youla–Kucera to Identification, Adaptive and Nonlinear Control*

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Key Words—Linear systems; nonlinear systems; identification; adaptive control; Youla–Kucera parameter.

Abstract—Some 20 years ago, formulae were presented for the set of all linear time-invariant controllers stabilizing a linear time-invariant plant. This paper traces the development of many ideas from these formulae, covering linear H_2 and H_∞ control, identification, adaptive control and nonlinear systems. © 1998 Elsevier Science Ltd. All rights reserved.

1. WHAT IS THIS PAPER ABOUT?

This paper is about a useful way of parametrizing plants and controllers — in the first instance linear plants and controllers. The idea had its origin decades ago, is still giving rise to theoretical developments, for example in problems of closed-loop identification, adaptive control, and nonlinear systems, and is being carried over into practice, one recent reported case being in controller design for flotation circuits.

We begin with motivating material, concerning the desirability of a parametrization of all stabilizing controllers for a prescribed linear time-invariant plant. The Youla–Kucera parametrization is presented, using fractional descriptions of plants and controllers via stable rational transfer functions. Examples of its applications are noted. We then turn to more recent developments. For linear systems, these include the incorporation of the parametrization into state-space formulas (with applications to approximate loop tranfer recovery, and to direct adaptive control); we also note the solution of H_2 and H_∞ problems with constrained pole positions.

A *dual parametrization* (of all plants stabilized by a fixed controller) is the basis for solving closed-loop identification problems by their reduction to standard open-loop problems, and the application of this concept to an adaptive control methodology (windsurfer adaptive control).

It is possible to consider Youla–Kucera parameters simultaneously associated with plant and controller. We explore this concept, considering how a small change in a plant (represented by introduction of a nonzero Youla–Kucera parameter) should give rise to a corresponding controller change, when the design issue is stability, H_2 optimality or H_∞ gain limiting.

Finally, we describe very recent work on nonlinear systems, including closed-loop identification, and Youla–Kucera parametrizations for general nonlinear plant-controller interconnections.

2. THE BASIC IDEA AND SOME OF ITS HISTORY

Let $P(s)$ be the rational transfer function of a stable open-loop plant with $P(\infty) = 0$. How can all stabilizing controllers be characterized? Knowing one stabilizing controller $C(s)$ and $P(s)$, we can define

$$Q(s) = \frac{C(s)}{1 + C(s)P(s)}, \tag{2.1}$$

while if $Q(s)$ and $P(s)$ are known, we could recover $C(s)$ by

$$C(s) = \frac{Q(s)}{1 - P(s)Q(s)}. \tag{2.2}$$

It is easily checked that if $C(s)$ is stabilizing for $P(s)$ and is proper, i.e. $C(\infty)$ is finite, then $Q(s)$ is stable and proper. Conversely, if $Q(s)$ is any stable proper transfer function, and $C(s)$ is *defined* by equation (2.1), then $C(s)$ is necessarily a stabilizing controller for $P(s)$.

Thus, *stabilizing controllers are parametrized in terms of the set of all stable proper transfer functions.*

The result probably does not seem very exciting. But there is an important bonus: the closed-loop

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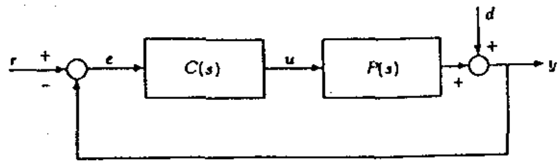

Fig. 1. Stabilized feedback loop showing plant $P(s)$ and controller $C(s)$.

transfer function $G(s)$ is given in terms of $P(s)$ and $Q(s)$ by

$$G(s) = P(s)Q(s). \tag{2.3}$$

The *linearity* in the free stable transfer function $Q(s)$ is what gives the added appeal. Also, $Q(s)$ does have intrinsic significance: it is the transfer function from r to u (see Fig. 1).

It is now well recognized that many closed-loop specifications are convex in $G(s)$ and hence convex in $Q(s)$, but not of course convex in $C(s)$; the convexity is of great assistance in design (Boyd and Barrett, 1991).

The idea of reparametrizing a controller design problem to obtain linearity in the design transfer function is probably an old one; for example, it appeared in (Newton *et al.* (1957). Formula (2.2) appears in Zames (1981) and is well known in chemical process control as "internal model control" — because the controller is viewed as an interconnection of $Q(s)$ in a forward loop and $P(s)$ in a positive feedback loop, see Morari and Zafiriou (1989).

As expressed above, the idea is limited to scalar plants, which are open-loop-stable. How can this substantial limitation be lifted?

3. YOULA–KUCERA PARAMETRIZATION

Youla *et al.* (1976) and Kucera (1975) appear to have independently explained how the ideas of Section 2 extend to MIMO plants which are *not* necessarily stable.

There are two key features in their idea. First, they assume one stabilizing controller is *a priori* known, so the task becomes one of characterizing all, already knowing one. [If the plant is *a priori* stable, $C(s) = 0$ is immediately known to be stabilizing.] Second, they describe plants using polynomial fractional representations.

Thus, if $P(s)$ is the rational transfer function of a scalar plant, one thinks of it as $n(s)/d(s)$, where n and d are polynomials, both coprime to avoid difficulties. If $P(s)$ is multivariable, it can be regarded as $N(s)D^{-1}(s)$ or $\tilde{D}^{-1}(s)\tilde{N}(s)$, where N, D, etc., are matrices of polymonials; a form of coprimeness condition is imposed.

It became evident that a tidier formulation was available if one moved away from polynomial fractional representations to *stable transfer function fractional representations.* Thus, a plant with transfer function $1/(s-1)$, instead of being regarded as a fraction with numerator and denominator 1 and $s-1$, respectively (both polynomials), can be regarded as a fraction with numerator $1/(s+1)$ and denominator $(s-1)/(s+1)$ (both stable rational transfer functions).

Because stable proper rational transfer functions form an algebraic entity known as a Euclidean domain [an observation certainly going back to Forney (1970) and exploited by Desoer *et al.* (1980), etc., and Vidyasagar (1985) for the present purposes] notions such as coprimeness, and greatest common divisor determination via a Euclidean algorithm and the like can be used.

Let $\mathcal{S}$ denote the set of stable proper rational transfer functions (in continuous or discrete time). Two matrices N, D with entries in $\mathcal{S}$ (henceforth written with some abuse of notation as $N, D \in \mathcal{S}$) and with the same number of columns are said to be right coprime if there exist matrices $\bar{X}, \bar{Y} \in \mathcal{S}$ with $\bar{X}N + \bar{Y}D = I$. The main result is as follows, see e.g. Vidyasagar (1985). As with the earlier result for scalar, open-loop stable plants, it characterizes all stabilizing controllers using as a parameter an arbitrary stable proper transfer function (matrix).

Theorem 1. Let a plant $P = ND^{-1}$, with N and D coprime over $\mathcal{S}$, be stabilized by a controller (in a negative feedback loop) $C = XY^{-1}$, with X, Y coprime over $\mathcal{S}$. Then the set of all stabilizing controllers for P is given by $\{(X + DQ)(Y - NQ)^{-1} : Q \in \mathcal{S}\}$.

The idea of Section 2 is included here: suppose P is scalar and stable. Then $N = P, D = 1$. [This pair (N, D) is coprime.] Also $C = 0$ is stabilizing, so $X = 0, Y = 1$. [The pair (X, Y) is coprime.] Then the formula yields $Q(1 - PQ)^{-1}$, as before in equation (2.2).

It is certainly not obvious that the above parametrization yields a closed-loop transfer function matrix which is linear in Q; the calculation is easy when P and C are scalar, and the closed-loop transfer function is computed to be

$$G = \frac{PC}{1 + PC} = \frac{NX}{NX + DY} + \frac{ND}{NX + DY}Q, \tag{3.1}$$

which is affine in Q—the next best thing to linear Q.

In the MIMO case, the result is

$$G = [N \quad 0]\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}\begin{bmatrix} D & -X \\ N & Y \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3.2}$$

and again the affine property holds. (The inverse will exist if the closed-loop is well-defined, which is the case if both P and C are proper with one strictly proper; the inverse is in $\mathscr{S}$ if the closed-loop is stable.) If $\tilde{X}$, $\tilde{Y}$, $\tilde{D}$ and $\tilde{N}$ are such that

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad (3.3)$$

then equation (3.2) simplifies significantly to

$$G = N(\tilde{X} + Q\tilde{D}). \quad (3.4)$$

Digression. Equation (3.3), which is known as the (double) Bezout identity (Vidyasagar, 1985), has embedded within it many ideas, a few of which we list here:

- $\tilde{D}^{-1}\tilde{N}$ and $\tilde{Y}^{-1}\tilde{X}$ equal ND^{-1} and XY^{-1}, and are left coprime fractional descriptions of the plant and initial stabilizing controller.
- The stability of the plant-controller loop is equivalent to the invertibility (over $\mathscr{S}$) of each of the matrices on the left of equation (3.3).
- $(X + DQ)(Y - NQ)^{-1} = (\tilde{Y} - Q\tilde{N})^{-1}(\tilde{X} + Q\tilde{D})$.
- For scalar plants and controllers, one can take $N = \tilde{N}$, $D = \tilde{D}$, $X = \tilde{X}$, $Y = \tilde{Y}$ and then the double Bezout identity is equivalent to $XN + YN = 1$.

Equations (3.2) and (3.4) show the affine dependence of the closed-loop transfer function (matrix) linking r to y on the parameter $Q(s) \in \mathscr{S}$. It is not hard to verify that the transfer function (matrix) linking either r or d to any of e, u and y are *similarly affine* in $Q(s)$. Of particular interest in many designs are the output sensitivity, viz., the transfer function matrix linking d to y:

$$S(s) = (I + PC)^{-1} = (Y - NQ)\tilde{D} \quad (3.5)$$

and the transfer function matrix from the external input r to the plant input u, viz.,

$$G_{ur}(s) = C(I + PC)^{-1} = (X + DQ)\tilde{D}. \quad (3.6)$$

The affine dependence carries over into even more general structures; for example, suppose that there exists one controller $C(s)$ producing internal stability for the set-up of Fig. 2. Here, w denotes the disturbance signals, z the regulated variables, u the actuator inputs and y the sensor outputs (normally including set-point data). Then the set of all controllers achieving internal stability can be parametrized very similarly to the set of Theorem 1 in terms of a free proper $Q(s) \in \mathscr{S}$, and the closed-loop transfer function (matrix) from w to z has the form

$$G_{zw} = T_1(s) + T_2(s)Q(s)T_3(s), \quad (3.7)$$

with T_1, T_2, and $T_3 \in \mathscr{S}$ [and independent of $Q(s)$].

Formula (3.7) is behind one approach to solving H_∞ problems where the task is to choose Q so that $\|G_{zw}\|_\infty$ lies below a nominated bound, or is minimized; see Zhou *et al.* (1995) and Green and Limebeer (1995). Equally, the formula can be used as a basis for tackling H_2 problems, where the task is to choose Q to minimize $\|G_{zw}\|_2$; see Doyle *et al.* (1992).

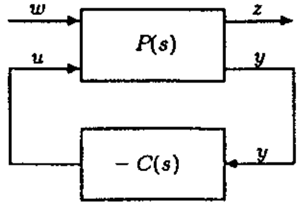

Fig. 2. More general plant-controller interconnection, with disturbances w and regulated variables z.

4. FLOTATION CIRCUIT DESIGN EXAMPLE

The Youla parametrization has a reputation of being a somewhat underutilized tool as far as practical control system design is concerned. In this section, as part counter to that idea, we summarize a practical example, in which the key is to exploit the affine occurrence of an adjustable Q in the closed-loop transfer function. The paper by Ross and Swartz (1995) describes a multivariable system application of the Q-parametrization of a controller to flotation circuit design, the model being like that of Fig. 2. In a flotation cell, see Fig. 3, a froth is produced through the injection of air, and the froth layer contains a higher proportion of valuable material and a lower proportion of gangue than the feed. It is used to generate the concentrate. Normally, a series of cells are connected, generally with feedback between them.

For an individual cell, the correspondences with Fig. 2 are

- u: 2-vector comprising air flow rate and additional water flow rate;
- w: change in total feed solids flow rate $(G_f + Z_f)$;
- z: concentrate valuable material;
- y: same as z.

A discrete-time formulation is used, and the $i - j$ element of the matrix Q is

$$Q_{ij} = \sum_{k=0}^{L} q_{ij}(k)z^{-k}. \quad (4.1)$$

Thus, an optimization is carried out over a finite-impulse-response (FIR) $Q(z^{-1})$. The quantity optimized is a time-weighted (weight = square of time) sum-of-squares-error of the outputs to a step in a feed disturbance

$$\Phi = \sum_{r=1}^{NZ} \sum_{s=1}^{ND} W_{rs} \sum_{k=1}^{L} k^2 [s_{setrs} - s_{rs}(k)]^2. \quad (4.2)$$

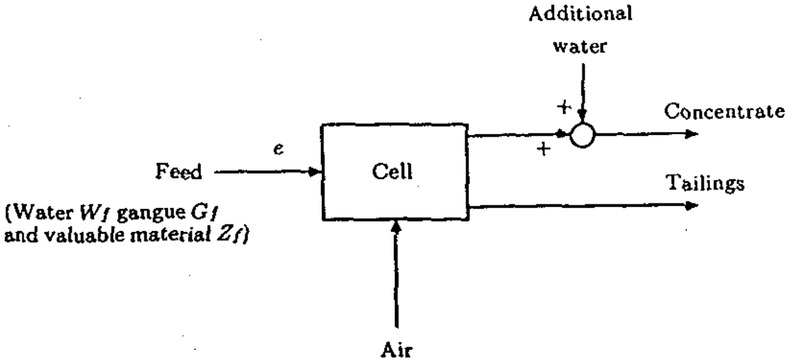

Fig. 3. Flotation cell.

Here NZ, ND refer to the dimensions of z and w (for all cells in a bank), W_{rs} is a weighting matrix, $s_{rs}(k)$ is the response at time k of z_r, to a step at time zero in w_s, and $s_{setrs}(k)$ is the corresponding reference value.

Normally, some cells will have their concentrates used as input to other cells. One or more cells will provide the final concentrate, and W_{rs} will reflect the importance attaching to these last cells.

Because of equation (3.7), the index (4.2) can be expressed in terms of the free parameters in $Q(z^{-1})$. Key to the optimization being so easy is the quadratic nature of equation (4.2) not just in $Q(z^{-1})$, but in the individual parameters $q_{ij}(k)$ making up $Q(z^{-1})$, as well as the finite-dimensional nature of the optimization.

5. A MORE ACADEMIC EXAMPLE

The advantage of the affine parametrization is brought out in the following example (Doyle *et al.*, 1992). The plant, which is to be stabilized with a series compensator in a negative feedback loop, is

$$P = \frac{1}{(s-1)(s-2)} = ND^{-1},$$

$$N = \frac{1}{(s+1)^2},$$

$$D = \frac{(s-1)(s-2)}{(s+1)^2}.$$

The problem is to find a proper $C(s)$ achieving closed-loop stability; also the final value of y is 1 when r is a unit step and $d = 0$, and the final value of y is zero when $r = 0$ and d is a sinusoid of 10 rad/s.

The Bezout equation $\tilde{X}N + \tilde{Y}D = 1$ is satisfied by

$$\tilde{X} = \frac{19s - 11}{s+1}, \qquad \tilde{Y} = \frac{s+6}{s+1}$$

and one stabilizing controller is given by XY^{-1} with $X = \tilde{X}$, $Y = \tilde{Y}$. The double Bezout identity holds by setting also $\tilde{N} = N$ and $\tilde{D} = D$.

A $Q(s) \in \mathcal{S}$ will ensure closed-loop stability for the controller $C = (X + DQ)(Y - NQ)^{-1}$. The unit step response requirement is that $G(0) = 1$, or by equation (3.4),

$$Q(0) = 6.$$

The requirement to suppress a sinusoidal output disturbance of 10 rad/s from equation (3.5) is

$$S(j10) = 0$$

or

$$Q(j10) = -94 + 70j.$$

Since $Q(s)$ is constrained by three real interpolation conditions, we can assume

$$Q(s) = \alpha_1 + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{(s+1)^2}$$

and choose α_i to ensure the interpolation data are fulfilled. This leads to a fourth-order controller

$$C(s) = \frac{X + DQ}{Y - NQ} = \frac{-60s^4 - 598s^3 + 2515s^2 - 1794s + 1}{s(s^2 + 100)(s + 9)}.$$

The denominator zeros of 0 and $\pm j10$ are consistent with the internal model principle and the data.

Professor S. Manabe has pointed out in a private communication that the choice of denominator for $N, D, \tilde{X}, \tilde{Y}$ and Q of $(s + 1)$ or $(s + 1)^2$ results in a very nonrobust system, while other choices may be far more satisfactory.

6. RECENT DEVELOPMENTS OF THE Q PARAMETRIZATION

In earlier sections, we have described material involving the Youla–Kucera parametrization that is reasonably well known. In this section, we look at

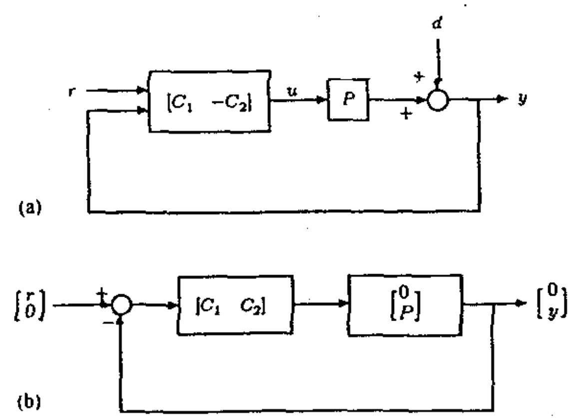

Fig. 4. (a) 2-Degree of freedom controller (b) Equivalent arrangment as in a.

some more recent developments, including two degrees-of-freedom controllers, state-variable versions of the earlier ideas, and controller design with constrained closed-loop pole positions.

6.1. *Two-degrees-of-freedom Controllers*

A two-degree-of-freedom arrangements is depicted in Fig. 4a; the specialization $C_1 = C_2 = C$ would recover the scheme of Fig. 1. Finding a $Q(s) \in \mathscr{S}$ parametrization of all stabilizing C is not hard, using a trick (Moore *et al.*, 1986).

Work with an augmented plant:

$$P_a(s) = \begin{bmatrix} 0 \\ P(s) \end{bmatrix} = N(s)D^{-1}(s) = \begin{bmatrix} 0 \\ N_2 \end{bmatrix} D^{-1} = \begin{bmatrix} I & 0 \\ 0 & \bar{D}_2^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \bar{N}_2 \end{bmatrix} = \bar{D}^{-1}\bar{N}. \quad (6.1)$$

Suppose a specific controller (in a negative feedback loop) for $P_a(s)$ is

$$[C_1^0(s) \quad C_2^0(s)] = \bar{Y}^{-1}[\bar{X}_1 \quad \bar{X}_2] = XY^{-1} \quad (6.2)$$

where equation (3.3) holds. From the fact that $\bar{N}X + \bar{D}Y = I$ and $\bar{N}$ and $\bar{D}$ are structured, we conclude that

$$Y = \begin{bmatrix} I & 0 \\ Y_{12} & Y_2 \end{bmatrix},$$

for some Y_2.

The set-up of Fig. 4a is exactly what is obtained when $P_a(s)$ of Fig. 4b has a series compensator (in a unity negative feedback loop), the first external input is r and the second external input is zero. The first output of $P_a(s)$ is always zero, and the second is y of Fig. 4a; see also Fig. 4b. The closed-loop of Fig. 4b is stable exactly when $C_2(s)$ stabilizes $P(s)$. The set of all stabilizing controllers $[C_1 \ C_2]$ is given, with $Q_1, Q_2 \in \mathscr{S}$, by

$$[\bar{Y} - (Q_1 \quad Q_2)\bar{N}]^{-1}[(\bar{X}_1 \quad \bar{X}_2) + (Q_1 \quad Q_2)\bar{D}] = [\bar{Y} - Q_2\bar{N}_2]^{-1}[\bar{X}_1 + Q_1 \quad \bar{X}_2 + Q_2\bar{D}_2]. \quad (6.3)$$

Thus, the set of $C_2(s)$ is the same as the set of $C(s)$ arising in the one degree of freedom of problem, viz., $(\bar{Y} - Q_2\bar{N}_2)^{-1}(\bar{X}_2 + Q_2\bar{D})$ for arbitrary $Q_2 \in \mathscr{S}$; the set of $C_1(s)$ is richer, and contains a further parameter, viz, $Q_1 \in \mathscr{S}$. The transfer function from r to y can be computed as

$$G(s) = N_2(\bar{X}_1 + Q_1) \quad (6.4)$$

and the transfer function from d to y is

$$S(s) = (Y_2 - N_2 Q_2)\bar{D}_2. \quad (6.5)$$

There is a helpful separation here: Q_1 affects tracking but not disturbance behaviour, and Q_2 affects disturbance but not tracking behaviour.

6.2. *Introducing $Q(s)$ into a controller in state-space form*

The idea of introducing $Q(s)$ into a controller obtained by combining a state estimator and a state feedback law goes back at least to Tay and Moore (1988). The key is to pick nice associated fractional descriptions of the plant and controller.

Consider a plant $P(s)$ with minimal realization

$$P(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]. \quad (6.6)$$

[The notation is to be understood as saying $P(s) = C(sI - A)^{-1}B$.] Let F be such that $\mathrm{Re}\,\lambda_i(A + BF) < 0$ and H such that $\mathrm{Re}\,\lambda_i(A + HC) < 0$. It is well-known that an observer-based stabilizing controller is defined by

$$\dot{x}_c = (A + HC)x_c - Hy + BFx_c, \quad (6.7)$$

$$u = Fx_c. \quad (6.8)$$

This controller is depicted in Fig. 5. Coprime factorizations $ND^{-1} = \bar{D}^{-1}\bar{N}$ of the plant and $XY^{-1} = \bar{Y}^{-1}\bar{X}$ of the controller obeying the Bezout identity (3.3) are given by (Nett *et al.*, 1984).

$$\begin{bmatrix} D(s) & -X(s) \\ N(s) & Y(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C & 0 & I \end{array}\right], \quad (6.9)$$

$$\begin{bmatrix} \bar{Y}(s) & -\bar{X}(s) \\ -\bar{N}(s) & \bar{Y}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A+HC & -B & H \\ \hline F & I & 0 \\ C & 0 & I \end{array}\right]. \quad (6.10)$$

(More complicated formulas hold if there are direct feedthrough terms present.) It is not hard to verify for Fig. 5 that

$$Y(s)\vartheta = [I - C(sI - A - BF)^{-1}H]\vartheta = -y,$$

$$X(s)\vartheta = F(sI - A - BF)^{-1}H\vartheta = u,$$

from which the right fractional representation YX^{-1} of the controller follows. The Youla–Kucera

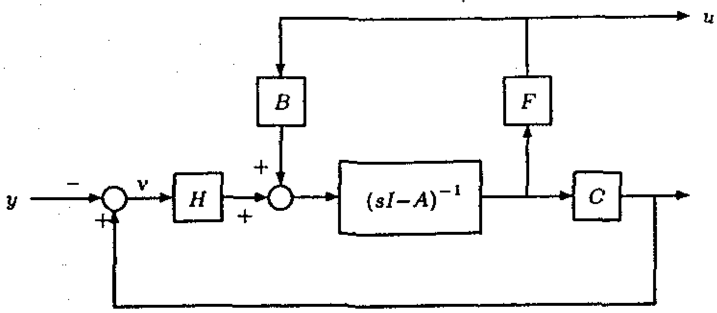

Fig. 5. Observer-based stabilizing controller.

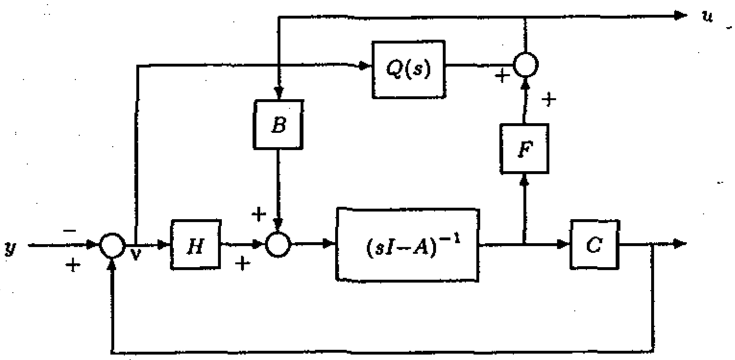

Fig. 6. Introduction of a Youla–Kucera parameter $Q(s) \in \mathscr{S}$ into an observer-based stabilizing controller.

parameter $Q(s) \in \mathscr{S}$ is introduced as shown in Fig. 6, and it is not hard to check that

$$[Y(s) - N(s)Q(s)]\vartheta = -y$$

$$[X(s) + D(s)Q(s)]\vartheta = u$$

so that

$$u = -[X + DQ][Y - NQ]^{-1}y.$$

Similarly, it can be argued that

$$u = -[\tilde{Y} - Q\tilde{N}]^{-1}[\tilde{X} + Q\tilde{D}]y.$$

A number of applications of this idea can be noted; for example:

- For a nonminimum phase plant, an LQG or H_2 design cannot achieve loop transfer recovery. However, a $Q(s) \in \mathscr{S}$ can be selected to trade off H_2 optimality and loop transfer recovery (Moore and Tay, 1989).
- Low-order adaptive adjustments can be made of a high-order controller connected to a nominal plant. An adaptive Q (of low-order) can be used to augment a fixed (possibly high order) controller connected to a nominal plant, where there are plant perturbations or uncertainties. In the discrete time case, Q can be taken as a finite impulse response transfer function; see Chakravarty and Moore (1986) and Tay and Moore (1991).

Above, for a controller of a particular structure, we identified in equation (6.2) left and right coprime fractional descriptions of the plant and controller, in state-space form and satisfying the Bezout identity (3.3). However, it is possible for *any* stabilizing controller to construct such fractional descriptions (Chakravarty and Moore, 1986). Suppose the controller is

$$C^0(s) \stackrel{s}{=} \left[\begin{array}{cc} A^c & B^c \\ C^c & D^c \end{array}\right]. \tag{6.11}$$

Let F and F^c be stabilizing state feedback gains such that $\operatorname{Re} \lambda_i(A + BF) < 0$ and $\operatorname{Re} \lambda_i(A^c + B^cF^c) < 0$. Then

$$\left[\begin{array}{cc} D(s) & -X(s) \\ N(s) & Y(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|cc} A + BF & 0 & B & 0 \\ 0 & A^c + B^cF^c & 0 & -B^c \\ \hline F & C^c - D^cF^c & I & -D^c \\ C & F^c & 0 & I \end{array}\right], \tag{6.12}$$

$$\left[\begin{array}{cc} \tilde{Y}(s) & \tilde{X}(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|cc} A + BD^cC & BC^c & B & -BD^c \\ -B^cC & A^c & 0 & -B^c \\ \hline -(F + D^cC) & C^c & I & D^c \\ C & -F^c & 0 & I \end{array}\right]. \tag{6.13}$$

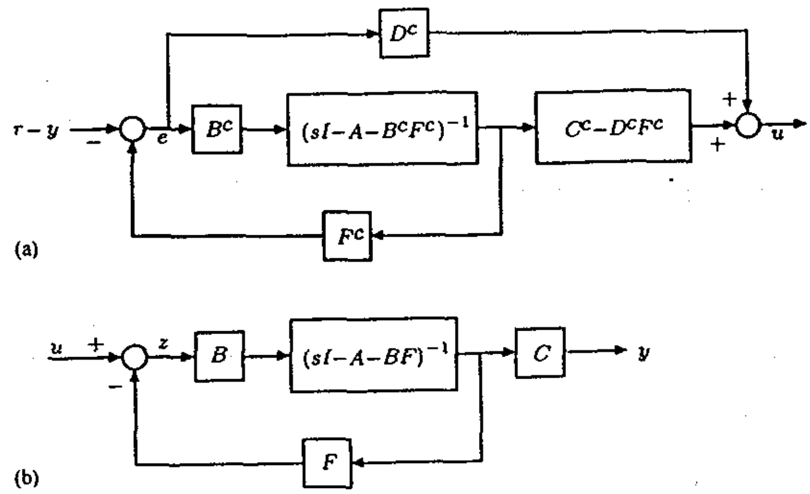

Fig. 7. (a) Controller $C^0(s)$, displaying right coprime factorization (b) Plant $P^0(s)$, displaying right coprime factorization.

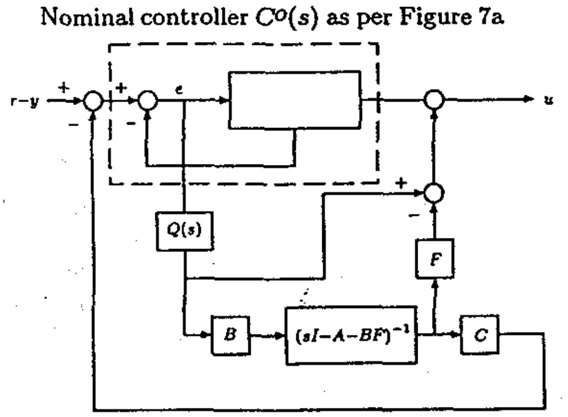

Fig. 8. Stabilizing controller, using Youla parameter $Q(s)$ and right coprime factorization of nominal controller and of plant.

There is a second pair of such definitions, using H and H^c such that $\mathrm{Re}\,\lambda_i(A+HC)<0$ and $\mathrm{Re}\,\lambda_i(A^c+C^cH^c)<0$.

If $C^0(s)$ is in fact $F(sI-A-HC-BF)^{-1}H$ then the choice $F^c=-C$ yields the pair (6.2).

Figures 7 and 8 illustrate the form of a general stabilizing controller, based on the use of equation (6.12). The order of a controller obtained when introducing a $Q(s)$ has the potential to be large. It may be that a controller order reduction step (Anderson and Moore, 1990) must be employed after determination of $Q(s)$.

6.3. *Pole positioning in LQG or H_∞ design*

Consider an LQG problem for the plant

$$\dot{x}=Fx+Gu+\Gamma w,$$

$$y=Hx+v,$$

where $\{F,G,H\}$ is minimal, and $[w'\ v']'$ is a zero mean, Gaussian white noise process with covariance

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\delta(t-s) \quad \text{where } R \text{ is nonsingular.}$$

Suppose one is interested in minimizing the expected value of the quadratic index

$$V=\int_{t_0}^{\infty}(x^TQx+u^TRu)\,dt,$$

where $Q=Q^T\geq 0$ and all the usual side conditions apply. Suppose also one wishes all closed-loop poles to lie in $\mathrm{Re}[s]<-\alpha$ for some $\alpha>0$. An old device is to replace the index by

$$V_\alpha=\int_{t_0}^{\infty}e^{2\alpha t}(x^TQx+u^TRu)\,dt;$$

see Anderson and Moore (1970). Minimizing the expected value of V_α ensures that all closed-loop poles lie in $\mathrm{Re}[s]<-\alpha$. One can then evaluate $E(V)$ for the design resulting from use of the V_α index. It turns out that on occasions much lower values of $E(V)$ could be achieved from a controller of prescribed order which achieves the closed-loop pole constraint; thus minimizing $E(V_\alpha)$ rather than $E(V)$ may lead to very unsatisfactory results as far as $E(V)$ is concerned.

The Youla–Kucera parametrization provides an alternative approach, which is closer to minimizing $E(V)$, and which allows a wider choice of closed-loop pole positions (DeBruyne *et al.*, 1995).

Suppose the closed-loop pole positions are required to lie in a domain $\mathcal{D}$, contained within $\mathrm{Re}[s]<0$, symmetric with respect to the real axis, and (for technical reasons) containing at least one point on the negative real axis. For example,

$$\mathcal{D}=\{s:\mathrm{Re}[s]<-\alpha,\ |\mathrm{Im}[s]|\leq\tan\theta|\mathrm{Re}[s]|,\ \alpha>0\}.$$

The plant can be represented as a coprime fraction ND^{-1}, where $N, D \in \mathcal{S}_D$, the set of proper rational transfer function matrices with poles in $\mathcal{D}$. Coprimeness is equivalent to the existence of $X, Y \in \mathcal{S}_D$ with $XN + YD = I$.

Now $\mathcal{S}_D \in \mathcal{S}$, the set of all stable proper transfer functions. Therefore, the result of solving the normal LQG problem with no closed-loop pole constraint other than simple stability will be a stabilizing controller $C(s)$ with fractional description representable as $(X + DQ)(Y - NQ)^{-1}$ for some $Q \in \mathcal{S}$.

Were $Q \in \mathcal{S}_D$, the closed-loop poles will be in $\mathcal{D}$. In general, $Q \notin \mathcal{S}_D$. However, one can approximate Q by some $Q_{\mathcal{G}} \in \mathcal{S}_D$, and secure thereby approximately the same value of $E(V)$, but with all closed-loop poles in $\mathcal{D}$. Indeed, if the degree of $Q_{\mathcal{G}}$ can be arbitrarily large, the approximation can be arbitrarily accurate. In general, we do not want the degree of $Q_{\mathcal{G}}$ and thus of $C(s)$ to be arbitrarily large.

The same idea is valid for H_∞ design.

7. PARAMETRIZATION OF THE PLANT AND CLOSED-LOOP IDENTIFICATION

In this section, we first consider the dual question to that treated earlier of how to parametrize all plants (in addition to a nominal one) which are stabilized by a prescribed controller.

In a unity feedback loop with series comparator, with a scalar plant, it is clear that $C(s)$ and $P(s)$ can be interchanged without affecting the closed-loop transfer function. This strongly suggests that there should be an analogous result to Theorem 1 dealing with the *set of all plants stabilized by a fixed controller*. Indeed, that is so:

Theorem 2. Let a plant $P = ND^{-1}$, with N and D coprime over $\mathcal{S}$, be stabilized by a (negative feedback) controller $C = XY^{-1}$, with X and Y coprime over $\mathcal{S}$. Then the set of all plants stabilized by the one controller C is given by $\{(N + YS)(D - XS)^{-1}: S \in \mathcal{S}\}$.

There is an extremely important application of this idea, to closed-loop identification in the presence of noise.

7.1. *Open and closed-loop identification*

To motivate the problem, first consider how open-loop identification can be carried out, (see Fig. 9). In a common scenario, the input process u and noise process n are assumed independent and stationary. The plant is scalar, time-invariant and stable, and we can write

$$y = Pu + n. \tag{7.1}$$

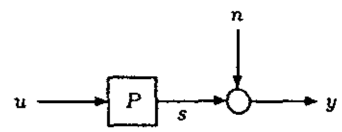

Fig. 9. Set-up for open-loop identification.

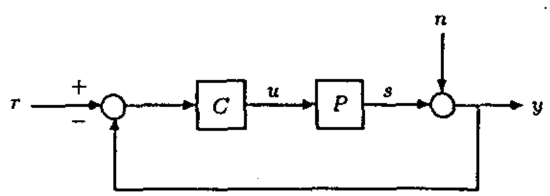

Fig. 10. Set-up for closed-loop identification.

Measurements of u and y are available. Many identification schemes for estimating P effectively amount to cross-correlating with u and then solving for P:

$$\Phi_{yu}(s) = P(s)\Phi_{uu}(s) + \Phi_{nu}(s) \tag{7.2}$$

$$= P(s)\Phi_{uu}(s). \tag{7.3}$$

Of course, equation (7.3) follows because u and n are made independent. In actual practice, the calculation may be done in the time domain, and sample estimates (obtained over a finite interval) may be used instead of true expectations, etc., but the basic principle is the same.

Now consider the stable closed loop of Fig. 10, with r and n independent processes. The task is to identify $P(s)$ from measurements which do not include its noiseless output s. Equations (7.1) and (7.2) remain true. However, equation (7.3) is replaced by

$$\Phi_{yu}(s) = P(s)\Phi_{uu}(s) - (1 + C^*P^*)^{-1}C^*\Phi_{nn}. \tag{7.4}$$

The superscript asterisk denotes replacement of s by $-s$, or complex conjugation on the $j\omega$ axis. Evidently, the task of obtaining $P(s)$ is considerably more complicated. Also, even if Φ_{nn} is very small, software packages may be predicated on an assumption that $P(s)$ is stable, which need not be so.

One can of course seek to identify the closed-loop transfer function $G = PC(1 + PC)^{-1}$ from

$$\Phi_{yr} = G\Phi_{rr} \tag{7.5}$$

and then since $P = G/C(1 - G)$, one chooses an estimate $\hat{P}$ of P in terms of an estimate $\hat{G}$ of G as

$$\hat{P} = \frac{\hat{G}}{C(1 - \hat{G})}. \tag{7.6}$$

These can be problems with this approach. For example, if C has an unstable pole, the loop comprising $\hat{P}$ and C will generally be unstable.

Also, parametrizations which are convenient for $\hat{G}$ may be convenient for $\hat{P}$ and *vice versa*, in view of the nonlinear connections; in some cases too, parameter space regions corresponding to stable $\hat{P}$, C pairs may contain disconnected sub-regions (Van Donkelaar and Van den Hof, 1996).

A clever resolution of these difficulties was obtained independently in Tay *et al.* (1989), Hansen (1989), Hansen *et al.* (1989) and Schrama (1991).

Suppose that the controller $C(s)$ is specified as $\tilde{Y}^{-1}\tilde{X}$ where $\tilde{X}, \tilde{Y} \in \mathscr{S}$ are coprime. Then by the coprimeness there exist $N, D \in \mathscr{S}$ with

$$\tilde{X}N + \tilde{Y}D = I. \tag{7.7}$$

This means that $P_0(s) = ND^{-1}$ is one plant (a "nominal" plant) stabilized by $C(s)$. [Alternatively, one may start with a representation ND^{-1} of the nominal $P_0(s)$ and then choose that particular representation $\tilde{Y}^{-1}\tilde{X}$ of the known stabilizing $C(s)$ so that equation (7.7) holds.] The set of all plants stabilized by $C(s)$ is given by

$$\{[N(s) + Y(s)S(s)][D(s) - X(s)S(s)]^{-1} : S \in \mathscr{S}\}. \tag{7.8}$$

Here, XY^{-1} is a coprime right fractional representation of $C(s)$.

Suppose that $\tilde{D}^{-1}\tilde{N}$ is a left coprime fractional description of the nominal $P_0(s)$ so that

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{7.9}$$

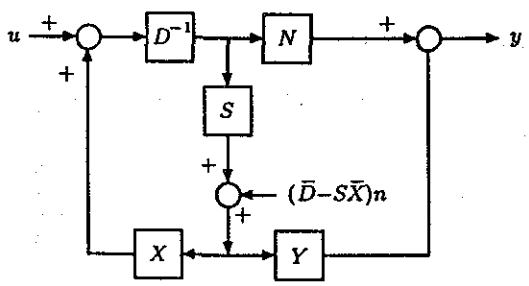

Fig. 11. Alternative representation of noise-contaminated plant.

Then it is not hard to check that the set-up of Fig. 11, with inputs u and n and output y, implies

$$y = Pu + n,$$

where P is given by equation (7.8).

Now the key identification idea is to identify S in the first instance, rather than P. The perhaps surprising and pleasing aspect is that the identification of S is *a standard open-loop identification problem.* To see this, observe from Fig. 12 that

$$\begin{aligned}(D - XS)x &= u + X(\tilde{D} - S\tilde{X})n \\ &= r_2 + \tilde{Y}^{-1}\tilde{X}r_1 - \tilde{Y}^{-1}\tilde{X}y \\ &\quad + X(\tilde{D} - S\tilde{X})n,\end{aligned}$$

and

$$(N + YS)x = y - Y(\tilde{D} - S\tilde{X})n.$$

Multiplying the first equation by $\tilde{Y}$ and the second by $\tilde{X}$ and adding yields

$$x = \tilde{X}r_1 + \tilde{Y}r_2. \tag{7.10}$$

Further

$$Dx = u + Xz,$$

$$Nx = y - Yz.$$

Multiplying the first equation by $\tilde{N}$, and the second by $\tilde{D}$ and subtracting yields

$$z = \tilde{D}y - \tilde{N}u, \tag{7.11}$$

while also direct inspection of Fig. 12 shows that

$$z = Sx + (\tilde{D} - S\tilde{X})n. \tag{7.12}$$

Observe in equation (7.12) that (i) x and z are available from measurements on the closed-loop systems (via equations (7.10) and (7.11)); (ii) if $n(\cdot)$ is independent of $r_1(\cdot)$ and $r_2(\cdot)$, then in equation (7.12), $x(\cdot)$ and $n(\cdot)$ are independent processes; and (iii) S is stable, via the Youla–Kucera theory. Hence, the identification of S is a standard open-loop identification problem. So, for that matter, is the identification of a shaping filter, which when driven by white noise, will generate n.

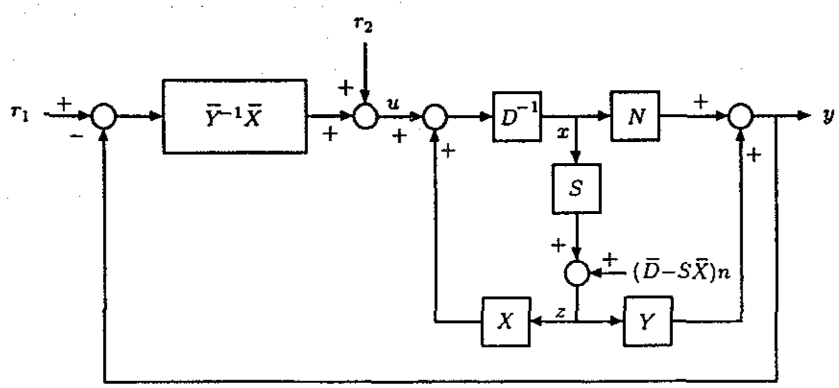

Fig. 12. Details of closed-loop identification.

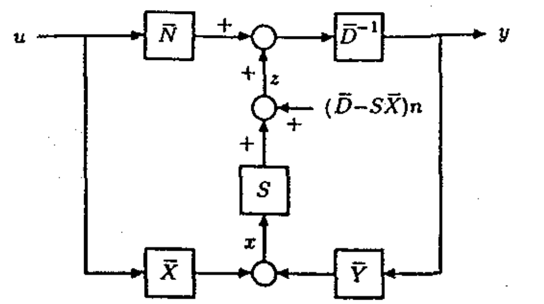

Fig. 13. Second alternative realization of noise-contaminated plant.

The blocks in Fig. 11, apart from S, are associated with right coprime fractional description of a nominal plant and a controller stabilizing the nominal and the true plant. It is also possible to work with left coprime realizations. Fig. 13 depicts a representation of the plant using $\bar{X}, \bar{Y}, \bar{N}$ and $\bar{D}$ instead of X, Y, N and D. The signals labelled x and z and blocks labelled S in Figs. 11 and 13 are the same, in that

$$\begin{aligned} P(s) &= (N + YS)(D - XS)^{-1} \\ &= (\bar{D} - S\bar{X})^{-1}(\bar{N} + S\bar{Y}) \end{aligned}$$

and for both schemes, equations (7.10) and (7.12) are valid when the plant is connected in a feedback loop with controller $XY^{-1} = \bar{Y}^{-1}\bar{X}$.

Since there is no convenient replacement of equations (7.10) and (7.12) in which entities associated with right rather than left coprime factorization appear, it is perhaps tidier to work with the scheme of Fig. 13 than that of Fig. 11.

A number of other points should be made:

(i) If $P(s)$ is known *a priori* to be strictly proper, and $P_0(s)$ is strictly proper, $S(s)$ must have this property, and conversely.

(ii) Suppose that for identification purposes a finitely parametrized model set $\{S_\alpha, \alpha \in A\}$ is adopted and one obtains S_{α^*} as some sort of best approximation to S. Let $P_{\alpha^*}(s)$ be the corresponding approximation to $P(s)$. Then it can be shown that

$$\begin{aligned} &PC(I + PC)^{-1} - P_{\alpha^*}C(I + P_{\alpha^*}C)^{-1} \\ &\quad = Y(S - S_{\alpha^*})\bar{X} \end{aligned} \tag{7.13}$$

and if $r_2 = 0$,

$$\begin{aligned} &[PC(I + PC)^{-1} - P_{\alpha^*}C(I + P_{\alpha^*}C)^{-1}]r \\ &\quad = Y(S - S_{\alpha^*})x. \end{aligned} \tag{7.14}$$

In both equations, the left-hand side is a modelling error associated with a closed-loop transfer function. The right-hand side is the open-loop modelling error associated with S, using a frequency weight Y.

A particular version of these formulas results when S_{α^*} is replaced by 0, corresponding to the nominal plant.

Then

$$PC(I + PC)^{-1} - P_0C(I + P_0C)^{-1} = YS\bar{X}, \tag{7.15}$$

$$[PC(I + PC)^{-1} - P_0C(I + P_0C)^{-1}]r = YSx. \tag{7.16}$$

The expressions relate S to a closed-loop quantity.

(iii) One can also note that

$$P - P_0 = \bar{D}^{-1}S(D - XS)^{-1}, \tag{7.17}$$

which shows that a frequency weighted version of S yields the open-loop error between P and P_0; one of the weights depends itself on $\mathscr{S}$.

(iv) The identification of a P through P_0 and then S can give rise to a possibly large degree for P. If subsequent identifications refine knowledge of P even further, the degree may increase at each identification step. For this reason, it often makes sense to do an order reduction of an identified P. Since it is the closed-loop obtained by using P with C which is relevant, one seeks a low-order $\hat{P}$ so that a quantity such as the following is minimized:

$$\|PC(I + PC)^{-1} - \hat{P}C(I + \hat{P}C)^{-1}\|_\infty.$$

This is approximately (neglecting second-order error terms)

$$\|(I + PC)^{-1}(P - \hat{P})C(I + PC)^{-1}\|_\infty$$

and a frequency weighted model reduction problem is to be tackled.

(v) It can be argued that a normalized coprime fractional description of P_0 is best used (Van den Hof *et al.*, 1995). One of the reasons for this is that it is possible to flag situations where controller redesign after identification (Zang *et al.*, 1995) may be problematic. We explain this point in more detail. Let N, D be a normalized right coprime fractional description of P_0, so that $N_*N + D_*D = I$. Let $\tilde{P}$ be any other plant, with any right coprime fractional description $\tilde{N}, \tilde{D}$. Define the directed distance from P_0 to P by

$$\delta(P_0, P) = \inf_{U \in \mathscr{S}} \left\| \begin{bmatrix} N \\ D \end{bmatrix} - \begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix} U \right\|_\infty.$$

Let

$$\left\| \begin{bmatrix} P_0C(I + P_0C)^{-1} & P_0(I + CP_0)^{-1} \\ (I + CP_0)^{-1}C & (I + CP_0)^{-1} \end{bmatrix} \right\|_\infty$$

$$= \gamma^{-1}.$$

Then C stabilizes all P with $\delta(P_0, P) < \gamma$ (Georgiou and Smith, 1990).

Now suppose that we use the identification procedure for P described above. In the presence of noise, it may not yield the correct S, but rather some $\hat{S}$. Let the associated plant be designated by $\hat{P}$. If (N, D) is normalized, we have

$$\delta(P_0, \hat{P}) = \inf_{U \in \mathcal{S}} \left\| \begin{bmatrix} N \\ D \end{bmatrix} - \begin{bmatrix} N + Y\hat{S} \\ D - X\hat{S} \end{bmatrix} U \right\|_\infty$$

$$\leq \left\| \begin{bmatrix} N \\ D \end{bmatrix} - \begin{bmatrix} N + Y\hat{S} \\ D - X\hat{S} \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{pmatrix} Y \\ X \end{pmatrix} \hat{S} \right\|_\infty$$

We know that C stabilizes P (by experiment), stabilizes P_0 (by calculation) and stabilizes $\hat{P}$ (by the theorem on Youla–Kucera parametrization). However, a condition like

$$\left\| \begin{pmatrix} Y \\ X \end{pmatrix} \hat{S} \right\|_\infty > \gamma^{-1}$$

is a flag that $\delta(P_0, \hat{P})$ may also exceed γ^{-1}, so that the difference between P_0 and $\hat{P}$ is in some sense substantial; redesign of a controller should therefore proceed cautiously; see Bitmead *et al.* (1997). Even controllers close to C resulting from a redesign process could have stability problems.

7.2. *Noisy identification*

We have explained that the basis of the identification problem is to use equation (7.12). It is important to consider at any one frequency what the signal-to-noise ratio is, and how it is related to quantities in the original actual and model closed-loop.

For simplicity, suppose $r_2 = 0$ and that plant and controller are scalar; thus $\tilde{D} = D$, $\tilde{N} = N$, etc. Then in equation (7.12), the signal-to-noise ratio is

$$\left| \frac{S}{D - SX} \right|^2 \frac{\Phi_{xx}}{\Phi_{nn}} = \left| \frac{SX}{D - SX} \right|^2 \frac{\Phi_{r_1 r_1}}{\Phi_{nn}}.$$

(Here Φ is a generic symbol used to denote a spectrum.) It is not hard to check the error between that part of the actual closed-loop output due to r_1 and the model output is

$$\left(\frac{PC}{1 + PC} - \frac{P_0 C}{1 + P_0 C} \right) r_1 = SXYr_1,$$

while the noisy component of the actual closed-loop output is $(1/(1 + PC))n = (D - XS)Yn$.

The associated signal to noise ratio is then

$$\left| \frac{\text{tracking error "signal"}}{\text{Noise}} \right|^2 = \left| \frac{SX}{D - SX} \right|^2 \frac{\Phi_{r_1 r_1}}{\Phi_{nn}},$$

which is the same quantity as the signal to noise ratio appearing in the identification step. This is an important observation; it means that despite the collection of filters and signal transformations which arise in setting up the open-loop identification, there are no choices that could improve or worsen the signal-to-noise ratio which constitutes a practical measure of the difficulty of identifying, and which is the SNR relevant in assessing the effectiveness of a closed-loop identification followed by loop unravelling.

7.3. *Windsurfer approach to adaptive control*

The ideas of Section 7.1 are crucial to the development of a methodology for adaptive control which is much less based on identifying numerator and denominator coefficients in plant or controller transfer functions than in an iterative identification and controller design scheme, tied to the frequency domain (Lee *et al.*, 1992, 1993, 1995).

The original motivation was to do adaptive control robustly, i.e. without incurring "temporary" instability in the learning phase. By way of example, conventional adaptive control can encounter substantial difficulty if there are high-frequency dynamics in the true plant not reflected in an *a priori* model of the plant, and possibly not even capable of being reflected by adjusting coefficients in the *a priori* model, due to too low a degree having been adopted for it.

Windsurfer adaptive control was first developed for open-loop-stable plants (permitted though to have a pole or poles at the origin). The key idea (as when a human learns windsurfing) is to initially use a controller defining a very small closed-loop bandwidth; design of this controller requires almost no more prior knowledge than the sign of the DC gain of the plant.

A series of redesigns of the controller is then effected, each pushing out the closed-loop bandwidth some more. At some stage, predicted and actual closed-loop performance becomes clearly different; at this point, the plant is better identified using the scheme of Section 7.1. This new identification will then cause the closed-loop transfer function obtained using the new model of the plant and that obtained using the actual model of the plant to be far more similar. In effect, knowledge of the plant is developed over a wider bandwidth than previously (with the quality of that knowledge linked to closed-loop rather than open-loop behaviour).

With the re-identified plant, the controller design is then adjusted, progressively expanding the closed-loop bandwidth until again there is divergence between behaviour as predicted by the model and the actual behaviour. A further (closed-loop) identification is executed, thereby further widening the bandwidth over which the plant is adequately described (for the purpose of closed-loop control). Once again, closed-loop bandwidth is expanded via a series of controller design, and so on.

The procedure thus gives rise to a sequence of (strictly) proper stable plant models, $\{G_0, G_1, \ldots\}$ and for each model G_i there is a sequence of controllers $K_i^0, K_i^1, \ldots K_i^f$ giving progressively larger bandwidths for the closed-loop transfer function $G_iK_i^j(I + G_iK_i^j)^{-1}$.

A great number of important questions arise:

- Is there a scheme for progressively increasing the closed-loop bandwidth? Yes, for stable plants at least. The IMC method (Morari and Zafiriou, 1989) is ideal, allowing direct control over closed-loop bandwidth.
- When can the closed-loop bandwidth be increased with safety, i.e. without losing stability, while retaining use of the (possibly inaccurate) model G? There can be no sudden onset of instability that is not preceded at a somewhat lesser design bandwidth by significant difference between predicted and actual closed-loop behaviour.
- What *would we like* to identify, in order that with the new model, performance of the closed-loop system can be improved through controller redesign? It turns out that we would like to replace the model G_i by a model G_{i+1} so that, with G the true plant and $\bar{T}_{i+1}$ the designed closed-loop,

$$\left\| \frac{G - G_{i+1}}{G_{i+1}} \bar{T}_{i+1} \right\|_\infty < 1$$

and

$$\left| \frac{G - G_{i+1}}{G_{i+1}} \bar{T}_{i+1} \right|$$

is small for all frequencies above the *lesser* of the bandwidth of $\bar{T}_{i+1}$ and the smallest frequency corresponding to zeros if G_{i+1} in $\mathrm{Re}[s] \geq 0$.
- What *can we identify* (as opposed to what would we like to identify)? As we know, we can identify a Youla–Kucera parameter S, albeit in the presence of noise. However, we can only identify it accurately if a certain signal-to-noise ratio is high. "Signal" is the error between that part of the true plant output which is due to input excitations and the model output (which is due solely to input excitations). "Noise" is that part of the plant output arising from disturbances. The signal-to-noise ratio is only likely to be high in certain frequency ranges. Further, as pointed out in the last subsection, it is this signal-to-noise ratio that is also directly relevant in the open-loop identification of the Youla–Kucera parameter S.
- *When will there be a significantly high signal-to-noise ratio allowing identification of* S *in a frequency band?* The instinctive answer is almost correct: there must be a failure of the model to correctly predict closed-loop performance of the controller with the true plant. But there is a surprise qualification. Nonminimum phase zeros in the passband of the plant and or model can mean that noise disturbances cannot be well cancelled by the controller. Then "signal" and noise can be high but the *ratio* may not be high, and reidentifiability is hard.

Roughly speaking it turns out that what we want to identify and what we can identify coincide, until nonminimum phase zeros appear in the closed-loop passband. Then the scheme will come to a stop in this situation; it may also stop when the open-loop bandwidth of the true plant is exceeded by the proposed design bandwidth, so that unreasonably large plant inputs are demanded.

Some important points should be made.

(i) The initial model may be very low order, even if it is suspected that the plant has high-frequency resonances.

(ii) The scheme does not permit instabilities to occur, nor for that matter adverse transient behaviour; the latter especially is a concern for conventional adaptive control schemes, especially if there are unknown high frequency resonances in the plant.

(iii) Generally, the Youla–Kucera parameter in the plant re-identification step need only be of order 2–4, since a modest adjustment of the model applicable over a limited bandwidth is what is required.

(iv) If the true plant is unstable, control of closed-loop bandwidth by adjustment of a single parameter in the IMC design scheme is not straightforward. Also, it may be hard to find an initially stabilizing controller. Only preliminary work has been done on the adaptive control scheme for such plants.

Why there is iteration in the first place? Starting with minimal information, the iterative approach, because it is gradual, allows an adaptive controller to be found without risking "transient instability", i.e. the occurrence of massive signals, in practical terms indistinguishable from those in an unstable system, during the learning phase of the adaption of the adaptive controller. Those "conventional" adaptive controllers have great difficulty in protecting against transient instability, especially those that presuppose an order for the unknown plant. There is a risk that the assumed order is too low to capture high-frequency resonances, and too high to allow inputs to be persistently exciting; either way, *trasient instability* can result.

New experiments are essential, to exceute progressively wider bandwidths. At early iterations, only low bandwidth inputs are applied to the plant. In later iterative steps, when higher-frequency behaviour is being learnt, higher-frequency plant excitation is required. This is achieved by ensuring that the closed-loop external input is of sufficient bandwidth and also the controller bandwidth is such as to let through signals of adequate bandwidth.

8. YOULA-KUCERA PARAMETERS IN PLANT AND CONTROLLER

We have so far dealt with a set of controllers stabilizing one plant, and a set of plants stabilized by one controller. We are now going to mix these themes.

8.1. *The stability issue*

We begin with a question. Consider a nominal plant-controller pair, $P^0(s) = N(s)D^{-1}(s) = \bar{D}^{-1}(s)\,\bar{N}(s)$ and $C^0(s) = X(s)Y^{-1}(s) = \bar{Y}^{-1}(s)\bar{X}(s)$. We assume that N, D, etc., are elements of $\mathcal{S}$ and

$$\begin{bmatrix} \bar{Y} & \bar{X} \\ -\bar{N} & \bar{D} \end{bmatrix}\begin{bmatrix} D & -X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (8.1)$$

For what pairs $Q(s), S(s) \in \mathcal{S}$ will the plant $P(s) = [N(s) + Y(s)S(s)][D(s) - X(s)S(s)]^{-1}$ and controller $C(s) = [X(s) + D(s)Q(s)][Y(s) - N(s)\ Q(s)]^{-1}$ define a stable closed-loop? The answer is (Tay *et al.*, 1989).

Theorem 3. Let $P^0 = ND^{-1} = \bar{D}^{-1}\bar{N}$ and $C^0 = XY^{-1} = \bar{Y}^{-1}\bar{X}$ with equation (8.1) holding and $N, D, \ldots, \in \mathcal{S}$ define a stable unity negative feedback closed loop. Then $P = [N + YS][D - XS]^{-1}$ and $C = [X + DQ][Y - NQ]^{-1}$ for S and $Q \in \mathcal{S}$ defines a stable unity negative feedback closed loop if and only if S and Q together define a stable loop.

Proof. P and C will form a stable loop if and only if

$$\begin{bmatrix} D - XS & -X - DQ \\ N + YS & Y - NQ \end{bmatrix}^{-1} \in \mathcal{S}$$

$$\Leftrightarrow \left\{\begin{bmatrix} D & -X \\ N & Y \end{bmatrix}\begin{bmatrix} I & -Q \\ S & I \end{bmatrix}\right\}^{-1} \in \mathcal{S}$$

$$\Leftrightarrow \left\{\begin{bmatrix} I & -Q \\ S & I \end{bmatrix}^{-1}\right\} \in \mathcal{S},$$

which holds if and only if S and Q together define a stable loop.

The fact that $S, Q \in \mathcal{S}$ is not directly used. Note also that it is not hard to vary the above argument to show that generically the closed-loop poles of the $P(s), C(s)$ system are the union of those of the $P^0(s), C^0(s)$ systems and the S, Q system (Tay *et al.*, 1989). □

The above result raises the following issue. Suppose a controller design has been achieved for a nominal plant, and then more information concerning the plant becomes available (i.e. $S(s)$ is learnt, at least approximately). How should the controller be adjusted? Equivalently, how should $Q(s)$ be chosen?

At the broadest level, the answer depends on the design criterion, and we look at two (LQG and H_∞) below. Of course, one could imagine repeating a whole design process with a new $P(s)$ (involving $S(s)$), obtaining $C(s)$, from which one could then devise $Q(s)$. This however is not the point of the question; the key idea is to see whether $Q(s)$ could be obtained from $S(s)$ via some simple formula. Thus, we could imagine that if the nominal $P^0(s)$ and $C^0(s)$ were of high order, 40 say, and $S(s)$ was second order (associated with the inclusion of a further resonant mode in $P(s)$ for example), then $Q(s)$ might be second order also. The discussion of closed-loop poles above shows that if their positions are the prime concerns, one could proceed in exactly this way, i.e. computing $Q(s)$ directly from $S(s)$.

8.2. *LQG design*

The problem of relating a plant perturbation through to a controller perturbation for LQ design is treated in Anderson *et al.* (1994). We shall indicate here simply the nature of the answer. First, it is only possible to get a simple relationship between S and Q when S is small. Then the form of the result is (for a scalar plant)

$$Q \simeq A^{-1}[A^{-*}S^*DD^*\Phi]_{\text{stable}}. \quad (8.2)$$

In this formula, A is the minimum phase spectral factor of an entity formed in solving the LQ problem for P^0 and C^0, $X^*(s)$ denotes $X(-s)$, Φ is the spectrum of the exciting noises (and may be white), and $[Z]_{\text{stable}}$ denotes the stable part of a partial fraction expansion of Z.

This formula has not been reworked using state variable calculations to try to relate the degrees of S and Q.

Retaining the assumption of small S, it is possible to obtain an expression for the change in quadratic cost. It is of the form

$$\text{cost change} \simeq \frac{1}{2\pi}\int [B_*Q + Q_*B]\,\mathrm{d}\omega. \quad (8.3)$$

where B depends on P^0 and C^0.

A second approach to LQ design is set out in Tay *et al.* (1989). A design for P^0 is done to yield C^0. Then a separate design for S is done to yield Q; the input and output of S appear in the index. No real guidance is given as to how to select weights for this second problem, although it is argued that for the second problem, having constant rather than freqency-dependent weights gives an index which is logical in terms of the original objective of minimizing through choice of $C(s)$ a quadratic index involving $P(s)$.

The degree of $Q(s)$ is effectively that of $S(s)$, which is an advantage. An example in Tay *et al.* (1989) shows the efficacy of the approach.

It would certainly be of interest to quantify the loss of optimality of a design achieved by the two-step procedure (P^0 gives C^0 and S gives Q) in comparison with an optimal design (P gives C).

8.3. H_∞ *design*

The set-up here starts with a generalized plant

$$P^0(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P^0_{22}(s) \end{bmatrix} \tag{8.4}$$

of the type shown in Fig. 2 and an associated H_∞ controller $C^0(s)$. We suppose that $P^0_{22} = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ and $C^0 = XY^{-1} = \tilde{Y}^{-1}\tilde{X}$, with the usual Bezout equation

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ -\tilde{N} & \tilde{D} \end{bmatrix}\begin{bmatrix} D & -X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

[Note the sign convention in Fig. 2 allows us to regard $P^0_{22}(s)$ and C^0 as forming a unity negative feedback loop.] All standard conditions for solvability of the H_∞ problem are assumed to be fulfilled and also in the original reference (Yan and Moore, 1993) P_{12} and P_{21} are assumed stable, though this seems unnecessary if all unstable modes of $P^0(s)$ appear in $P^0_{22}(s)$.

Suppose now that P^0_{22} is replaced by $P_{22} = (N + YS)(D - XS)^{-1}$ and let $P(S)$ denote the corresponding generalized plant. Then there is a clever way of choosing $Q(s)$, as follows. Define a second generalized plant involving S alone, by

$$P^S = \begin{bmatrix} P^S_{11}(s) & P^S_{12}(s) \\ P^S_{21}(s) & P^S_{22}(s) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} S \\ I \end{bmatrix} \\ [I - S] & S \end{bmatrix}. \tag{8.5}$$

Let $F_L(P, C)$ denote the system arising from interconnecting a generalized plant P with a controller C, as illustrated in Fig. 2. Then (Yan and Moore, 1993)

$$\|F_L(P(S), C(Q)) - F_L(P^0, C^0)\|_\infty \le \alpha \|F_L(P^S, Q)\|_\infty, \tag{8.6}$$

where

$$\alpha = \|P_{12}[D \quad -X]\|_\infty \left\| \begin{bmatrix} \tilde{X} \\ \tilde{D} \end{bmatrix} P_{21} \right\|_\infty. \tag{8.7}$$

Evidently, the idea is to choose Q to keep the right-hand side of equation (8.6) small; then there is supposed to be good matching on the left-hand side. The expression $F_L(P^S, Q)$ evaluates as

$$F_L(P^S, Q) = \begin{bmatrix} SQ(I - SQ)^{-1} & (I - SQ)^{-1}S \\ Q(I - SQ)^{-1} & -(I - QS)^{-1}QS \end{bmatrix}.$$

It is undoubtedly clear that if $\|S\|_\infty$ is small then Q can be chosen with $\|Q\|_\infty$ small and $\|F_L(P^S, Q)\|_\infty$ also small. What is not clear is how much damage the multiplier α could do. An examination of Yan and Moore (1993) shows that α is a crude bounding quantity; it may be possible to improve on the bound, and/or it may be possible to identify structures where α is guaranteed to be small.

The above has one clear advantage; finding Q knowing S does not involve $P^0(s)$ or $C^0(s)$, and the degrees of Q and S will be comparable if Q is found via H_∞ methods.

8.4. *Indirect adaptive control and iterative controller design*

The potential for indirect adaptive control using the ideas of this and the previous section should be clear. Methods tackling closed-loop identification by reduction to an open-loop problem allow correction of a nominal plant model $P_0(s)$ by a stable $S(s)$; then a pole-positioning, LQG or H_∞ approach allows derivation of a $Q(s)$ (Tay *et al.*, 1989). Of course, $S(s)$ can be slowly varying, requiring on-line technology. Then $Q(s)$ must be subject also to update.

In one sense, iterative controller design (Zang *et al.*, 1995) is a slowed-up form of indirect adaptive control. In a recent development of the ideas of Zang *et al.* (1995) and Bitmead *et al.* (1997), a clever formula has been derived which neatly encapsulates the effects of plant and controller variation. In the scalar case, suppose

$$P_0 = D_0^{-1}N_0, C_0 = Y_0^{-1} X_0$$

with

$$D_0Y_0 + N_0X_0 = 1.$$

Suppose further that

$$P_1 = (D_0 + Q_1X_0)^{-1}(N_0 - Q_1Y_0) = D_1^{-1}N_1$$

is stabilized by C_0 and P_1 is also stabilized by

$$C_1 = (Y_0 + S_1N_1)^{-1}(X_0 - S_1D_1)$$

(Of course, D_0, N_0, etc., are all stable transfer functions). Then

$$\mathcal{T}(P_0, C_0) = \begin{bmatrix} \frac{1}{1+P_0C_0} & \frac{C_0}{1+P_0C_0} \\ \frac{P_0}{1+P_0C_0} & \frac{P_0C_0}{1+P_0C_0} \end{bmatrix} = \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} [Y_0 \quad X_0]$$

and

$$\mathcal{T}(P_1, C_1) = \begin{pmatrix} X_0 & D_0 \\ -Y_0 & N_0 \end{pmatrix} \begin{pmatrix} Q_1 \\ 1 \end{pmatrix} (S \quad 1 - S_1Q_1) \times \begin{pmatrix} N_0 & -D_0 \\ Y_0 & X_0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} X_0 & D_0 \\ -Y_0 & N_0 \end{pmatrix}\begin{pmatrix} N_0 & -D_0 \\ Y_0 & X_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

8.5. *Open issues*

A key question raised at the start of this section was: how should the adjustment $Q(s)$ of a controller be determined, knowing an adjustment $S(s)$ of the plant. The answers to this point have been incomplete:

- *Pole positioning*. The incompleteness arises because, firstly, pole positioning by itself is a limited design objective and, secondly, the result is only generic.
- *LQG*. The incompleteness also arises because either one stays with the original performance index and then only gets simplification, albeit limited, when adjustments are small; or one uses a different performance index for the S–Q optimization, which cannot give an overall optimal result.
- H_∞: The incompleteness arises because of an inequality in the key result, and an appearance of a constant α with rather unclear intuitive content. In particular, even if an H_∞ controller $C(Q)$ exists for a plant $P(S)$ (to achieve internal stability and a given performance level), there is no guarantee that one can find it by using an index based on a plant P^S (determined solely by S) and a feedback controller Q.

Stabilizing $P(S)$ with $C(Q)$ knowing that P^0 is stabilized by C^0 is completely equivalent to stabilizing S with Q; thus there is a decoupling of the stability problem for $P(S)$, $C(Q)$ into two, namely P^0, C^0 and S, Q. Unfortunately, there is less tidy decoupling as soon as one introduces other goals of control apart from stability, and there may still be much scope for improving the decoupling. Two ideas which should be explored are the use of normalized coprime realizations, and consideration of S which have significant values of $\|S(j\omega)\|$ only where $P^0(j\omega)$ is approximately constant or even very small, thereby offering scope for a decoupling in the frequency domain that could feed through into the performance objectives.

8.6. *Broader issues again*

There are some linear system problems that Youla–Kucera parametrizations cannot address. We give two examples. First, suppose the controller is structured, e.g. decentralized, or hierarchical in some specific way; there is not a simple parametrization of all such stabilizing controllers. Second, suppose that a continuous-time plant is controlled by a sampled-data controller, and the plant is to be identified in closed loop. Once again, the theory is in trouble.

These deficiences are the more serious in that they point to an apparent non-universality of the Youla–Kucera parametrization idea, and its non-applicability to classes of control systems of increasing importance (multi-agent/hybrid systems). The situation is however partly rescued by the fact that the idea does carry over to nonlinear systems, as we now explore.

9. NONLINEAR SYSTEMS

Many of the ideas applicable to linear systems can be carried over to nonlinear systems, but a number cannot. Research aimed at establishing just what is possible is in fact very active.

Before illustrating the role of a Youla–Kucera parameter, it is however necessary to understand how the concepts of right and left coprime realization arise.

We begin with right coprime factorizations; see Hammer (1985, 1987) and Verma (1988), and we shall drive that idea as far as possible before introducing the appropriate nonlinear generalization of a left coprime realization. We shall see that Youla– Kucera parametrization results using right coprime factorization of nonlinear systems clearly fall short of those for the linear case.

9.1. *Right coprime factorizations*

We begin with a system Σ, possessing a set of input signals $\mathcal{U}$ (a subset of the time functions from $[0, \infty)$ to R^k, k being the system input dimension) and a set of output signals y (a subset of the time functions from $[0, \infty)$ to R^l, l being the output dimension). We shall identify a subset of $\mathcal{U}$, call it $\mathcal{U}^s$, with the set of stable input signals; often $\mathcal{U}^s$ is $L_2^k[0, \infty)$. $\mathcal{Y}^s$ is similarly defined. Stable systems are those which map $\mathcal{U}^s$ to $\mathcal{Y}^s$ (causally) for any internal condition.

We say that Σ has a (stable) right factorization if there exists a set of time functions $\mathcal{W}$ with stable subset $\mathcal{W}^s$ and stable operators $D: \mathcal{W} \to \mathcal{U}$ and $N: \mathcal{W} \to \mathcal{Y}$ with D invertible such that $\Sigma = ND^{-1}$. If the initial state is not zero, Σ should be parametrized by $x(0) = x_0$.

The factorization is termed right coprime if for all unbounded $w \in \mathcal{W}$, either Nw or Dw is also unbounded, or equivalently if $Nw \in \mathcal{Y}^s$ and $Dw \in \mathcal{U}^s$ imply $w \in \mathcal{W}^s$.

A sufficient condition for this property is that there exists a bounded operator $L: \mathcal{U} \times \mathcal{Y} \to \mathcal{W}$ such that

$$L\begin{bmatrix} N \\ D \end{bmatrix} = I. \tag{9.1}$$

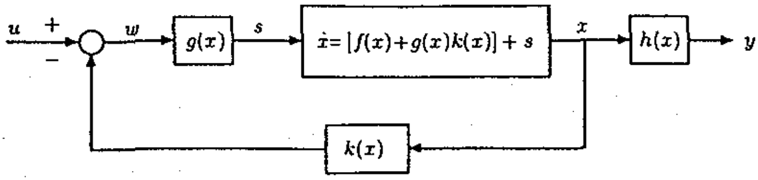

Fig. 14. Representation of nonlinear system to allow generation of right factorization.

Note there is no requirement for L to be separable, in the sense that

$$L(u, y) = L_1(u) + L_2(y).$$

Such separability would imply

$$L_1N + L_2D = I,$$

which is reminiscent of a necessary and sufficient condition for coprimeness in the linear case. Of course, if L is known to be a linear operator, it automatically has the separability property.

Example (*Based on Van der Schaft*, 1996). Consider the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{9.2}$$

with $f(0) = 0, h(0) = 0$, and suppose that the control law $u = k(x) + w$ yields a stable closed-loop system from w to y. Figure 14 is a redrawing of the system, adding and subtracting a feedback $k(x)$. The operators N and D are defined by

$$\begin{aligned} D^{-1}: u \to w.\ \dot{x} &= f(x) + g(x)u, \\ w &= u - k(x) \end{aligned} \tag{9.3}$$

or

$$\begin{aligned} D: w \to u.\ \dot{x} &= [f(x) + g(x)k(x)] + g(x)w, \\ u &= k(x) + w \end{aligned} \tag{9.4}$$

and

$$\begin{aligned} N: w \to y.\ \dot{x} &= [f(x) + g(x)k(x)] + g(x)w, \\ y &= h(x). \end{aligned} \tag{9.5}$$

Is the factorization coprime? Under an observability condition, it is. Suppose w is unknown, but the associated $u(\cdot) = Dw(\cdot)$ and $y(\cdot) = Nw(\cdot)$ are known. Observability allows the recovery of $x(\cdot)$ via a stable operator from $u(\cdot)$ and $y(\cdot)$; then $w(\cdot)$ can be recovered — see the second equation of equation (9.3). Conversely, if a stable L exists, w can be recovered in a stable manner from u and y. Then the first equation of (9.4) allows x to be recovered in a stable fashion. Thus, observability is equivalent to the existence of the stable operator L in equation (9.1).

Example (*Scherpen and Van der Schaft*, 1994; *Van der Schaft et al.*, 1995). Consider the same system as that of the previous Example, and suppose that there is a state feedback law which minimizes the index

$$\int_0^{\infty} \frac{1}{2}[u^{\mathrm{T}}u + h^{\mathrm{T}}(x)h(x)]\,\mathrm{d}t$$

for the system. Suppose further that with initial condition x_0, the minimized value of the index is $V(x_0) \geq 0$, with $V(0) = 0$ and

$$\frac{\partial V}{\partial x}f - \frac{1}{2}\frac{\partial V}{\partial x}gg^{\mathrm{T}}\frac{\partial^{\mathrm{T}} V}{\partial x} + \frac{1}{2}hh^{\mathrm{T}} = 0. \tag{9.6}$$

(This is the standard Hamilton–Jacobi equation associated with the problem).

Then we define $[{}^N_D]$ by

$$\begin{aligned} \dot{x} &= f(x) - g(x)g^{\mathrm{T}}(x)\frac{\partial^{\mathrm{T}} V}{\partial x} + g(x)w, \\ y &= h(x), \\ u &= -g^{\mathrm{T}}(x)\frac{\partial^{\mathrm{T}} V}{\partial x} + w. \end{aligned} \tag{9.7}$$

Notice that this is a special case of the previous Example; with the observability assumption that $\dot{x} = f(x)$ and $h(x) \equiv 0$ implies $x \equiv 0$, the stabilizing property of the feedback law can be proved using the Hamilton–Jacobi equation. [The Lyapunov function for the unforced system is $V(x)$.] The Hamilton–Jacobi equation also is the basis for establishing that

$$\int_0^{\infty}(u^{\mathrm{T}}u + y^{\mathrm{T}}y)\,\mathrm{d}t = \int_0^{\infty} w^{\mathrm{T}}w\,\mathrm{d}t.$$

This establishes coprimeness of the factorization, and in fact establishes that the factorization is a nonlinear equivalent of normalized realization, with unity gain from w to $[u^{\mathrm{T}}\ \ y^{\mathrm{T}}]^{\mathrm{T}}$.

Defining an explicit operator L satisfiying equation (9.1) requires a further assumption, regarding the dual Hamilton–Jacobi equation (which is associated with the problem of transferring from $x(-\infty) = 0$ to a prescribed x_0, and minimizing $\int_{-\infty}^{0}\frac{1}{2}[u^{\mathrm{T}}u + h^{\mathrm{T}}(x)h(x)]\,\mathrm{d}t$). Let $W(x) \geq 0$, $W(0) = 0$ satisfy

$$\frac{\partial W}{\partial x}f + \frac{1}{2}\frac{\partial W}{\partial x}gg^{\mathrm{T}}\frac{\partial W^{\mathrm{T}}}{\partial x} - \frac{1}{2}h^{\mathrm{T}}h = 0. \tag{9.8}$$

Assume additionally that there exists a $k_e(\cdot)$ such that

$$\frac{\partial W}{\partial x} k_e(x) = h^{\mathrm{T}}(x). \tag{9.9}$$

We use it to define an operator L by

$$\dot{p} = [f(p) - k_e(p)h(p)] + g(p)u + k_e(p)y,$$
$$\zeta = g^{\mathrm{T}}(p)\frac{\partial V^{\mathrm{T}}}{\partial p}(p) + u, \tag{9.10}$$
$$p(0) = x(0).$$

The stability of L is established using the Hamilton–Jacobi equation for W, with $W(p)$ a Lyapunov function for the unforced equation. The cascade of equations (9.7) and (9.10) results in $p(t) = x(t)$ for all T and then $\zeta(t) = w(t)$ for all t.

Note that while the signals $u(\cdot)$ and $y(\cdot)$ enter L linearly, the nonlinearity of L ensures that we cannot decompose the output $L(u, y)$ as $L_1(u) + L_2(y)$.

The examples reveal that a wide class of finite-dimensional nonlinear systems have a right coprime factorization. This motivates us to consider interconnections of plants and controllers, both with right coprime factorization.

9.2. *Plant-controller interconnections*

Figure 15 illustrates a plant-controller interconnection, which we assume to be well-posed, i.e. w_1, w_2 exist and depend causally on r_1, r_2 when the latter are bounded time functions. Evidently,

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} I & -C \\ P & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and the closed-loop is stable (in the obvious sense) just when

$$\begin{pmatrix} I & -C \\ P & I \end{pmatrix}^{-1}$$

exists and is bounded. (Existence of the inverse is a requirement for w_1, w_2 to exist; obviously, we could have chosen as outputs the outputs of C and P instead of w_1, w_2 without significantly changing the conclusion.)

The following result connects the stability property to right coprime factorization.

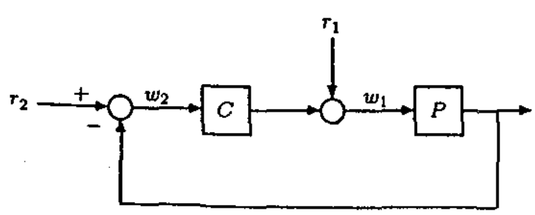

Fig. 15. Nonlinear plant-controller interconnection.

Theorem 4 (*Verma*, 1988; *Paice et al.*, 1992). Suppose that $P = ND^{-1}$ and $C = XY^{-1}$ are right coprime factorizations. The the closed-loop is well posed if

$$\begin{bmatrix} D & -X \\ N & Y \end{bmatrix}^{-1}$$

exists and the closed loop is stable if and only if this inverse is a bounded operator. (If the inverse is known to be stable for any plant and controller right factorizations, which are not necessarily coprime, *coprimeness of the factorizations is automatically implied*.)

The proof is very simple and most can be included. Observe that

$$\begin{pmatrix} I & -C \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} D & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} D & -X \\ N & Y \end{pmatrix}^{-1}.$$

If the second operator on the right is bounded, that on the left must be bounded. For the converse, suppose that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} D & -X \\ N & Y \end{pmatrix}^{-1}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

with r_i bounded, one of z_i at least unbounded (say z_1) and $w_1 = Dz_1$ and $w_2 = Yz_2$ are bounded. Since $Nz_1 + Yz_2 = r_2$, Nz_1 is bounded; with Nz_1 and Dz_1, bounded, comprimeness requires z_1 to be bounded, a contradiction.

9.3. *Characterizing the stabilizing controllers for an open-loop-stable plant*

We can now exhibit a Youla parameter for an open-loop-stable plant, using ideas of Paice and Van der Schaft (1995a). The result is just like that for the linear case, given in Section 2.

Consider the scheme of Fig. 16; the controller is composed of an interconnection of an arbitrary stable operator Q and a model of the plant P. It is assumed that the initial state of the model and of the plant itself are the same. It is then evident that $y = PQr$, so that C, given by

$$C = Q(I - PQ)^{-1} \tag{9.11}$$

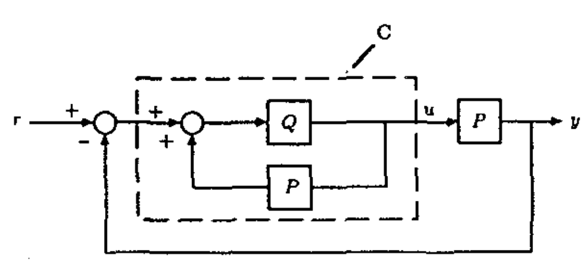

Fig. 16. With stable P, all stabilizing controllers are given by stable Q.

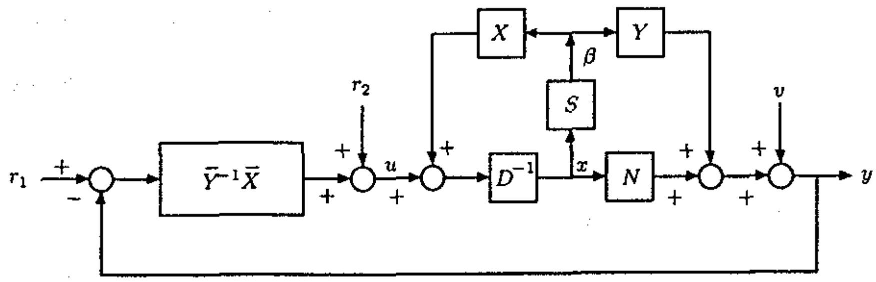

Fig. 17. Noisy identification problem with nominal nonlinear plant and linear stabilizing controller $XY^{-1} = \bar{Y}^{-1}\bar{X}$.

is stabilizing. (*Note*: the derivation of this formula is somewhat nontrivial in the nonlinear case.) To express Q in terms of C and P, we have

$$Q = C(I + PC)^{-1}. \tag{9.12}$$

[This formula is not straightforward to derive by manipulation from equation (9.11); it can be substituted into equation (9.11) and the result verified. Alternatively, one can observe using Fig. 16 that Q follows from C by connecting P in a negative feedback loop, to undo the P in the positive feedback loop generating C from Q.]

To establish that any stabilizing C necessarily can be represented in the manner depicted, observe that if C is stabilizing, then $\hat{Q} = C(I + PC)^{-1}$ is stable (this is the operator from r to u). This formula can be inverted to yield $C = \hat{Q}(I - P\hat{Q})^{-1}$, as required.

Actually, there is a substantial concealed deficiency in the above argument. Suppose there is an additional external input r_1 entering between C and P. It is not possible to ensure that the state of the actual plant and that of the model within the controller are the same. Accordingly, the earlier argument cannot be carried through.

All is not lost however. If P has a Lipschitz continuity property, one can write (with u the controller output)

$$P(u + r_1) = Pu + y_1, \tag{9.13}$$

where y_1 obeys $\|y_1\| \le K\|r_1\|$ for some K. Then the scheme of Fig. 16 applies again, with r replaced by $r - y_1$.

9.4. *Characterizing the set of stabilizing non-linear controllers for linear plants*

When the plant is linear, and we have one stabilizing (in general nonlinear) controller, it is again easy to parametrize all stabilizing controllers.

Theorem 5 (*Verma*, 1988). Let $P = ND^{-1}$ define a linear right coprime factorization of a linear plant and let $C = XY^{-1}$ define a right coprime factorization of a possibly nonlinear stabilising controller used in a negative feedback loop of the form of Fig. 15. Then the set of all stabilizing controllers is given by $C = (X - DQ)(Y + NQ)^{-1}$ where Q is a stable operator, such that $(Y + NQ)^{-1}$ is well defined.

9.5. *Identification using right coprime realizations*

In this subsection, we will consider a nonlinear version of the earlier result on identification.

The key is to base the identification on a dual of the idea of Section 9.4. We suppose that a *linear* controller $C = XY^{-1} = \bar{Y}^{-1}\bar{X}$ stabilizes a nominal nonlinear plant $P^0 = ND^{-1}$; with noisy measurements on the true plant P connected in a negative feedback loop with C, we seek to find stable S such that the plant is $(N - YS)(D + XS)^{-1}$.

Figure 17 illustrates the arrangement. As in the linear case, we aim to produce an open-loop identification problem, based on x, β and S.

We would like to be able to compute β and x from external measurements (of r_1, r_2, u and y); using β and x, we would hope to infer S from $\beta = Sx$. In the presence of noise, it turns out, as we now establish, that x and β cannot be exactly computed from measurements, but only noisy approximations of them.

To assist the calculation one shall introduce a preliminary simplification. Because of the closed-loop stability of the nominal system, it follows that

$$R = \bar{Y}D + \bar{X}N \tag{9.14}$$

is stably invertible. Let us replace the fractional description ND^{-1} of the nominal plant by $NR(DR)^{-1}$. Then we obtain

$$I = \bar{Y}D + \bar{X}N \tag{9.15}$$

Computation of a noisy approximation of x. The plant structure implies

$$\begin{aligned} Dx - X\beta &= u \\ Nx + Y\beta &= y - v. \end{aligned} \tag{9.16}$$

Multiply the first equation by $\bar{Y}$, the second by $\bar{X}$ and add. The linearity of $\bar{Y}$ and $\bar{X}$ must be used, and there results [using equation (9.15)].

$$x = (\bar{Y}D + \bar{X}N)x = \bar{Y}u + \bar{X}y - \bar{X}v. \tag{9.17}$$

An alternative expression is

$$x = \bar{Y}r_2 + \bar{X}r_1 - \bar{X}v. \qquad (9.18)$$

Computation of a noisy approximation of β. Because X, Y are coprime and linear, there are linear stable operators K and L such that $-KX + LY = I$. Now equations (9.16) and (9.18) yield

$$-X\beta = u - D(\bar{Y}r_2 + \bar{X}r_1 - \bar{X}v),$$
$$Y\beta = y - v - N(\bar{Y}r_2 + \bar{X}r_1 - \bar{X}v)$$

and so

$$\beta = K[u - D(\bar{Y}r_2 + \bar{X}r_1 - \bar{X}v)], + L[y - N(\bar{Y}r_2 + \bar{X}r_1 - \bar{X}v] - Lv \qquad (9.19)$$

Obtaining S. A "normal" open-loop identification problem to find an operator S linking x and β via $\beta = Sx$ would involve noiseless measurement of x and noisy measurement of β (typically with the noise additive). Our knowledge of r_1, r_2, u and y shows that we have a noisy measurement of x, viz., $\bar{Y}r_2 + \bar{X}r_1$ as shown by equation (9.18), and a noisy measurement of β, viz., $K[u - D(\bar{Y}r_2 + \bar{X}r_1)] + L(Y - N(\bar{Y}r_2 + \bar{X}r_1)]$ as shown by equation (9.19). The same noise v is perturbing input and output measurements x and β, and it perturbs the output measurements in a nonlinear way.

In a high signal to noise situation, it is possible to obtain a more conventional identification problem. Let S_L, D_L, and N_L denote the linearizations of S, D and N about the operating trajectory induced by the input $\bar{X}r_1 + \bar{Y}r_2$. Then $\beta = Sx$ and equation (9.18) implies

$$\beta = S(\bar{X}r_1 + \bar{Y}r_2) - S_L\bar{X}v,$$

while also equation (9.19) implies

$$\beta = \{Ku + Ly - KD(\bar{Y}r_2 + \bar{X}r_1) - LN(\bar{Y}r_2 + \bar{X}r_1)\} + (KD_L + LN_L)\bar{X}v - Lv$$
$$= \text{known signal} + (KD_L + LN_L)\,\bar{X}v - Lv.$$

The "known signal" is the contents of $\{\ldots\}$, and is known because r_1, r_2, u and y are presumed measurable, and the operators K, L, D, etc., generating the signal are known. Combining these two equations yields

$$\text{known signal} = S(\bar{X}r_1 + \bar{Y}r_2) - (KD_L + LN_L + S_L)\bar{X}v + Lv$$

with a v independent of r_1 and r_2. Determination of S is now conventional open-loop (albeit nonlinear) identification problem.

The above ideas are simple when P^0 itself is linear, and are explained in Dasgupta and Anderson (1996). Extensions can be found in Linard and Anderson (1996, 1997).

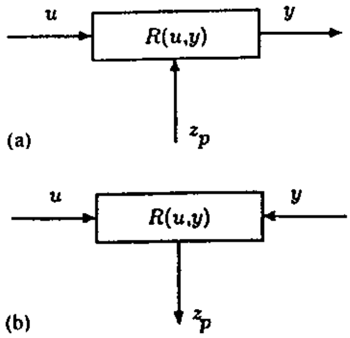

Fig. 18. (a) Stable kernel representation, with u and y determining z via a stable mapping (b) Plant represented by a stable kernel representation, so that u and z determine y; $z = 0$ gives normal plant description.

9.6. *Kernel representations*

In general, left fractional representations do not exist. As explained earlier, the fundamental reason is that one cannot write $L(u, y) = L_1(u) + L_2(y)$ in general, when L is a nonlinear operator. There is however a form of representation that carries many such properties; this has been demonstrated in a series of papers; see (Paice and Van der Schaft, 1994a, b; 1995b), as well as in a recent book by Van der Schaft (1996).

We call an operator $R: u \times y \to z$ *a stable kernel representation* of a system when R is a stable mapping, and the equation $R(u, y) = z$ for a prescribed x_0, $u(\cdot)$ and $z(\cdot)$ is solvable for y; further when $z = 0$, the system I/O mapping $y(\cdot) = \sum_{x_0} u(\cdot)$ is to be recovered. It is logical to use two different diagrammatic representations for $R(u, y) = z$; see Fig. 18a and b, depending on whether u and y are determining z, or u and z are determining y.

Example. Let $\dot{x} = f(x) + g(x)u \quad y = h(x)$ be a stable plant. Then

$$R: \quad \dot{x} = f(x) + g(x)u, \quad z = y - h(x), \qquad (9.20)$$

is a stable kernel representation.

Put another way, if a plant P is stable, and thus has a right coprime factorization $N(I)^{-1}$, it has a kernel representation.

$$R(u, y) = y - Pu = z. \qquad (9.21)$$

Example. Suppose that $k(x)$ has the property that

$$L: \quad \dot{x} = f(x) - k(x)h(x) + g(x)u + k(x)y, \quad z = y - h(x), \qquad (9.22)$$

is a stable system, with $u(\cdot)$, $y(\cdot)$ arbitrary inputs. Now set $z = 0$. Then this forces

$$\dot{x} = f(x) + g(x)u,$$
$$y = h(x).$$

Thus L is a stable kernel representation.

[In the second example of Section 9.1, we gave a procedure which will generally, but not infallibly, yield such a $k(x)$, given an observability property. When $k(x)$ is chosen according to that procedure, L has unity L_2 gain from $[y^T\ u^T]^T$ to $[z^T\ \bar{z}^T]^T$, where $\bar{z} = u - g^T\,(\partial w/\partial x)$.

Example. Let P be a linear plant, with left coprime fractional representation $\bar{D}^{-1}\bar{N}$. Then $\bar{D}y - \bar{N}u = 0$ and the kernel representation is

$$[\bar{D} - \bar{N}]\begin{bmatrix} y \\ u \end{bmatrix} = z.$$

A kernel representation is called *coprime* if it is stable and has a stable right inverse. A right inverse for L in equation (9.10) is provided by

$$\begin{aligned} L^{-1}:\quad \dot{x} &= f(x) - g(x)g^T(x)\nabla V^T(x) + k(x)\varsigma \\ x(0) &= p(0), \\ u &= -g^T(x)\nabla V^T(x), \\ y &= h(x) + \varsigma. \end{aligned} \tag{9.23}$$

Notice that L^{-1} can be expressed as $[^X_Y]$, even though L cannot be expressed as $[L_1\ L_2]$. Also, if $p(0) \neq x(0)$, but L forgets the initial state as time evolves, then as $t \to \infty$, $L_{p(0)}L^{-1}_{x(0)}$ behaves more and more like the identity operator.

Feedback is described in the following way. Consider a plant P with stable kernel representation $R_P: \mathscr{U} \times \mathscr{Y} \to \mathscr{Z}_p$ and a controller C with stable kernel representation $R_C: \mathscr{Y} \times \mathscr{U} \to \mathscr{Z}_c$ which are interconnected, for the moment with no external inputs. (For convenience, sign inversions are assumed to be incorporated in the plant or controller, so the interconnection is a positive feedback interconnection.) Then the loop is said to be *null-well-posed* if the equations

$$\begin{aligned} R_P(u, y) &= z_p, \\ R_C(y, u) &= z_c, \end{aligned} \tag{9.24}$$

have unique solutions for all z_p, z_c; if the solutions are stable for stable z_p, z_c the system is said to be null-stable. Note that the operators R_P, R_C can be initial condition dependent.

It is possible to show

- if a closed-loop system using stable kernel representation is null-stable, the kernel representations are coprime and there exist right coprime factorizations of plant and controller, and
- if a plant and controller have right coprime factorization and form a stable feedback system, coprime stable kernel representation can be found which together are null-stable. (They are not unique.)

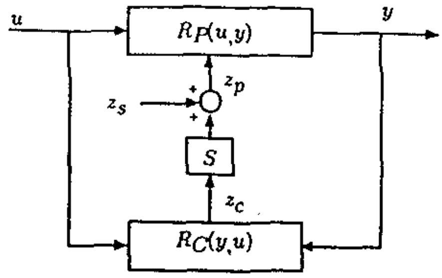

Fig. 19. Construction of kernel representation of P_S from kern representations of P and C (differently viewed) and from t Youla–Kucera parameter S.

Although the definition of null-stability intr duces certain extra signals, viz., z_p and z_c, these a not the same as the external signals r_1 and r_2 of sa Fig. 15, and the concept of null-stability is formall a different concept than stability.

Youla–Kucera parametrizations are introduce as follows:

- Assume $P = ND^{-1}$ and $C = XY^{-1}$ are describe by right coprime factorizations, and togethe form a stable closed loop. (For convenienc assume the connection is a positive feedbac controller)
- Form stable kernel representations of P and C viz., $R_P(u, y) = z_p$, $R_C(y, u) = z_c$.
- Let $S: \mathscr{Z}_c \to \mathscr{Z}_p$ and suppose S has a kerne representation $R_S(z_c, z_p) = z_s$.
- The modified plant P_S is defined by the kerne representation

$$R_{P_s}(u, y) = z_s = R_S(R_C(y, u), R_P(u, y)).$$

Figure 19 shows the arrangement, probably muc more clearly. Notice that the subblocks R_P an R_C require the interpretations of both Fig. 18b an a, respectively. The parallel with Fig. 13 (with nois signal absent) should also be noted.

The main conclusions are

- The set of all plants stabilized by C is obtained by letting S range over all stable operators. [In this case, one may take $R_S(z_c, z_p)$ as $-Sz_c + z_p$ and $R_{P_s}(u,y) = -SR_C(y, u) + R_P(u, y)$.]
- The set of all controllers stabilizing P is obtained by an obvious dual characterisation involving a $Q: z_p \to z_c$.
- P_S is stabilized by C_Q if and only if S is stabilized by Q.

Example. Suppose P and C are linear, with left coprime factorizations $\bar{D}^{-1}\bar{N}$ and $\bar{Y}^{-1}\bar{X}$. Then $R_P(u, y) = \bar{D}y - \bar{N}u = z_p$ and $R_C(y,u) = \bar{Y}u - \bar{X}y = z_C$. Now $R_S(z_c, z_p)$ is $-Sz_c + z_p = z_s$, which implies $(\bar{D} - S\bar{X})y - (\bar{N} + S\bar{Y})u = z_s$ for $R_{P_s}(u, y)$, and the left factorization is $(\bar{D} - S\bar{X})^{-1}(\bar{N} + S\bar{Y})$.

These results can be pushed to handle plants with external inputs. By way of general comment, it turns out that

- Closed-loop stability is most easily handled with right coprime fractional representations.
- Youla–Kucera parametrizations are most easily handled with stable kernel representations.
- One requires results on the ability to switch between the two types of representation.

10. CRITIQUE

In this brief concluding section, we aim to record some of the issues that are not well addressed by the Youla–Kucera parameter idea and its variants:

- *Decentralized controller design.* There is no explicit parametrization of all stabilizing decentralized controllers for a prescribed plant.
- *Linear time-invariant plant identification in a sampled-data loop.* It is clear that substantial modification of the earlier theory would be required to handle identification of a continuous-time plant when a nominal model is known and a sampled-data controller is connected.
- *Degree explosion*: $P(s)$ can have degree equal to degree P^0 + degree C^0 + degree S.
- *Q design given a plant S, and nominal P^0 and C^0.* This point has been well described.
- *Keeping S simple when unknownness is simple.* Suppose that a plant is completely known except for the value assumed by two physical parameters appearing in the equation of the plant. It is generally not straightforward to have a very simple S capturing this unknownness. Again, suppose P comprises a memoryless saturating nonlinearity (unknown) followed by a known linear dynamic part. It is not straightforward to have S memoryless.
- More generally, utility of the nonlinear results has yet really to be established.

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