Pergamon

PII: S0005-1098(97)00207-0

Automatica, Vol. 34, No. 3, pp. 369–374, 1998

0005-1098/98 $19.00 + 0.00

Brief Paper

A Complete and Simple Parametrization of Controllers for a Nonstandard H_∞ Control Problem*

TSUTOMU MITA,† JEREMY B. MATSON‡ and BRIAN D. O. ANDERSON§

Key Words—Nonstandard H_∞ control problem; zeros; parametrization of H_∞ controllers; Riccati equation; redundant inputs and outputs.

Abstract—A complete and simple parametrization of all H_∞ controllers is derived for a class of nonstandard H_∞ control problems with D_{12} and D_{21} being of full row and column ranks, respectively, by making use of the Youla parametrization and standard H_∞ control theory. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

The H_∞ control problem has been extensively studied for the last decade and Glover *et al.* (1988) and Doyle *et al.* (1989) solved the so-called standard problem, where the number of control inputs is less than or equal to the number of controlled outputs, and the number of measurement outputs is less than or equal to the number of external inputs.

In this paper, we deal with a nonstandard H_∞ control problem with a greater number of control inputs than controlled outputs and/or a greater number of measurement outputs than external inputs, under the conditions that certain direct feedthrough matrices D_{12} and D_{21} [defined below, see equation (1)] are of full row rank and full column rank, respectively. In fact, we often face a plant having a greater number of measurement outputs than external inputs in industrial applications. In this nonstandard problem, we can expect to have better control performance compared to the standard case, since every H_∞ controller will have more free parameters than the standard H_∞ controller (Doyle *et al.*, 1989). However, in order to use the free parameters effectively for other design purposes, we first need an explicit and simple formula of all controllers ensuring the closed-loop H_∞ norm bound. In this paper, we will provide a direct derivation of the solvability condition via Riccati equations and give a simple parametrization of all H_∞ controllers for the nonstandard H_∞ control problem.

Sampei *et al.* (1990) solved an H_∞ control problem without the standard assumptions on D_{12} or D_{21} by using algebraic Riccati inequalities, and they derived one H_∞ controller. Stoorvogel (1991, 1996) treated the same nonstandard H_∞ control problem where the solvability condition was reduced to checking the satisfaction of two elegant quadratic matrix inequalitiesin addition to some rank conditions. Moreover, a proper H_∞ controller was developed via the solution of the almost disturbance decoupling problem. However, no controller parametrization was shown. Actually, for our nonstandard H_∞ control problem also, though Stoorvogel's results can provide the solvability condition, they cannot immediately give the explicit parametrization of all H_∞ controllers. Scherer (1992) also gave the solvability condition of general H_∞ control problems, where $G_{12}(s)$ and $G_{21}(s)$ [defined below, see equation (1)] have imaginary-axis zeros including zeros at infinity, by using algebraic Riccati inequalities. He proposed an algorithm to compute an H_∞ controller. However, no controller parametrization was provided.

As for our nonstandard problems, a parametrization of H_∞ controllers for a singly nonstandard problem, i.e. one only of D_{12} and D_{21} is nonstandard in terms of dimensions, was provided in Zhang and Hosoe (1993) using relatively complicated expressions. Kimura *et al.* (1992) also solved the singly nonstandard problem under the condition that $G_{21}(s)$ has no unstable zeros. Le and Safonov (1992) displayed the freedom in the Youla parameter Q for a doubly nonstandard problem, i.e. both D_{12} and D_{21} are nonstandard. However, the final form of the H_∞ controller was not given (and is not straightforward to obtain). In this paper, which is a modified version of Mita *et al.* (1993), we obtain (we believe for the first time) a complete and simple parametrization of all H_∞ controllers for the doubly nonstandard problem.

In what follows, we express the star product of M_1 and M_2 by $M = M_1 * M_2$ so that $F_l(M_1, F_l(M_2, K)) = F_l(M, K)$ holds, where $F_l(*, *)$ denotes the standard lower linear fractional map (Zhou *et al.*, 1995). RH^∞ is the set of proper and stable rational functions and BH^∞ is the set of functions in RH^∞ whose H_∞ norm is less than unity (<1).

2. *Preliminaries*

2.1. *Assumptions.* In the H_∞ design, we first need generalized plants which describe input–output relations for given control problems. The generalized plant in this paper is given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{12} u, \\ y &= C_2 x + D_{21} w, \end{aligned} \tag{1}$$

or

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \tag{2}$$

where $x \in \mathbb{R}^n$ is the state variable; $w \in \mathbb{R}^r$ the external input; $u \in \mathbb{R}^p$ the control input; $z \in \mathbb{R}^m$ the controlled output; and $y \in \mathbb{R}^q$ the measurement output. Via the control law:

$$u = K(s)y, \tag{3}$$

* Received 1 December 1993; revised 20 July 1997; received in final form 29 October 1997. The preliminary version of this paper was presented at the 12th IFAC World Congress, which was held in Sydney, Australia, during 18–23 July 1993. This paper was recommended for publication in revised form by Associate Editor Y. Yamamoto under the direction of Editor Ruth F. Curtain. *Corresponding author* Professor T. Mita. Tel. + 81-3-5734-2544; Fax + 81-3-5734-3596; E-mail mita@ctrl.titech.ac.jp.

† Department of Control and Systems Engineering, Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo, 152, Japan.

‡ Department of Engineering, Mill Lane, Cambridge, CB2 1RX, U.K.

§ Research School of Information Sciences and Engineering, Australian National University, Canberra, ACT, 0200, Australia.

the control purpose is to stabilize the closed-loop system internally and ensure the H_∞ norm of the closed-loop transfer function

$$G_{zw} = F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{4}$$

is less than unity, i.e.

$$G_{zw}(s) \in BH^\infty. \tag{5}$$

In this paper, except for Section 4, we assume that $p \geq m$, $q \geq r$ with

(A1) D_{12} is of full row rank and D_{21} is of full column rank,

which is the opposite of the standard assumption. Under this assumption, we define $D_{12}^\dagger$, $D_{12}^\perp$, $D_{21}^\dagger$ and $D_{21}^\perp$ to satisfy the following:

$$\begin{bmatrix} D_{12} \\ (D_{12}^\perp)^T \end{bmatrix} (D_{12}^\dagger, D_{12}^\perp) = I_p, \qquad \begin{bmatrix} D_{21}^\dagger \\ D_{21}^\perp \end{bmatrix} (D_{21}, (D_{21}^\perp)^T) = I_q. \tag{6}$$

When D_{12} and D_{21} are square, we define $D_{12}^\perp = 0$ and $D_{21}^\perp = 0$. We also need the following assumptions, which are the same as the standard ones.

(A2) (A, B_2, C_2) is stabilizable and detectable;
(A3) $G_{12}(s)$ and $G_{21}(s)$ have no $j\omega$-axis invariant zeros.

Using the condition (A1), we can show that the assumption (A3) is equivalent to requiring that

$$(A - B_2D_{12}^\dagger C_1, B_2D_{12}^\perp), \ (A - B_1D_{21}^\dagger C_2, D_{21}^\perp C_2), \tag{7}$$

have no $j\omega$-axis uncontrollable and unobservable modes, respectively.

The purpose of this paper is to find conditions under which a solution exists to this nonstandard H_∞ control problem and to give a simple parametrization of all H_∞ controllers which solve the problem.

2.2. *Youla parametrization.* Before proceeding, we review the Youla parametrization (Zhou *et al.*, 1995) for all stabilizing controllers, which is independent of the ranks or dimensions of D_{12} and D_{21}.

Lemma 1. Every controller $K(s)$ which internally stabilizes $G(s)$ is given by

$$K(s) = F_l(M(s), Q(s)), \tag{8}$$

where $Q(s) \in RH^\infty$ is a free parameter and

$$M = \left[\begin{array}{c|cc} A_F + HC_2 & -H & B_2 \\ \hline F & 0 & I_p \\ C_2 & -I_q & 0 \end{array}\right], \tag{9}$$

with F and H being any matrices which stabilize:

$$A_F = A + B_2F, \qquad A_H = A + HC_2, \tag{10}$$

respectively.

In terms of this parametrization, $G_{zw}(s)$ in equation (4) is described by

$$G_{zw}(s) = F_l(G_Q(s), Q(s)) \tag{11}$$

where

$$G_Q = \left[\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ 0 & A_H & B_H & 0 \\ \hline C_F & -D_{12}F & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right], \tag{12}$$

with

$$C_F = C_1 + D_{12}F, \qquad B_H = B_1 + HD_{21} \tag{13}$$

In accordance with the assumption (A3) on the invariant zeros of $G_{12}(s)$, we introduce the following similarity transformation T displaying the uncontrollable part of the pair $(A - B_2D_{12}^\dagger C_1, B_2D_{12}^\perp)$:

$$T^{-1}(A - B_2D_{12}^\dagger C_1)T = \begin{bmatrix} A_w & 0 \\ A_{2w} & A_2 \end{bmatrix}, \qquad T^{-1}B_2D_{12}^\perp = \begin{bmatrix} 0 \\ \beta_F \end{bmatrix}, \tag{14}$$

where (A_2, β_F) is controllable and A_w has no $j\omega$-axis eigenvalues due to assumption (A3). We also define

$$T^{-1}B_2D_{12}^\dagger = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}. \tag{15}$$

Then assumption (A2) leads to the condition that (A_w, γ_1) is stabilizable.

Based on this consideration, let us choose F in Lemma 1 as

$$F = -D_{12}^\dagger C_1 + D_{12}^\dagger E_F + D_{12}^\perp L_F, \tag{16}$$

with

$$E_FT = (E_1, 0), \qquad L_FT = (L_1, L_2), \tag{17}$$

where E_1 and L_2 are any matrices that stabilize $A_w + \gamma_1E_1$ and $A_2 + \beta_FL_2$, respectively, and L_1 is free.

Then

$$T^{-1}A_FT = \begin{bmatrix} A_w + \gamma_1E_1 & 0 \\ A_{2w} + \gamma_2E_1 + \beta_FL_1 & A_2 + \beta_FL_2 \end{bmatrix}, \qquad C_FT = E_FT = (E_1, 0) \tag{18}$$

and A_F is stable.

The following lemma shows the advantage of this choice of F.

Lemma 2. With F chosen as in equation (16),

$$\left[\begin{array}{c|c} A_F & B_2 \\ \hline C_F & D_{12} \end{array}\right] = \left[\begin{array}{c|c} A_F & B_2D_{12}^\dagger \\ \hline C_F & I_m \end{array}\right] D_{12} \tag{19}$$

holds, moreover

$$\left[\begin{array}{c|c} A_F & B_2D_{12}^\dagger \\ \hline C_F & I_m \end{array}\right] \tag{20}$$

has no $j\omega$-axis invariant zeros.

Proof. From equations (14)–(18), we have

$$\left[\begin{array}{c|c} A_F & B_2D_{12}^\perp \\ \hline C_F & 0 \end{array}\right] = \left[\begin{array}{cc|c} * & 0 & 0 \\ * & * & \beta_F \\ \hline E_1 & 0 & 0 \end{array}\right] = 0, \tag{21}$$

which together with $B_2 = B_2[D_{12}^\dagger D_{12} + D_{12}^\perp(D_{12}^\perp)^T]$ lead to (19). The invariant zeros of (20) are given by all the eigenvalues of

$$T^{-1}(A_F - B_2D_{12}^\dagger C_F)T = \begin{bmatrix} A_w & 0 \\ A_{2w} + \beta_FL_1 & A_2 + \beta_FL_2 \end{bmatrix}, \tag{22}$$

because equation (20) is square. Since A_w has no $j\omega$-axis eigenvalues and $A_2 + \beta_FL_2$ is stable, equation (20) has no $j\omega$-axis invariant zeros. □

Dually, in accordance with assumption (A3) on the invariant zeros of $G_{21}(s)$, we consider the following similarity transformation S displaying for the unobservable part of the pair $(A - B_1D_{21}^\dagger C_2, D_{21}^\perp C_2)$:

$$S(A - B_1D_{21}^\dagger C_2)S^{-1} = \begin{bmatrix} A_1 & 0 \\ A_{1z} & A_z \end{bmatrix}, \qquad D_{12}^\perp C_2S^{-1} = (\beta_H, 0), \tag{23}$$

where (A_1, β_H) is observable and A_z has no $j\omega$-axis eigenvalues due to assumption (A3). In accordance with equation (23), define

$$D_{21}^{\dagger}C_2S^{-1} = (\delta_1, \delta_2). \quad (24)$$

Then assumption (A2) leads to the condition that (A_z, δ_2) is detectable.

Hence, A_H becomes stable and the dual of Lemma 2 holds if we choose H as

$$H = -B_1D_{21}^{\dagger} + E_HD_{21}^{\dagger} + L_HD_{21}^{\perp}, \quad (25)$$

with

$$SE_H = \begin{bmatrix} 0 \\ \tilde{E}_2 \end{bmatrix}, SL_H = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix}, \quad (26)$$

where $\tilde{E}_2$ and $\tilde{L}_1$ are any matrices which stabilize $A_z + \tilde{E}_2\delta_2$ and $A_1 + \tilde{L}_1\beta_H$, respectively, and $\tilde{L}_2$ is free.

3. *Main result and proof*

3.1. *AREs and main result.* Define the following algebraic Riccati equation (ARE):

$$X(A - B_2D_{12}^{\dagger}C_1 + B_2D_{12}^{\perp}L_F) + (A - B_2D_{12}^{\dagger}C_1 + B_2D_{12}^{\perp}L_F)^{\mathrm{T}}X$$
$$+ X(B_1B_1^{\mathrm{T}} - B_2D_{12}^{\dagger}(D_{12}^{\dagger})^{\mathrm{T}}B_2^{\mathrm{T}})X = 0, \quad (27)$$

where L_F as earlier is any matrix which stabilizes the controllable subspace of $(A - B_2D_{12}^{\dagger}C_1, B_2D_{12}^{\perp})$. The stabilizing solution to equation (27) is given by an X which stabilizes

$$A_X := A - B_2D_{12}^{\dagger}C_1 + B_2D_{12}^{\perp}L_F + (B_1B_1^{\mathrm{T}} - B_2D_{12}^{\dagger}(D_{12}^{\dagger})^{\mathrm{T}}B_2^{\mathrm{T}})X. \quad (28)$$

Dually, define the second ARE:

$$Y(A - B_1D_{21}^{\dagger}C_2 + L_HD_{21}^{\perp}C_2)^{\mathrm{T}} + (A - B_1D_{21}^{\dagger}C_2 + L_HD_{21}^{\perp}C_2)Y$$
$$+ Y(C_1^{\mathrm{T}}C_1 - C_2^{\mathrm{T}}(D_{21}^{\dagger})^{\mathrm{T}}D_{21}^{\dagger}C_2)Y = 0, \quad (29)$$

where L_H as earlier is any matrix which stabilizes the observable subspace of $(A - B_1D_{21}^{\dagger}C_2, D_{21}^{\perp}C_2)$. The stabilizing solution to equation (29) is given by a Y which stabilizes

$$A_Y := A - B_1D_{21}^{\dagger}C_2 + L_HD_{21}^{\perp}C_2 + Y(C_1^{\mathrm{T}}C_1 - C_2^{\mathrm{T}}(D_{21}^{\dagger})^{\mathrm{T}}D_{21}^{\dagger}C_2). \quad (30)$$

Note that such L_F and L_H can be chosen according to equations (17) and (26) so that L_2 and $\tilde{L}_1$ stabilize $A_2 + \beta_FL_2$ and $A_1 + \tilde{L}_1\beta_H$, respectively.

We show some properties of the stabilizing solutions first.

Lemma 3. (1) When equation (27) has a stabilizing solution $X \geq 0$, $A + B_2F_\infty$ is stable where

$$F_\infty := -D_{12}^{\dagger}C_1 - D_{12}^{\dagger}(D_{12}^{\dagger})^{\mathrm{T}}B_2^{\mathrm{T}}X + D_{12}^{\perp}L_F. \quad (31)$$

Moreover, the stabilizing solution satisfies $XB_2D_{12}^{\perp} = 0$.

(2) When equation (29) has a stabilizing solution $Y \geq 0$, $A + L_\infty C_2$ is stable where

$$L_\infty := -B_1D_{21}^{\dagger} - YC_2^{\mathrm{T}}(D_{21}^{\dagger})^{\mathrm{T}}D_{21}^{\dagger} + L_HD_{21}^{\perp}. \quad (32)$$

Moreover, the stabilizing solution satisfies $D_{21}^{\perp}C_2Y = 0$.

Proof. We only prove the first assertion. The second follows by duality.

Since

$$A_X = A + B_2F_\infty + B_1B_1^{\mathrm{T}}X \quad (33)$$

is stable, $(A + B_2F_\infty, B_1^{\mathrm{T}}X)$ is detectable. Moreover (27) can be written as:

$$X(A + B_2F_\infty) + (A + B_2F_\infty)^{\mathrm{T}}X + XB_1B_1^{\mathrm{T}}X$$
$$+ XB_2D_{12}^{\dagger}(D_{12}^{\dagger})^{\mathrm{T}}B_2^{\mathrm{T}}X = 0. \quad (34)$$

These facts, together with $X \geq 0$ imply the stability of $A + B_2F_\infty$. Since

$$T^{-1}(A - B_2D_{12}^{\dagger}C_1 + B_2D_{12}^{\perp}L_F)T = \begin{bmatrix} A_w & 0 \\ A_{2w} + \beta_FL_1 & A_2 + \beta_FL_2 \end{bmatrix}, \quad (35)$$

holds for some $L_FT = (L_1, L_2)$ where $A_2 + \beta_FL_2$ is stable, it follows from Lemma A1 (see Appendix A) with $V = (0, I)^{\mathrm{T}}$ and $A_{\mathrm{sub}} = A_2 + \beta_FL_2$ that the stabilizing solution to equation (27) has the form

$$X = (T^{-1})^{\mathrm{T}}\begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \quad (36)$$

where X_{11} is the stabilizing solution to

$$X_{11}A_w + A_w^{\mathrm{T}}X_{11} + X_{11}(B_{11}B_{11}^{\mathrm{T}} - \gamma_1\gamma_1^{\mathrm{T}})X_{11} = 0, \quad (37)$$

with

$$T^{-1}B_1 := \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}. \quad (38)$$

Then equation (36) together with the second relation in equation (14) yield $XB_2D_{12}^{\perp} = 0$. □

The main result of this paper now follows.

Theorem 1. Suppose that assumptions (A1)–(A3) hold. Let L_F and L_H be any matrices which stabilize the controllable subspace of $(A - B_2D_{12}^{\dagger}C_1, B_2D_{12}^{\perp})$ and observable subspace of $(A - B_1D_{21}^{\dagger}C_2, D_{21}^{\perp}C_2)$, respectively. Then an H_∞ controller exists if and only if equations (27) and (29) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(YX) < 1$.* Under such conditions, every H_∞ controller $K_\infty(s)$ can be parametrized as

$$K_\infty(s) = F_l\left(M_\infty(s), \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\right), \quad (39)$$

where

$$N(s) \in BH_{m\times r}^{\infty}, \quad W_1(s) \in RH_{m\times(q-r)}^{\infty},$$
$$W_2(s) \in RH_{(p-m)\times r}^{\infty}, \quad W_3(s) \in RH_{(p-m)\times(q-r)}^{\infty},$$

and

$$M_\infty(s) := \left[\begin{array}{c|ccc} \hat{A} & -ZL_\infty & Z\hat{B}_2D_{12}^{\dagger} & B_2D_{12}^{\perp} \\ \hline F_\infty & 0 & D_{12}^{\dagger} & D_{12}^{\perp} \\ D_{21}^{\dagger}\hat{C}_2 & -D_{21}^{\dagger} & 0 & 0 \\ D_{21}^{\perp}C_2 & -D_{21}^{\perp} & 0 & 0 \end{array}\right], \quad (40)$$

with

$$\hat{A} := A + B_1B_1^{\mathrm{T}}X + B_2F_\infty + ZL_\infty\hat{C}_2,$$
$$\hat{B}_2 = B_2 + YC_1^{\mathrm{T}}D_{12}, \qquad \hat{C}_2 = C_2 + D_{21}B_1^{\mathrm{T}}X, \quad (41)$$
$$Z = (I - YX)^{-1}.$$

We will prove this theorem by the following two steps.

1. Applying Lemmas 1 and 2 to an FI (full information) system, we first prove that equation (27) has a stabilizing solution $X \geq 0$. Then, by the dual argument, equation (29) has a stabilizing solution $Y \geq 0$.
2. We apply a lossless decomposition (Doyle *et al.*, 1989) to $G(s)$ to obtain $G_{\mathrm{tmp}}(s)$, then we again apply a lossless decomposition to $G_{\mathrm{tmp}}^{\mathrm{T}}(s)$ to get $S_{\mathrm{tmp}}^{\mathrm{T}}(s)$, which is a G_{tmp} system for $G_{\mathrm{tmp}}^{\mathrm{T}}(s)$. In this step, we prove $\rho(XY) < 1$ and derive the controller parametrization by applying Lemma 1 to $S_{\mathrm{tmp}}(s)$.

**Remark.* The stability of A_X and existence of L_F are equivalent requirements to the condition that rank $(L_\gamma^{\mathrm{T}}(X, s), F_\gamma^{\mathrm{T}}(X))^{\mathrm{T}} = n + \text{normalrank } G_{12}(s)\ \forall s \in C_0 \cup C_+$ (Stoorvogel, 1991).

3.2. *Existence of the stabilizing solutions.* We will now show that equations (27) and (29) have stabilizing solutions $X \geq 0$ and $Y \geq 0$, respectively.

As is well known (Doyle *et al.*, 1989), the output feedback control problem has a solution only if the FI problem is solvable, where the generalized plant $G^{FI}(s)$ for the FI problem is given by equation (1) with C_2 and D_{21} being replaced by

$$C_2^{FI} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \qquad D_{21}^{FI} = \begin{bmatrix} 0 \\ I_r \end{bmatrix}. \tag{42}$$

We apply Lemma 1 to $G^{FI}(s)$ to simplify the problem.

To this end, we first set the matrix F in Lemma 1 as given in equation (16). Then we choose H as

$$H = (F, \ -B_1), \tag{43}$$

so that $B_H = B_1 + HD_{21}^{FI} = 0$ and $A_H = A + HC_2^{FI} = A_F$ hold. Then, from equations (11), (12) and Lemma 2, we have

$$G_{zw}(s) = F_l\left(\left[\begin{array}{c|cc} A_F & B_1 & B_2 \\ \hline C_F & 0 & D_{12} \\ C_2 & D_{21}^{FI} & 0 \end{array}\right], \ Q(s)\right)$$

$$= F_l\left(\left[\begin{array}{c|cc} A_F & B_1 & B_2 D_{12}^\dagger \\ \hline C_F & 0 & I_m \\ 0 & D_{21}^{FI} & 0 \end{array}\right], \ D_{12}Q(s)\right). \tag{44}$$

Since the existence of an H_∞ control solution is independent of any particular Youla parametrization and equation (20) has no $j\omega$-axis invariant zeros, standard H_∞ control theory (Glover *et al.*, 1988) can be applied to conclude that the following ARE:

$$X(A_F - B_2 D_{12}^\dagger C_F) + (A_F - B_2 D_{12}^\dagger C_F)^T X$$
$$+ X(B_1 B_1^T - B_2 D_{12}^\dagger (D_{12}^\dagger)^T B_2^T) X = 0, \tag{45}$$

must have a stabilizing solution $X \geq 0$ for the FI problem to be solved. Equation (45) is just equation (27). By duality, equation (29) also must have a stabilizing solution $Y \geq 0$.

3.3. *Controller parametrization*

3.3.1. *Introduction of* G_{tmp}. Knowing that equation (27) has a stabilizing solution, let us define

$$\Theta(s) = \left[\begin{array}{c|cc} A + B_2 F_\infty & B_1 & B_2 D_{12}^\dagger \\ \hline -(D_{12}^\dagger)^T B_2^T X & 0 & I_m \\ -B_1^T X & I_r & 0 \end{array}\right] := \left[\begin{array}{c|c} A_\Theta & B_\Theta \\ \hline C_\Theta & D_\Theta \end{array}\right],$$

$$G_{tmp}(s) = \left[\begin{array}{c|cc} A + B_1 B_1^T X & B_1 & B_2 \\ \hline -D_{12} F_\infty & 0 & D_{12} \\ \hat{C}_2 & D_{21} & 0 \end{array}\right], \tag{46}$$

$$\hat{C}_2 = C_2 + D_{21} B_1^T X,$$

for our generalized plant with D_{12} of full row rank. Then the following relation holds:

$$\Theta(s) * G_{tmp}(s) = G(s). \tag{47}$$

Moreover, we can prove that $\Theta_{21}^{-1}(s) \in RH^\infty$, $\Theta_{22}(\infty) = 0$ and

$$XA_\Theta + A_\Theta^T X + C_\Theta^T C_\Theta = 0,$$
$$B_\Theta^T X + D_\Theta^T C_\Theta = 0, \qquad D_\Theta^T D_\Theta = I_{m+r}. \tag{48}$$

Therefore, $\Theta(s)$ is a lossless matrix.

Then it follows from Doyle *et al.* (1989) that the original H_∞ control problem reduces to finding a $K(s)$ that satisfies

$$F_l(G_{tmp}(s), K(s)) \in BH^\infty, \tag{49}$$

or, equivalently,

$$F_l(G_{tmp}^T(s), K^T(s)) \in BH^\infty, \tag{50}$$

where

$$G_{tmp}^T(s) = \left[\begin{array}{c|cc} (A + B_1 B_1^T X)^T & -F_\infty^T D_{12}^T & \hat{C}_2^T \\ \hline B_1^T & 0 & D_{21}^T \\ B_2^T & D_{12}^T & 0 \end{array}\right]. \tag{51}$$

3.3.2. *Proof of* $\rho(XY) < 1$ *and introduction of* S_{tmp}. Since D_{21}^T is of full row rank and $G_{tmp_{12}}(s)$ has no $j\omega$-axis invariant zeros due to the assumption (A3) for $G_{21}(s)$, G_{tmp}^T can be treated in manner completely analogous to the way that $G(s)$ was treated in the previous section.

Following the above reasoning, we first choose L_H^T to stabilize the controllable part of the pair consisting of

$$(A + B_1 B_1^T X)^T - \hat{C}_2^T (D_{21}^\dagger)^T B_1^T = (A - B_1 D_{21}^\dagger C_2)^T, \tag{52}$$

and

$$\hat{C}_2^T (D_{21}^\perp)^T = (D_{21}^\perp C_2)^T, \tag{53}$$

i.e. L_H stabilizes the observable part of the pair $(A - B_1 D_{21}^\dagger C_2$, $D_{21}^\perp C_2)$. Then, for equation (50) to be solvable, the ARE:

$$WA_{ZH}^T + A_{ZH}W + W(F_\infty^T D_{12}^T D_{12} F_\infty - \hat{C}_2^T (D_{21}^\dagger)^T D_{21}^\dagger \hat{C}_2)W = 0 \tag{54}$$

must have a stabilizing solution $W \geq 0$ which satisfies $D_{21}^\perp \hat{C}_2 W = D_{21}^\perp C_2 W = 0$, where

$$A_{ZH} := A - B_1 D_{21}^\dagger C_2 + L_H D_{21}^\perp C_2. \tag{55}$$

Since $D_{21}^\perp C_2 W = 0$, it follows from equation (54) that

$$WA_{ZH}^T + A_{ZH}W + W(F_\infty^T D_{12}^T D_{12} F_\infty - \hat{C}_2^T (D_{21}^\dagger)^T D_{21}^\dagger \hat{C}_2$$
$$+ XL_H D_{21}^\perp C_2 + C_2^T (D_{21}^\perp)^T L_H^T X)W = 0, \tag{56}$$

also has a stabilizing solution, which means that $H_W \in \mathrm{Dom(Ric)}$ where

$$H_W := \begin{bmatrix} A_{ZH}^T, & F_\infty^T D_{12}^T D_{12} F_\infty - \hat{C}_2^T (D_{21}^\dagger)^T D_{21}^\dagger \hat{C}_2 + XL_H D_{21}^\perp C_2 + C_2^T (D_{21}^\perp)^T L_H^T X \\ 0, & -A_{ZH} \end{bmatrix}. \tag{57}$$

Then, we obtain (see Appendix B):

$$H_Y = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} H_W \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix};$$

$$H_Y = \begin{bmatrix} A_{ZH}^T, & C_1^T C_1 - C_2^T (D_{21}^\dagger)^T D_{21}^\dagger C_2 \\ 0, & -A_{ZH} \end{bmatrix}, \tag{58}$$

where H_Y is the Hamiltonian matrix corresponding to the ARE equation (29). This leads to

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \tag{59}$$

for the stabilizing solutions $Y = Y_2Y_1^{-1} \geq 0$ and $W = W_2W_1^{-1} \geq 0$. From equation (59), we can prove the following using well-known manipulations (Doyle *et al.*, 1989):

$$W = (I - YX)^{-1}Y, \ \rho(XY) < 1. \tag{60}$$

Since W is a stabilizing solution, we can construct

$$S_{\rm tmp}^{\rm T}(s) = \left[\begin{array}{c|cc} (A + B_1B_1^{\rm T}X)^{\rm T} + F_\infty^{\rm T}D_{12}^{\rm T}D_{12}F_\infty W & -F_\infty^{\rm T}D_{12}^{\rm T} & \hat{C}_2^{\rm T} \\ \hline B_1^{\rm T} + D_{21}^{\dagger}\hat{C}_2W & 0 & D_{21}^{\rm T} \\ B_2^{\rm T} - D_{12}^{\rm T}D_{12}F_\infty W & D_{12}^{\rm T} & 0 \end{array}\right], \tag{61}$$

as the $G_{\rm tmp}$ system for equation (51) by observing the relationship between $G(s)$ and $G_{\rm tmp}(s)$ in equation (46). Then, take the transpose of equation (61) as follows:

$$S_{\rm tmp}(s) = \left[\begin{array}{c|cc} A + B_1B_1^{\rm T}X + WF_\infty^{\rm T}D_{12}^{\rm T}D_{12}F_\infty & B_1 + W\hat{C}_2^{\rm T}(D_{21}^{\dagger})^{\rm T} & B_2 - WF_\infty^{\rm T}D_{12}^{\rm T}D_{12} \\ \hline -D_{12}F_\infty & 0 & D_{12} \\ \hat{C}_2 & D_{21} & 0 \end{array}\right]. \tag{62}$$

Now satisfying equation (50) reduces to finding a $K(s)$ that satisfies

$$\Phi(s) = F_l(S_{\rm tmp}(s), K(s)) \in BH^\infty. \tag{63}$$

3.3.3. *Parametrization of all controllers.* In order to display the solution set, we first define

$$Z := (I - YX)^{-1}, \tag{64}$$

which leads to

$$W = ZY, \qquad Z = I + WX. \tag{65}$$

We now apply the Youla parametrization as given in Lemma 1 to $S_{\rm tmp}$ to simplify $\Phi(s)$ in equation (63). To this end, we first choose

$$F = F_\infty. \tag{66}$$

Then

$$A_F = A + B_1B_1^{\rm T}X + WF_\infty^{\rm T}D_{12}^{\rm T}D_{12}F_\infty + (B_2 - WF_\infty^{\rm T}D_{12}^{\rm T}D_{12})F_\infty = A_X \tag{67}$$

which guarantees the stability of A_F.

On the other hand, since equation (56) has a stabilizing solution W with $D_{21}^{\perp}C_2W = 0$,

$$A_W := A_{ZH} + W(F_\infty^{\rm T}D_{12}^{\rm T}D_{12}F_\infty - \hat{C}_2^{\rm T}(D_{21}^{\dagger})^{\rm T}D_{21}^{\dagger}\hat{C}_2 + XL_HD_{21}^{\perp}C_2) \tag{68}$$

is stable. Therefore, the choice:

$$H = -B_1D_{21}^{\dagger} - W\hat{C}_2^{\rm T}(D_{21}^{\dagger})^{\rm T}D_{21}^{\dagger} + ZL_HD_{21}^{\perp}, \tag{69}$$

and $Z = I + WX$ show that

$$A_H = A + B_1B_1^{\rm T}X + WF_\infty^{\rm T}D_{12}^{\rm T}D_{12}F_\infty + H\hat{C}_2 = A_W, \tag{70}$$

which guarantees the stability of A_H for $S_{\rm tmp}(s)$.

Moreover, we obtain

$$C_F = -D_{12}F_\infty + D_{12}F = 0,$$
$$B_H = B_1 + W\hat{C}_2^{\rm T}(D_{21}^{\dagger})^{\rm T} + HD_{21} = 0. \tag{71}$$

Therefore, application of equations (8)–(12) to $S_{\rm tmp}(s)$ give

$$K(s) = F_l(M(s), Q(s)), \tag{72}$$

where

$$M(s) = \left[\begin{array}{c|cc} A_X + H\hat{C}_2 & -H & B_2 - WF_\infty^{\rm T}D_{12}^{\rm T}D_{12} \\ \hline F_\infty & 0 & I_p \\ \hat{C}_2 & -I_q & 0 \end{array}\right], \tag{73}$$

and

$$\Phi(s) = D_{12}QD_{21} \in BH^\infty. \tag{74}$$

Observe that equation (74) gives the following general solution set of $Q(s) \in RH^\infty$:

$$Q(s) = [D_{12}^{\dagger}, \ D_{12}^{\perp}]\begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\begin{bmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{bmatrix}, \tag{75}$$

where $N \in BH^\infty$ and $W_i \in RH^\infty$. Moreover, we can rearrange some parameters in M as follows:

$$H = -(I + WX)B_1D_{21}^{\dagger} - WC_2(D_{21}^{\dagger})^{\rm T}D_{21}^{\dagger} + ZL_HD_{21}^{\perp}$$
$$= Z[-B_1D_{21}^{\dagger} - YC_2(D_{21}^{\dagger})^{\rm T}D_{21}^{\dagger} + L_HD_{21}^{\perp}] = ZL_\infty$$
$$B_2 - WF_\infty^{\rm T}D_{12}^{\rm T}D_{12} = B_2 + WC_1^{\rm T}D_{12} + WXB_2[I - D_{12}^{\perp}(D_{12}^{\perp})^{\rm T}]$$
$$= (I + WX)B_2 + WC_1^{\rm T}D_{12}$$
$$= Z[B_2 + YC_1^{\rm T}D_{12}] = Z\hat{B}_2. \tag{76}$$

Therefore, equations (72)–(76) together with

$$Z\hat{B}_2D_{12}^{\perp} = B_2D_{12}^{\perp}, \qquad D_{21}^{\perp}\hat{C}_2 = D_{21}^{\perp}C_2 \tag{77}$$

lead to equation (40).

4. *Discussions*

The case where both D_{12} and D_{21} are of full column rank (i.e. D_{12} is standard but D_{21} is nonstandard) can be solved by changing some of the treatments in the previous section.

We first define $D_{12}^{\dagger}$, $D_{12}^{\perp}$ and E_{12} to satisfy

$$\begin{bmatrix} D_{12}^{\dagger} \\ D_{12}^{\perp} \end{bmatrix}(D_{12}, (D_{12}^{\perp})^{\rm T}) = I_m, \ E_{12} = D_{12}^{\rm T}D_{12}. \tag{78}$$

Then, we change equation (27) to the standard ARE:

$$X(A - B_2D_{12}^{\dagger}C_1) + (A - B_2D_{12}^{\dagger}C_1)^{\rm T}X + X(B_1B_1^{\rm T} - B_2E_{12}^{-1}B_2^{\rm T})X + C_1^{\rm T}(D_{12}^{\perp})^{\rm T}D_{12}^{\perp}C_1 = 0, \tag{79}$$

and replace equation (31) by

$$F_\infty = -D_{12}^{\dagger}C_1 - E_{12}^{-1}B_2^{\rm T}X. \tag{80}$$

Then we have the following theorem.

Theorem 2. Suppose that both D_{12} and D_{21} are of full column rank and the assumptions (A2) and (A3) hold. Let L_H be any matrix which stabilizes the observable subspace of $(A - B_1D_{21}^{\dagger}C_2, D_{21}^{\perp}C_2)$. Then an H_∞ controller exists if and only if equations (79) and (29) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.

Under this condition, every H_∞ controller is parametrized by

$$K_\infty(s) = F_l(M_\infty(s), [N(s), W_1(s)]), \tag{81}$$

where $N(s) \in BH_{p\times r}^\infty$, $W_1(s) \in RH_{p\times(q-r)}^\infty$ and

$$M_\infty(s) := \left[\begin{array}{c|cc} \hat{A} & -ZL_\infty & Z\hat{B}_2E_{12}^{-1/2} \\ \hline F_\infty & 0 & E_{12}^{-1/2} \\ D_{12}^{\dagger}\hat{C}_2 & -D_{21}^{\dagger} & 0 \\ D_{21}^{\perp}C_2 & -D_{21}^{\perp} & 0 \end{array}\right]. \tag{82}$$

Proof. Knowing that equation (79) has a stabilizing solution $X \geq 0$, we can start from the following $\hat{G}_{\rm tmp}^{\rm T}(s)$ which is transposition of the $G_{\rm tmp}$ system for the standard $G(s)$ with D_{12} of full column rank (Doyle *et al.*, 1989):

$$\hat{G}^{\mathrm{T}}_{\mathrm{tmp}}(s) = \left[\begin{array}{c|cc} (A + B_1B_1^{\mathrm{T}}X)^{\mathrm{T}} & -F_\infty^{\mathrm{T}}E_{12}^{1/2} & \hat{C}_2^{\mathrm{T}} \\ \hline B_1^{\mathrm{T}} & 0 & D_{12}^{\mathrm{T}} \\ B_2^{\mathrm{T}} & E_{12}^{1/2} & 0 \end{array}\right]. \qquad (83)$$

Then replacing $D_{12}^{\mathrm{T}}D_{12}$ by E_{12} in equations (54), (56) and (57), equations (52)–(61) are all true and we have

$$S_{\mathrm{tmp}}(s) = \left[\begin{array}{c|cc} A + B_1B_1^{\mathrm{T}}X + WF_\infty^{\mathrm{T}}E_{12}F_\infty & B_1 + W\hat{C}_2^{\mathrm{T}}(D_{21}^{\dagger})^{\mathrm{T}} & B_2 - WF_\infty^{\mathrm{T}}E_{12} \\ \hline -E_{12}^{1/2}F_\infty & 0 & E_{12}^{1/2} \\ \hat{C}_2 & D_{21} & 0 \end{array}\right] \qquad (84)$$

in the present case. Moreover, the same choices of F and H as in equations (66) and (69), respectively, give

$$\Phi(s) = E_{12}^{1/2}QD_{21} \in BH^\infty \qquad (85)$$

and equation (73) with $D_{12}^{\mathrm{T}}D_{12} = E_{12}$. Then, the general solution to equation (85):

$$Q = E_{12}^{-1/2}(N, W_1)\begin{bmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{bmatrix}, \qquad N \in BH^\infty,\ W_1 \in RH^\infty \qquad (86)$$

together with the same treatments as in equations (76) and (77) give equations (81) and (82). □

The solution to the case where both D_{12} and D_{21} are of full row rank (i.e. D_{12} is nonstandard but D_{21} is standard) can be given by the dual of Theorem 2. First, we define $D_{21}^{\dagger}$, $D_{21}^{\perp}$ and E_{21} to satisfy

$$\begin{bmatrix} D_{21} \\ (D_{21}^{\perp})^{\mathrm{T}} \end{bmatrix}(D_{21}^{\dagger}, D_{21}^{\perp}) = I_r, \qquad E_{21} = D_{21}D_{21}^{\mathrm{T}}. \qquad (87)$$

Then, we change equation (29) to the standard ARE:

$$\begin{aligned} &Y(A - B_1D_{21}^{\dagger}C_2)^{\mathrm{T}} + (A - B_1D_{21}^{\dagger}C_2)Y \\ &\quad + Y(C_1^{\mathrm{T}}C_1 - C_2^{\mathrm{T}}E_{21}^{-1}C_2)Y + B_1D_{21}^{\perp}(D_{21}^{\perp})^{\mathrm{T}}B_1^{\mathrm{T}} = 0, \end{aligned} \qquad (88)$$

and replace equation (32) by

$$L_\infty = -B_1D_{21}^{\dagger} - YC_2^{\mathrm{T}}E_{21}^{-1}. \qquad (89)$$

Then we have the following theorem.

Theorem 3. Suppose that both D_{12} and D_{21} are of full row rank and the assumptions (A2) and (A3) hold. Let L_F be any matrix which stabilizes the controllable subspace of $(A - B_2D_{12}^{\dagger}C_1, B_2D_{12}^{\perp})$. Then an H_∞ controller exists if and only if equations (27) and (88) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.

Under this condition, every H_∞ controller is parametrized by

$$K_\infty(s) = F_l\left(M_\infty(s), \begin{bmatrix} N(s) \\ W_2(s) \end{bmatrix}\right), \qquad (90)$$

where $N(s) \in BH^\infty_{m\times q}$, $W_2(s) \in RH^\infty_{(p-m)\times q}$ and

$$M_\infty(s) := \left[\begin{array}{c|ccc} \hat{A} & -ZL_\infty & Z\hat{B}_2D_{12}^{\dagger} & B_2D_{12}^{\perp} \\ \hline F_\infty & 0 & D_{12}^{\dagger} & D_{12}^{\perp} \\ E_{21}^{-1/2}\hat{C}_2 & -E_{21}^{-1/2} & 0 & 0 \end{array}\right]. \qquad (91)$$

5. *Conclusions*

We have derived a complete and simple parametrization for all H_∞ controllers which solve a class of nonstandard H_∞ problems. Future work might consider use of the free parameters to meet other control performance objectives in addition to the objective $\|G_{zw}\|_\infty < 1$.

References

Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis (1989). State space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Automat Control*, **34**(8), 831–847.

Glover, K. and J. C. Doyle (1988). State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity. *System Control Lett.*, **11**, 167–172.

Kimura, H., Y. Lu and T. Fujii (1992). Parametrizations of H_∞ controllers for non-standard plant. *Proc. 1st IFAC meeting on Design Method of Control Systems*, pp. 318–323.

Le, V. X. and M. G. Safonov (1992). Rational matrix GCD's and design of squaring-down compensators—a state space theory. *IEEE Trans. Automat Control*, **37**(3), 384–392.

Mita, T., J. B. Matson and B. D. O. Anderson (1993). A singular H_∞ control problem. *Proc. IFAC'93*, pp. 121–126.

Sampei, M., T. Mita and M. Nakamichi (1990). An algebraic approach to H_∞ output feedback control problems. *System Control Lett.*, **14**, 13–24.

Scherer, S. (1992). H_∞-optimization without assumptions on finite or infinite zeros. *SIAM J. Control Optim.*, **30**(1), 143–166.

Stoorvogel, A. A. (1991). The singular H_∞ control problem with dynamic measurement feedback. *SIAM J. Control Optim.*, **29**, 160–184.

Stoorvogel, A. A. (1996). The H_∞ control problem with zeros on the boundary of the stability domain. *Int. J. Control*, **63**, 1029–1053.

Zhang, F. and S. Hosoe (1993). H_∞ control problem with partial measurement states. *Trans. SICE*, **29**(2), 169–178 (in Japanese).

Zhou, K., J. C. Doyle and K. Glover (1995). *Robust and Optimal Control.* Prentice-Hall, Englewood Cliffs, NJ.

Appendix A: A property of a Riccati equation

Lemma A1. Suppose that $AV = VA_{\mathrm{sub}}$ with A_{sub} stable. Then, the stabilizing solution to

$$XA + A^{\mathrm{T}}X + XRX = 0 \qquad (A1)$$

satisfies $XV = 0$.

Proof. Multiplication by V on the right of equation (A1) yields

$$XVA_{\mathrm{sub}} + (A + RX)^{\mathrm{T}}XV = 0. \qquad (A2)$$

Since A_{sub} and $A + RX$ are stable, we have $XV = 0$. □

Appendix B: Proof of equation (58)

Equation (58) is equivalent to

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}H_W - H_Y\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = 0. \qquad (A3)$$

All the entries except the (1, 2)th are apparently 0. Using $XB_2D_{12}^{\perp} = 0$ and equation (27), it is easy to confirm that (1, 2)th entry is also 0. Even D_{12} is of full column rank, the same procedure with equation (79) proves equation (A3).