A Complete and Simple Parametrization of Controllers for a Nonstandard $H_\infty$ Control Problem

TSUTOMU MITA,† JEREMY B. MATSON‡ and BRIAN D. O. ANDERSON§

Abstract—A complete and simple parametrization of all $H_\infty$ controllers is derived for a class of nonstandard $H_\infty$ control problems with $D_{12}$ and $D_{21}$ being of full row and column ranks, respectively, by making use of the Youla parametrization and standard $H_\infty$ control theory. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The $H_\infty$ control problem has been extensively studied for the last decade and Glover and Glover (1988) and Doyle et al. (1989) solved the so-called standard problem, where the number of control inputs is less than or equal to the number of controlled outputs, and the number of measurement outputs is less than or equal to the number of external inputs.

In this paper, we deal with a nonstandard $H_\infty$ control problem with a greater number of control inputs than controlled outputs and/or a greater number of measurement outputs than external inputs, under the conditions that certain direct feedthrough matrices $D_{12}$ and $D_{21}$ [defined below, see equation (1)] are of full row rank and full column rank, respectively. In this way, we often face a plant having a greater number of measurement outputs than external inputs in industrial applications. In this nonstandard problem, we can expect to have better control performance compared to the standard case, since every $H_\infty$ controller will have more free parameters than the standard $H_\infty$ controller (Doyle et al., 1989). However, in order to use the free parameters effectively for other design purposes, we first need an explicit and simple formula of all controllers ensuring the closed-loop $H_\infty$ norm bound. In this paper, we will provide a direct derivation of the solvability condition via Riccati equations and give a simple parametrization of all $H_\infty$ controllers for the nonstandard $H_\infty$ control problem.

Sempé et al. (1990) solved an $H_\infty$ control problem with the standard assumptions on $D_{12}$ or $D_{21}$ by using algebraic Riccati inequalities, and they derived one $H_\infty$ controller. Stoorvogel (1991, 1996) treated the same nonstandard $H_\infty$ control problem where the solvability condition was reduced to checking the satisfaction of two elegant quadratic matrix inequalities in addition to some rank conditions. Moreover, a proper $H_\infty$ controller was developed via the solution of the almost disturbance decoupling problem. However, no controller parametrization was shown. Actually, for our nonstandard $H_\infty$ control problem also, though Stoorvogel's results can provide the solvability condition, they cannot immediately give the explicit parametrization of all $H_\infty$ controllers. Scherer (1992) also gave the solvability condition of general $H_\infty$ control problems, where $G_{12}(s)$ and $G_{21}(s)$ [defined below, see equation (1)] have imaginary-axis zeros including zeros at infinity, by using algebraic Riccati inequalities. He proposed an algorithm to compute an $H_\infty$ controller. However, no controller parametrization was provided.

As for our nonstandard problems, a parametrization of $H_\infty$ controllers for a singly nonstandard problem, i.e., one only of $D_{12}$ and $D_{21}$ is nonstandard in terms of dimensions, was provided in Zhang and Hooke (1993) using relatively complicated expressions. Kimura et al. (1993) solved the simply nonstandard problem under the condition that $G_{12}(s)$ has no unstable zeros. Le and Safonov (1992) solved the freedom of Youla parameter $Q$ for a doubly nonstandard problem, i.e., both $D_{12}$ and $D_{21}$ are nonstandard. However, the final form of the $H_\infty$ controller was not given (and is not straightforward to obtain). In this paper, which is a modified version of Mita et al. (1993), we obtain (we believe for the first time) a complete and simple parametrization of all $H_\infty$ controllers for the doubly nonstandard problem.

In what follows, we express the star product of $M_L$ and $M_r$ by $M = M_L ** M_r$ so that $F(M, M_r, K) = F(M, K)$ holds, where $F(\star, \ast)$ denotes the standard lower linear fractional map (Zhou et al., 1993). $RH^\infty$ is the set of proper and stable rational functions and $BH^\infty$ is the set of functions in $RH^\infty$ whose $H_\infty$ norm is less than unity ($<1$).

2. Preliminaries

2.1. Assumptions. In the $H_\infty$ design, we first need generalized plants which describe input-output relations for given control problems. The generalized plant in this paper is given by

\[ \dot{x} = Ax + B_1 w + B_2 y, \]
\[ z = C_1 x + D_{12} w, \]
\[ y = C_2 x + D_{21} y, \]

or

\[ G(\omega) = \begin{bmatrix} G_{11}(\omega) & G_{12}(\omega) \\ G_{21}(\omega) & G_{22}(\omega) \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \]

where $x \in \mathbb{R}^n$ is the state variable; $w \in \mathbb{R}^m$ the external input; $u \in \mathbb{R}^m$ the control input; $z \in \mathbb{R}^p$ the controlled output; and $y \in \mathbb{R}^q$ the measurement output. Via the control law:

\[ u = K(y), \]
the control purpose is to stabilize the closed-loop system internally and ensure the $H_\infty$ norm of the closed-loop transfer function

$$G_{sv} = F_s(G, K) = G_{11} + G_{12}K(U - G_{22}K)^{-1}G_{21}$$

is less than unity, i.e.

$$G_{sv}(s) \in \mathcal{H}_\infty^\infty.$$  

(5)

In this paper, except for Section 4, we assume that $p \geq m, q \geq r$ with

(A1) $D_{12}$ is of full row rank and $D_{21}$ is of full column rank,

which is the opposite of the standard assumption. Under this assumption, we define $D_{12}^T, D_{21}^T, D_{13}^T$, and $D_{23}^T$ to satisfy the following:

$$\begin{bmatrix} D_{12}^T \\ D_{21}^T \end{bmatrix} = L_p, \quad \begin{bmatrix} D_{13}^T \\ D_{23}^T \end{bmatrix} = L_p$$

(6)

When $D_{12}$ and $D_{21}$ are square, we define $D_{12} = 0$ and $D_{21} = 0$.

We also need the following assumptions, which are the same as the standard ones:

(A2) $(A, B, C)$ is stabilizable and detectable;

(A3) $G_{sv}(s)$ and $G_{zv}(s)$ have no $j\omega$-axis invariant zeros.

Using the condition (A1), we can show that the assumption (A3) is equivalent to requiring that

$$A - B_2D_{12}^T(C_1, D_{12}^T, C_1, D_{12}^T)$$

have no $j\omega$-axis uncontrollable and unobservable modes, respectively.

The purpose of this paper is to find conditions under which a solution exists to this nonstandard $H_\infty$ control problem and to give a simple parametrization of all $H_\infty$ controllers which solve the problem.

2.2. Youla parametrization. Before proceeding, we review the Youla parametrization (Zhou et al., 1995) for all stabilizing controllers, which is independent of the ranks or dimensions of $D_{12}$ and $D_{21}$.

Lemma 1. Every controller $K(s)$ which internally stabilizes $G(s)$ is given by

$$K(s) = F_s(M(s), Q(s)),$$

where $Q(s) \in \mathcal{R}^m$ is a free parameter and

$$M = \begin{bmatrix} A_p + HC_2 & \begin{bmatrix} I_p \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ I_p \end{bmatrix} & -B_2 \end{bmatrix},$$

(9)

with $F$ and $H$ being any matrices which stabilize $A_p = A + B_2F$, $A_H = A + HC_2$,

$$A_p = A + B_2F, \quad A_H = A + HC_2,$$

(10)

respectively.

In terms of this parametrization, $G_{sv}(s)$ in equation (4) is described by

$$G_{sv}(s) = F_s(G_s(s), Q(s)),$$

where

$$G_s(s) = \begin{bmatrix} A_p - B_2F & B_1 \\ 0 & A_H \end{bmatrix},$$

(11)

$$C_F = C_1 + D_{12}F, \quad B_H = B_1 + HD_{21},$$

(12)

with

In accordance with the assumption (A3) on the invariant zeros of $G_{sv}(s)$, we introduce the following similarity transformation $T$ displaying the uncontrollable part of the pair $(A - B_2D_{12}^T, C_1, B_2D_{21}^T)$:

$$T^{-1}(A - B_2D_{12}^T)C_1 = \begin{bmatrix} A_\omega & 0 \\ A_{2\omega} & A_2 \end{bmatrix},$$

(14)

$$T^{-1}B_2D_{21}^T = \begin{bmatrix} 0 \\ \beta_p \end{bmatrix},$$

(15)

where $(A_\omega, \beta_p)$ is controllable and $A_\omega$ has no $j\omega$-axis eigenvalues due to assumption (A3). We also define

$$T^{-1}B_2D_{21}^T = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

(16)

Then assumption (A2) leads to the condition that $(A_\omega, \gamma_1)$ is stabilizable. Based on this consideration, let us choose $F$ in Lemma 1 as

$$F = -D_{12}^T C_1 + D_{13}^T E_1 + D_{12}^T L_1,$$

with

$$E_1 T = (E_1, 0), \quad L_1 T = (L_1, L_2),$$

(17)

where $E_1$ and $L_1$ are any matrices that stabilize $A_\omega$, $\gamma_1 E_1$, and $A_2 + \beta_L L_2$, respectively, and $L_1$ is free.

Then

$$T^{-1}A_\omega T = \begin{bmatrix} A_\omega + \gamma_1 E_1 & 0 \\ A_{2\omega} + \beta_L L_1 & A_2 + \beta_L L_2 \end{bmatrix}$$

(18)

and $A_\omega$ is stable.

The following lemma shows the advantage of this choice of $F$.

Lemma 2. With $F$ chosen as in equation (16),

$$\begin{bmatrix} A_p & B_2D_{12}^T \\ C_F & I_m \end{bmatrix}$$

holds, moreover

$$\begin{bmatrix} A_p & B_2D_{12}^T \\ C_F & I_m \end{bmatrix}$$

(19)

has no $j\omega$-axis invariant zeros.

Proof. From equations (14)-(18), we have

$$\begin{bmatrix} A_p & B_2D_{12}^T \\ C_F & I_m \end{bmatrix}$$

(20)

which together with $B_2 = B_2[D_1 D_1^T + D_{12}^T D_{12}^T]$ lead to (19). The invariant zeros of (20) are given by all the eigenvalues of

$$T^{-1}(A - B_2D_{12}^T)C_1:$$

(21)

because equation (20) is square. Since $A_\omega$ has no $j\omega$-axis eigenvalues and $A_2 + \beta_L L_2$ is stable, equation (20) has no $j\omega$-axis invariant zeros.

Dually, in accordance with assumption (A3) on the invariant zeros of $G_{sv}(s)$, we consider the following similarity transformation $S$ displaying the unobservable part of the pair $(A - B_1D_{12}^T, C_2, D_{23}^T)$:

$$S(A - B_1D_{12}^T C_2) S^{-1} = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix},$$

(22)

$$D_{23}^T C_2 S^{-1} = (\beta_L 0).$$

(23)
where \((A_{A}, \beta_{A})\) is observable and \(A_{A}\) has no j-axis eigenvalues due to assumption (A3). In accordance with equation (23), define
\[
D_{A_{A}}C_{A}S^{-1} = (\delta_{A}, \delta_{A}). \tag{24}
\]
Then assumption (A2) leads to the condition that \((A_{A}, \delta_{A})\) is detectable.
Hence, \(A_{A}\) becomes stable and the dual of Lemma 2 holds if we choose \(H\) as
\[
H = -B_{1}D_{F_{1}}C_{1} + B_{1}D_{F_{1}}L_{1} + L_{4}D_{F_{1}}, \tag{25}
\]
with
\[
SB_{H} = \begin{bmatrix} 0 & e_{2} \end{bmatrix}, \quad SL_{H} = \begin{bmatrix} \bar{L}_{1} \end{bmatrix}. \tag{26}
\]
where \(e_{2}\) and \(\bar{L}_{1}\) are any matrices which stabilize \(A_{A} + e_{2}\beta_{A}\) and \(A_{A} + \bar{L}_{1}\beta_{A}\), respectively, and \(L_{2}\) is free.

3. Main result and proof

3.1. AREs and main result. Define the following algebraic Riccati equation (ARE):
\[
X(A - B_{1}D_{F_{1}}C_{1} + B_{1}D_{F_{1}}L_{1} + (A - B_{1}D_{F_{1}}C_{1} + B_{1}D_{F_{1}}L_{1})^{T}X + X(B_{1}B_{1}^{T}) - B_{1}D_{F_{1}}(D_{F_{1}}D_{F_{1}}^{T})B_{1})X = 0, \tag{27}
\]
where \(L_{P}\) as earlier is any matrix which stabilizes the controllable subspace of \((A - B_{1}D_{F_{1}}C_{1}, B_{1}D_{F_{1}}L_{1})\). The stabilizing solution to equation (27) is given by an \(X\) which stabilizes
\[
A_{X} := A - B_{1}D_{F_{1}}C_{1} + B_{1}D_{E_{1}}L_{1} + (B_{1}B_{1}^{T} - B_{1}D_{F_{1}}(D_{F_{1}}D_{F_{1}}^{T})B_{1})X. \tag{28}
\]
Dually, define the second ARE:
\[
Y(A - B_{1}D_{F_{1}}C_{2} + L_{P}D_{F_{1}}C_{2}Y + (A - B_{1}D_{F_{1}}C_{2} + L_{P}D_{F_{1}}C_{2})Y + Y(C_{1}C_{1} - C_{1}(D_{F_{1}}^{T})C_{1})Y = 0, \tag{29}
\]
where \(L_{P}\) as earlier is any matrix which stabilizes the observable subspace of \((A - B_{1}D_{F_{1}}C_{2}, L_{P}D_{F_{1}}C_{2})\). The stabilizing solution to equation (29) is given by a \(Y\) which stabilizes
\[
A_{Y} := A - B_{1}D_{F_{1}}C_{2} + L_{P}D_{F_{1}}C_{2} + Y(C_{1}C_{1} - C_{1}(D_{F_{1}}^{T})C_{1})C_{1}. \tag{30}
\]
Note that such \(L_{P}\) and \(L_{P}\) can be chosen according to equations (17) and (26) so that \(L_{3}\) and \(\bar{L}_{1}\) stabilize \(A_{A} + \beta_{L_{3}}\) and \(A_{A} + \bar{L}_{1}\beta_{A}\), respectively. We show some properties of the stabilizing solutions first.

Lemma 3. (1) When equation (27) has a stabilizing solution \(X \geq 0\) and \(A + B_{2}F_{2}\) is stable where
\[
F_{2} := -D_{F_{1}}C_{1} - D_{F_{1}}(D_{F_{1}}^{T})B_{1}X + D_{F_{1}}L_{1}. \tag{31}
\]
Moreover, the stabilizing solution satisfies \(XB_{1}B_{1}^{T}X = 0\).
(2) When equation (29) has a stabilizing solution \(Y \geq 0\), and \(A + L_{P}C_{2}\) is stable where
\[
L_{P} := -B_{1}D_{F_{1}} - YC_{1}(D_{F_{1}}^{T})B_{1} + L_{P}D_{F_{1}}. \tag{32}
\]
Moreover, the stabilizing solution satisfies \(D_{P}C_{2}Y = 0\).

Proof. We only prove the first assertion. The second follows by duality. Since
\[
A_{X} = A + B_{2}F_{2} + B_{2}B_{2}X, \tag{33}
\]
is stable, \((A + B_{2}F_{2}, B_{2}B_{2})\) is detectable. Moreover (27) can be written as:
\[
X(A + B_{2}F_{2}) + (A + B_{2}F_{2})^{T}X + XB_{2}B_{2}X + XB_{2}D_{F_{1}}D_{F_{1}}^{T}B_{2}X = 0. \tag{34}
\]
These facts, together with \(X \geq 0\) imply the stability of \(A + B_{2}F_{2}\). Since
\[
T^{-1}(A - B_{1}D_{F_{1}}C_{1} + B_{1}D_{F_{1}}L_{1})T = \begin{bmatrix} A_{2} + \beta_{L_{2}}A_{2} + \beta_{L_{2}}L_{2} \end{bmatrix}, \tag{35}
\]
holds for some \(L_{2}T = (L_{1}, L_{2})\) where \(A_{2} + \beta_{L_{2}}L_{2}\) is stable, it follows from Lemma A1 (see Appendix A) with \(V = (0, I)^{T}\) and \(A_{A_{2}} = A_{2} + \beta_{L_{2}}L_{2}\) that the stabilizing solution to equation (27) has the form
\[
X = (T^{-1})^{T}X_{11}X_{11} + (T^{-1})^{T}X_{12}X_{21} = 0, \tag{36}
\]
where \(X_{11}\) is the stabilizing solution to
\[
X_{11}A_{2} + (A_{2}^{T}X_{11} + X_{11}B_{2}B_{2})X_{11} = 0, \tag{37}
\]
with
\[
T^{-1}B_{1} := \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}. \tag{38}
\]
Then equation (39) together with the second relation in equation (14) yield \(XB_{2}B_{2}X = 0\).

The main result of this paper now follows.

Theorem 1. Suppose that assumptions (A1)–(A3) hold. Let \(L_{P}\) and \(L_{P}\) be any matrices which stabilize the controllable subspace of \((A - B_{1}D_{F_{1}}C_{1}, B_{1}D_{F_{1}}L_{1})\) and observable subspace of \((A - B_{1}D_{F_{1}}C_{2}, L_{P}D_{F_{1}}C_{2})\), respectively. Then an \(H_{A}\) controller exists if and only if equations (27) and (29) have stabilizing solutions \(X \geq 0\) and \(Y \geq 0\) which satisfy \(\rho(YX) < 1\). Under such conditions, every \(H_{A}\) controller \(K_{A}(s)\) can be parametrized as
\[
K_{A}(s) = M_{A}(s) \begin{bmatrix} N(s) \\ W_{2}(s) \\ W_{3}(s) \end{bmatrix}, \tag{39}
\]
where
\[
N(s) \in BH_{\infty}, \quad W_{2}(s) \in RH_{\infty}, \quad W_{3}(s) \in RH_{\infty}. \tag{40}
\]
and
\[
M_{A}(s) := \begin{bmatrix} A & -ZL_{A} & ZB_{2}D_{F_{1}} \\ 0 & D_{F_{1}} & 0 \\ D_{F_{1}}^{T}C_{2} & 0 & 0 \\ 0 & D_{F_{1}} & 0 \\ 0 & 0 & C_{2} + D_{2}B_{2}X \end{bmatrix}. \tag{41}
\]
with
\[
X := \begin{bmatrix} \bar{L}_{2} + YC_{1}B_{2}X \quad \bar{L}_{1} \end{bmatrix} \quad \bar{L}_{2} = B_{2} + YC_{1}D_{F_{1}} \quad C_{2} = C_{2} + D_{2}B_{2}X. \tag{42}
\]
\[
Z = (I - YY^{-1})^{-1}. \tag{43}
\]
We will prove this theorem by the following two steps.

1. Applying Lemmas 1 and 2 to an FI (full information) system, we first prove that equation (27) has a stabilizing solution \(X \geq 0\). Then, by the dual argument, equation (29) has a stabilizing solution \(Y \geq 0\).
2. We apply a lossless decomposition (Doyle et al., 1989) to \(G(s)\) to obtain \(G_{\text{mp}}(s)\), then we again apply a lossless decomposition to \(G_{\text{mp}}(s)\) to get \(G_{\text{mp}}(s)\), which is a \(G_{\text{mp}}\) system for \(G_{\text{mp}}(s)\). In this step, we prove \(p(XX^{-1}) < 1\) and derive the controller parametrization by applying Lemma 1 to \(G_{\text{mp}}(s)\).

Remark. The stability of \(A_{X}\) and existence of \(L_{P}\) are equivalent requirements to the condition that \(\text{rank} (L_{P}^{T}(X, A), P) = s + \text{nullrank} G_{12}(s) \forall s \in \mathbb{C}_{0} \cup \mathbb{C}_{-} \) (Stoorvogel, 1991).
3.2. Existence of the stabilizing solutions. We will now show that equations (27) and (29) have stabilizing solutions $X \geq 0$ and $Y \geq 0$, respectively.

As is well known (Doyle et al., 1989), the output feedback control problem has a solution only if the FI problem is solvable, where the generalized plant $G^+(s)$ for the FI problem is given by equation (1) with $C_2$ and $D_{21}$ being replaced by

$$
C^+_2 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad D^+_2 = \begin{bmatrix} 0 \\ I_n \end{bmatrix}.
$$

(42)

We apply Lemma 1 to $G^+(s)$ to simplify the problem. To this end, we first set the matrix $F$ in Lemma 1 as given in equation (16). Then we choose $H$ as

$$
H = (F, -B_1),
$$

(43)

so that $B_0 = B_1 + HD^+_2 = 0$ and $A_H = A + HC^+_2 = A_F$ hold. Then, from equations (11), (12) and Lemma 2, we have

$$
G_{en}(s) = F_I \begin{bmatrix} A_F & B_1 \\ C_F & 0 & D_{12} \\ B_2 & 0 & 0 \end{bmatrix} \cdot Q(s) \begin{bmatrix} A_F & B_1 \\ C_F & 0 & D_{12} \\ B_2 & 0 & 0 \end{bmatrix} F_I^T.
$$

(44)

Since the existence of an $H_\infty$ control solution is independent of any particular Youla parametrization and equation (20) has no jo-axis invariant zeros, standard $H_\infty$ control theory (Glover et al., 1988) can be applied to conclude that the following ARE:

$$
X(A_F - B_2D_{12}C_2)^T + (A_F - B_2D_{12}C_2)^T X + X(B_1B_1^T - B_1D_{12}D_{12}^T B_1^T)X = 0,
$$

(45)

must have a stabilizing solution $X \geq 0$ for the FI problem to be solved. Equation (45) is just equation (27). By duality, equation (29) also must have a stabilizing solution $Y \geq 0$.

3.3. Controller parametrization

3.3.1. Introduction of $G_{en}$. Knowing that equation (27) has a stabilizing solution, let us define

$$
\Theta(s) = \begin{bmatrix} A + B_1F_0 & B_1 & B_2 D_{12} \\ -D_{12}^T D_{12} & 0 & I_n \\ -B_2^T X & I_n & 0 \end{bmatrix} \cdot \begin{bmatrix} A_F & B_1 \\ C_F & 0 & D_{12} \\ B_2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} A + B_1F_0 & B_1 & B_2 D_{12} \\ -D_{12}^T D_{12} & 0 & I_n \\ -B_2^T X & I_n & 0 \end{bmatrix}^{T},
$$

(46)

$$
G_{en}(s) = \begin{bmatrix} A + B_1B_1^T & B_1 & B_2 \\ -D_{12}^T D_{12} & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix},
$$

(47)

$$
\Theta(s)^T \Theta(s) = G(s),
$$

(48)

for our generalized plant with $D_{12}$ of full row rank. Then the following relation holds:

$$
\Theta(s) \cdot G_{en}(s) = G(s).
$$

(49)

Moreover, we can prove that $\Theta_2(s) = \Theta(s)X$ and $\Theta_2(s) = 0$ and

$$
X A_F + A_F^T X + C_F^T C_F = 0,
$$

$$
B_1^T X + D_{12}^T C_F = 0, \quad D_{12}^T D_{12} = I_{n+}.
$$

(50)

Therefore, $\Theta(s)$ is a lossless matrix.

Then it follows from Doyle et al. (1989) that the original $H_\infty$ control problem reduces to finding a $K(s)$ that satisfies

$$
F_I(G_{en}(s), K(s)) \in BH_\infty,
$$

(51)

or, equivalently,

$$
F_I(G_{en}(s), K(s)) \in BH_\infty,
$$

(52)

where

$$
G_{en}(s) = \begin{bmatrix} (A + B_1B_1^T)^T & -F_I D_{12}^T & C_2 \\ B_1^T & 0 & 0 \\ F_I^T & D_{12} & 0 \end{bmatrix}.
$$

(53)

3.3.2. Proof of $\rho(XY) < 1$ and introduction of $S_{en}$. Since $D_{12}$ is of full row rank and $G_{en}(s)$ has no jo-axis invariant zeros due to the assumption (A3) for $G_1(s)$, $G_{en}(s)$ can be treated in a manner completely analogous to the way that $G(s)$ was treated in the previous section.

Following the above reasoning, we first choose $L_H$ to stabilize the controllable part of the pair consisting of

$$
(A + B_1B_1^T)^T - C_2^T D_{12}^T B_1 = (A - B_1D_{12}C_2)^T.
$$

(54)

and

$$
C_2^T D_{12}^T = (D_{12}^T C_2)^T
$$

(55)

i.e. $L_H$ stabilizes the observable part of the pair $(A - B_1D_{12}C_2, D_{12}C_2)$. Then, for equation (50) to be solvable, the ARE

$$
W A_{en} + A_{en} W + W(F_I^T D_{12}^T D_{12} F_0 - C_2^T D_{12}^T D_{12} C_2)W = 0
$$

(56)

must have a stabilizing solution $W \geq 0$ which satisfies $D_{12}C_2W = W D_{12}C_2 = 0$, where

$$
A_{en} := A - B_1D_{12}C_2 + L_H D_{12}C_2.
$$

(57)

Since $D_{12}C_2W = 0$, it follows from equation (54) that

$$
W A_{en} + A_{en} W + W(F_I^T D_{12}^T D_{12} F_0 - C_2^T D_{12}^T D_{12} C_2 + X L_H D_{12} C_2 + C_2^T D_{12}^T L_H X) = 0
$$

(58)

also has a stabilizing solution, which means that $H_w \in Dom(Ric)$ where

$$
H_w := \begin{bmatrix} A_{en} & \\
0 & A_{en} \end{bmatrix},
$$

(59)

$$
F_I^T D_{12}^T D_{12} F_0 - C_2^T D_{12}^T D_{12} C_2 + X L_H D_{12} C_2 + C_2^T D_{12}^T L_H X
$$

(60)

Then, we obtain (see Appendix B):

$$
H_F = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} H_H \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix},
$$

(61)

$$
H_F = \begin{bmatrix} A_{en} & C_2 \end{bmatrix} - \begin{bmatrix} C_2^T D_{12}^T D_{12} C_2 \end{bmatrix},
$$

(62)

where $H_F$ is the Hamiltonian matrix corresponding to the ARE equation (29). This leads to
for the stabilizing solutions \( Y = Y_2 Y_1^{-1} \geq 0 \) and \( W = W_2 W_1^{-1} \geq 0 \). From equation (59), we can prove the following using well-known manipulations (Doyle et al., 1989):

\[
W = (I - XY)^{-1} Y, \quad p(XY) < 1. 
\]

(60)

Since \( W \) is a stabilizing solution, we can construct

\[
S_{\text{imp}}(s) = \begin{bmatrix}
(A + B_1 B_1^T) + F_0^T D_{12} D_{12} F_0 & -F_0^T D_{12}^T C_2^T \\
B_1 + W C_2^T D_{12}^T & 0 \\
B_1^T - D_{12}^T D_{12} F_0 & D_{12}^T 0
\end{bmatrix}
\]

(61)

as the \( G_{\text{imp}} \) system for equation (51) by observing the relationship between \( G(s) \) and \( G_{\text{imp}}(s) \) in equation (46). Then, take the transpose of equation (61) as follows:

\[
S_{\text{imp}}(s) = \begin{bmatrix}
A + B_1 B_1^T X + W F_0^T D_{12} D_{12} F_0 & B_1 + W C_2^T D_{12}^T \\
- D_{12} F_0 & C_2 \\
- D_{12}^T F_0 & 0
\end{bmatrix}
\]

(62)

Now satisfying equation (59) reduces to finding a \( K(s) \) that satisfies

\[
\Phi(s) = F_0(S_{\text{imp}}(s), K(s)) \in BH^w. 
\]

(63)

### 3.3. Parametrization of all controllers.
In order to display the solution set, we first define

\[
Z = (I - XY)^{-1} Y, 
\]

(64)

which leads to

\[
W = ZY, \quad Z = I + WX. 
\]

(65)

We now apply the Youla parameterization as given in Lemma 1 to \( S_{\text{imp}} \) to simplify \( \Phi(s) \) in equation (63). To this end, we first choose

\[
F = F_0. 
\]

(66)

Then

\[
A_F = A + B_1 B_1^T X + W F_0^T D_{12} D_{12} F_0 + (B_2 - W F_0^T D_{12} D_{12} F_0) - A_X 
\]

(67)

which guarantees the stability of \( A_F \).

On the other hand, since equation (56) has a stabilizing solution \( W_2 = W_2 \), we have

\[
A_w = A_2 + W F_0^T D_{12} D_{12} F_0 - C_2^T D_{12}^T F_0 - C_2 + X L_0 D_{12} C_2
\]

(68)

is stable. Therefore, the choice:

\[
H = -B_1 D_{12} - W C_2^T D_{12}^T F_0 + Z L_0 D_{12} 
\]

(69)

and \( Z = I + WX \) show that

\[
A_H = A + B_1 B_1^T X + W F_0^T D_{12} D_{12} F_0 + H C_2 - A_w, 
\]

(70)

which guarantees the stability of \( A_H \) for \( S_{\text{imp}}(s) \).

Moreover, we obtain

\[
C_H = -D_{12} F_0 + D_{12} F_0 = 0, 
\]

(71)

\[
B_H = B_1 + W C_2^T D_{12}^T F_0 + H D_{12} = 0. 
\]

(72)

Therefore, application of equations (8)-(12) to \( S_{\text{imp}}(s) \) give

\[
K(s) = F_0(M(s), Q(s)), 
\]

(73)

where

\[
M(s) = \begin{bmatrix}
A_H + H C_2 & -H B_2 - W F_0^T D_{12} D_{12}
\end{bmatrix} 
\]

(74)

\[
Q(s) = D_{12} Q D_{21} \in BH^w, 
\]

(75)

Observe that equation (74) gives the following general solution set of \( Q(s) \in RH^w \):

\[
Q(s) = [D_{12}^T - D_{12}^T] N(s) W_4(s) [D_{12}^T - D_{12}^T] 
\]

(76)

where \( N \in BH^w \) and \( W_4 \in RH^w \). Moreover, we can rearrange some parameters in \( M \) as follows:

\[
H = -(I + W X) B_1 D_{12} - W C_2^T D_{12}^T D_{12} + Z L_0 D_{12} 
\]

(77)

\[
B_2 - W F_0^T D_{12} D_{12} - B_2 + W C_2^T D_{12} + W X B_1 [I - D_{12}^T D_{12}]^T 
\]

(78)

\[
= (I + W X) B_1 + W C_2 D_{12} 
\]

(79)

\[
= Z[B_2 + Y C_2 D_{12}] = Z B_2. 
\]

(80)

Therefore, equations (72)-(76) together with

\[
Z B_2 D_{12} = B_1 D_{12}, \quad D_{12} C_2 = D_{12} C_2 
\]

(81)

lead to equation (40).

### 4. Discussions
The case where both \( D_{12} \) and \( D_{21} \) are of full column rank (i.e. \( D_{12} \) is standard but \( D_{21} \) is nonstandard) can be solved by changing some of the treatments in the previous section.

We first define \( D_{12} \) and \( D_{21} \) to satisfy

\[
[DIz,(D:z)'] = I2, 
\]

(78)

Then, we change equation (27) to the standard ARE:

\[
X(A - B_1 D_{12} C_1) + (A - B_1 D_{12} C_1)' X 
\]

(79)

+ \( X(B_1 B_1') - B_2 E_1 E_1 B_1') X + C_1^T D_{12}^T D_{12} C_1 = 0, \]

(80)

and replace equation (31) by

\[
F_0 = -D_{12} C_1 - E_1 E_1 B_1'). X 
\]

(81)

Then we have the following theorem.

**Theorem 2** Suppose that both \( D_{12} \) and \( D_{21} \) are of full column rank and the assumptions (A2) and (A3) hold. Let \( L_0 \) be any matrix which stabilizes the observable subspace of \( \langle A - B_1 D_{12} C_1, D_{21} C_2 \rangle \). Then an \( H_\infty \) controller exists if and only if equations (79) and (29) have stabilizing solutions \( X \geq 0 \) and \( Y \geq 0 \) which satisfy \( p(XY) < 1 \).

Under this condition, every \( H_\infty \) controller is parametrized by

\[
K(s) = F_0(M(s), \phi(s)), 
\]

(82)

where \( N(s) \in BH^w \), \( W_4(s) \in RH^w \), and

\[
M(s) = \begin{bmatrix}
A & -Z L_0 & Z B_2 E_1 E_1^T \\
F_0 & 0 & E_1 E_1^T \\
D_{12} C_2 & -D_{12} & 0 \\
D_{21} C_2 & -D_{21} & 0
\end{bmatrix}
\]

(83)

**Proof.** Knowing that equation (79) has a stabilizing solution \( X \geq 0 \), we can start from the following \( C_{\text{imp}}(s) \) which is transpose of the \( G_{\text{imp}} \) system for the standard \( G(s) \) with \( D_{12} \) of full
column rank (Doyle et al., 1989):

\[
\hat{G}_{\text{imp}}(s) = \begin{bmatrix}
(A + B_1 B_2 X)^T & -F_1^T E_1^T  \\
B_1^T & 0 & D_{12}^T
\end{bmatrix} = \begin{bmatrix}
A + B_1 B_2 X + W F_1^T E_1^T & F_\omega & B_1 + W C_1^T(D_1^T)^T  \\
-E_1^T F_\omega & 0 & E_1^T
\end{bmatrix},
\]  \hspace{1cm} (83)

Then replacing \(D_{12}^T E_{21}^T\) by \(E_{21}^T\) in equations (54), (56) and (57), equations (52)–(61) are all true and we have

\[
S_{\text{mp}}(s) = \begin{bmatrix}
A + B_1 B_2 X + W F_1^T E_1^T F_\omega & F_\omega & B_1 + W C_1^T(D_1^T)^T  \\
-E_1^T F_\omega & 0 & E_1^T
\end{bmatrix},
\]  \hspace{1cm} (84)

in the present case. Moreover, the same choices of \(F\) and \(H\) as in equations (66) and (69), respectively, give

\[
\Phi(\theta) = E_1^T Q D_{21} \in RH^\infty
\]  \hspace{1cm} (85)

and equation (73) with \(D_{12}^T E_{12}^T = E_{21}\). Then, the general solution to equation (85):

\[
Q = E_1^T Q D_{21} (N, W_2) = N \in RH^\infty, \quad W_1 \in RH^\infty
\]  \hspace{1cm} (86)

together with the same treatments as in equations (76) and (77) give equations (81) and (82).

The solution to the case where both \(D_{12}\) and \(D_{21}\) are of full row rank (i.e. \(D_{12}\) is nonstandard but \(D_{21}\) is standard) can be given by the dual of Theorem 2. First, we define \(D_{21}^T E_{21}^T\) and \(E_{21}^T D_{21}^T\) to satisfy

\[
\begin{bmatrix} D_{21}^T \end{bmatrix} (D_{21}^T D_{21}^T) = I_n, \quad E_{21}^T = D_{21}^T E_{21}^T
\]  \hspace{1cm} (87)

Then, we change equation (29) to the standard ARE:

\[
Y(A - B_1 D_{21} C_2)Y + (A - B_1 D_{21} C_2) Y + B_1 D_{21} (D_{21}^T )B] = 0,
\]  \hspace{1cm} (88)

and replace equation (32) by

\[
L_x = -B_1 D_{21}^T - Y C_1^T E_{21}^T
\]  \hspace{1cm} (89)

Then we have the following theorem.

Theorem 3. Suppose that both \(D_{12}\) and \(D_{21}\) are of full row rank and the assumptions (A2) and (A3) hold. Let \(L_x\) be any matrix which stabilizes the controllable subspace of \((A - B_1 D_{12}^T C_2, B_1 D_{12}^T )\). Then an \(H_\infty\) controller exists if and only if equations (27) and (88) have stabilizing solutions \(X \geq 0\) and \(Y \geq 0\) which satisfy \(p(XY) < 1\).

Under this condition, every \(H_\infty\) controller is parametrized by

\[
K_{\text{imp}}(s) = F_\omega \left( M_{\text{imp}}(s), \begin{bmatrix} N(\theta) \\ W_2(\theta) \end{bmatrix} \right),
\]  \hspace{1cm} (90)

where \(N(\theta) \in BH_F\), \(W_2(\theta) \in RH^{m-n}\) and

\[
M_{\text{imp}}(s) = \begin{bmatrix}
A & -Z L_w \quad Z B_1 D_{12}^T & B_1 D_{12}^T  \\
F_\omega & 0 & D_{12}  \\
-E_1^T F_\omega & 0 & 0
\end{bmatrix},
\]  \hspace{1cm} (91)

5. Conclusions

We have derived a complete and simple parametrization for all \(H_\infty\) controllers which solve a class of nonstandard \(H_\infty\) problems. Future work might consider use of the free parameters to meet other control performance objectives in addition to the objective \(|G_{\text{imp}}| < 1\).

References