Relating H_2- and H_∞-Norm Bounds for Sampled-Data Systems

Shinji Hara, Brian D. O. Anderson, and Hisaya Fujioka

***Abstract*— We relate the H_∞ and H_2 norms for multi-input/multi-output sampled-data feedback control systems, where a continuous-time plant is controlled by a digital compensator with hold and sampler. Upper bounds on both H_2 and H_∞ norms are obtained based on fundamental relations derived by two different approaches, namely the hybrid state-space approach and the fast sampling and lifting approach.**

***Index Terms*—H_2 norm, H_∞ norm, norm bounds, sampled-data system.**

I. Introduction

The H_2 and H_∞ norms are the most popular performance measures for control system analysis and synthesis, and the theory of optimal H_2 and H_∞ control has been widely developed in the last ten years as an advanced design methodology. H_∞-control synthesis relates to a worst case synthesis, taking account of the robustness of the feedback system to plant perturbations, while H_2 control is a generalization of LQG optimal control. The only obvious connection between the associated two norms is that, for discrete-time transfer functions, the H_2 norm is bounded by the H_∞ norm

$$\|G_d[z]\|_2^2 \leq \min\{m, p\}\|G_d[z]\|_\infty^2 \tag{1}$$

which holds for any stable discrete-time transfer function $G_d[z]$ with m inputs and p outputs. Recently, it has been shown for both single-input/single-output (SISO) continuous and discrete-time systems that given precise or certain partial knowledge of the poles of a scalar transfer function, it is possible to obtain an upper bound for the H_∞ norm as a function of the H_2 norm [3]. Also, given bandwidth information for a continuous-time system, it is possible to obtain an upper bound for the H_2 norm as a function of the H_∞ norm. Note that results connecting other different norms are rather scarce. Some inequalities connecting the l_1-norm and the H_∞ norm can be found in [5], while [5] and [9] establish inequalities between the H_∞ norm and the Hankel singular values of the system.

Sampled-data feedback control has also been paid a lot of attention in the area of control system design, motivated by rapid progress of computer and digital technologies. Several different approaches have been introduced to analyze and design sampled-data feedback systems, where a continuous-time plant is controlled by a digital compensator with appropriate hold and sample devices [1]–[2], [6], [8], [11]–[22]. Most notably, H_∞-type and H_2-type optimization problems for sampled-data feedback systems have been investigated

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in order to take into account intersampling behavior, as opposed to just the response values at sampling instants. The problem settings [6], [15], [18] are natural extensions of the H_2- and H_∞-optimization problems for continuous-time systems. It is known that the analysis and synthesis of H_2 and H_∞ problems can be reduced to equivalent discrete-time problems for computation and synthesis of H_2 and H_∞ problems, respectively (H_2: [6], [18], H_∞: [2], [11]–[15], [21]). However, relationships among them have not yet been clarified at all.

The purpose of this paper is to relate the H_2 and H_∞ norms for multi-input/multi-output (MIMO) sampled-data feedback systems with continuous-time plant, discrete-time controller, sampler, and hold device. The following are our motivations for the work.

- m1) The first motivation is simple theoretical interest. If we know general relationships between the H_2 and H_∞ norms, then they help us to understand the performance of sampled-data feedback systems and may become a basis of future investigations, like the ideas of [5] and [9].
- m2) The second one is for the analysis and synthesis of a sampled-data feedback system. For example, the relation will give an answer to the following primitive question: *How can we evaluate the achievable H_∞ norm just by solving an easier problem, the H_2-optimal control problem?*
- m3) The last one arises from computational considerations. We need iteration for the computation and optimization of the H_∞ norm, while no iteration is required for the H_2 norm calculation. Hence, the bounds obtained from H_2 calculation can be used as upper or lower bounds in the bisection type (γ-iteration) procedures for the H_∞-norm computation.

The paper is organized as follows: Section II gives fundamental results on the H_2 and H_∞ norms for hybrid systems by two different approaches, namely the hybrid state-space approach [15] in Section II-A and the fast sampling and lifting approach [16] in Section II-B. An upper bound for the H_∞ norm is obtained in Section III, where we also treat MIMO discrete-time systems as a preliminary. An upper bound for the H_2 norm is derived in Section IV. The upper bounds in Sections III and IV bear on our motivations m2) and m3).

II. Sampled-Data System

We here give fundamental results on the H_2 and H_∞ norms of sampled-data systems by two different approaches, namely the hybrid state-space model approach and fast sampling and lifting approach.

A. Hybrid State-Space Model Approach

We consider a linear sampled-data system $\mathcal{G}_h$ expressed as a hybrid state-space representation [14], [15]

$$\begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix} = \begin{bmatrix} A_c + A_{cs}(t)\mathcal{S}_\tau & A_{cd}(t) \\ A_{ds}\mathcal{S}_\tau & A_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} w(t)$$

$$z(t) = \begin{bmatrix} C_c + C_{cs}(t)\mathcal{S}_\tau & C_{cd}(t) \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} \tag{2}$$

where $k\tau \le t < (k+1)\tau$. $x_c(t) \in \mathcal{R}^{n_c}$ and $x_d[k] \in \mathcal{R}^{n_d}$ denote the analog and discrete-state variables, respectively, $w(t) \in \mathcal{R}^m$ is the piecewise continuous input, and $z(t) \in \mathcal{R}^p$ is the continuous output. $\mathcal{S}_\tau$ denotes the sampling operator with sampling period τ satisfying $(\mathcal{S}_\tau v)[k] = v(k\tau)$ for any $v(t)$. A_c, A_{ds}, A_d, B_c, and C_c are constant matrices and $A_{cs}(t), A_{cd}(t), C_{cs}(t)$, and $C_{cd}(t)$ are τ-periodic matrices of appropriate dimensions.

Formula (2) is a reduced version of the general one which encompasses any sampled-data system [15]. We use (2) instead of a more general equation to ensure the output $z(t)$ lies in L_2 for either an L_2 input $w(t)$ and/or an impulse input $w(t)$. In this sense, the model is general enough for representing sampled-data systems having finite H_2 and H_∞ norms. For example, a general sampled-data feedback control system depicted in Fig. 1 with state-space realizations of $P(s)$ and $K[z]$ is in this class under mild reasonable assumptions.

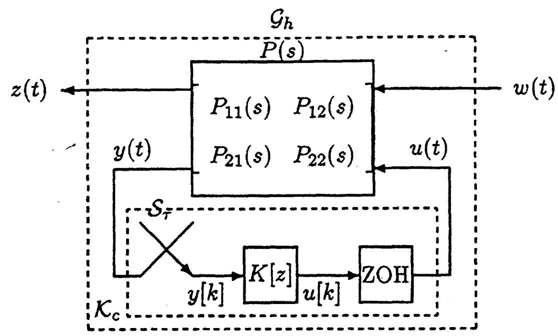

Fig. 1. Sampled-data feedback control system.

The next lemma gives the closed-loop stability condition.

Lemma 1 [14]: System (2) is stable if and only if all the eigenvalues of the matrix $\mathcal{A}_d$ lie in the inside of the unit disk, where

$$\mathcal{A}_d := \begin{bmatrix} e^{A_c\tau} + \int_0^\tau e^{A_c(\tau-\xi)} A_{cs}(\xi)\, d\xi & \int_0^\tau e^{A_c(\tau-\xi)} A_{cd}(\xi)\, d\xi \\ A_{ds} & A_d \end{bmatrix}. \tag{3}$$

We can define the H_∞ norm for the sampled-data system (with some abuse of terminology) as the L_2-induced norm in time domain.

Definition 1: The H_∞ norm of the stable sampled-data system $\mathcal{G}_h$ is defined by

$$\|\mathcal{G}_h\|_\infty = \sup_{w \in L_2^*, w \ne 0} \frac{\|\mathcal{G}_h w\|_2}{\|w\|_2} = \sup_{w \in L_2^*, w \ne 0} \frac{\|z\|_2}{\|w\|_2} \tag{4}$$

where L_2^* denotes the space of piecewise continuous square integrable functions.

We can also define the H_2 norm in the time domain as follows.

Definition 2 [18]: The H_2 norm of the stable sampled-data system $\mathcal{G}_h$ of the form (2) is defined by

$$\|\mathcal{G}_h\|_2 := \left(\frac{1}{\tau} \int_0^\tau \sum_{i=1}^m \|\mathcal{G}_h \delta(t-\nu) e_i\|_2^2 \, d\nu \right)^{1/2} \tag{5}$$

where $\delta(t)e_i$ denotes an impulse at the ith component of the exogenous input.

This H_2 norm can be interpreted as the square root of the average of the integral square of the impulse response with averaging over the time of impulse input, $\nu \in [0, \tau]$. Definition 2 is also equivalent to a stochastic definition of the H_2 norm for sampled-data systems as mentioned in [18].

It has been shown in [14] and [15] that the H_∞ norm of the sampled-data system is smaller than γ if and only if a certain discrete-time linear time-invariant (LTI) system depending on γ has H_∞ norm smaller than γ. Let $\hat{\gamma}_0 > 0$ be defined as

$$\hat{\gamma}_0 := \sup_{w \in L_2[0,\tau], w \ne 0} \left(\int_0^\tau z^T(t) z(t)\, dt \Big/ \int_0^\tau w^T(t) w(t)\, dt \right)^{1/2}.$$

Then, we can readily see that $\hat{\gamma}_0$ gives a lower bound of $\|\mathcal{G}_h\|_\infty$.

The following two lemmas relate to the computations of the H_∞ and H_2 norms.

Lemma 2 [14], [15]: Consider a stable sampled-data system $\mathcal{G}_h$ represented by (2). The H_∞ norm of $\mathcal{G}_h$ defined in (4) is less than $\gamma > \hat{\gamma}_0$, i.e.,

$$\|\mathcal{G}_h\|_\infty < \gamma \tag{6}$$

if and only if the H_∞ norm of a fictitious discrete-time LTI plant $\hat{G}_\gamma[z]$ defined below is less than γ, i.e.,

$$\|\hat{G}_\gamma[z]\|_\infty < \gamma, \qquad \|\hat{G}_\gamma[z]\|_\infty := \sup_{\hat{w}\in l_2, \hat{w}\neq 0} \frac{\|\hat{G}_\gamma \hat{w}\|_2}{\|\hat{w}\|_2}. \tag{7}$$

The realization of $\hat{G}_\gamma[z]$ has the form

$$\hat{G}_\gamma[z] := \left[\begin{array}{c|c} \mathcal{A}_d & \begin{bmatrix}\hat{B}_\gamma \\ 0\end{bmatrix} \\ \hline [\hat{C}_{\gamma 1} \quad \hat{C}_{\gamma 2}] & \hat{D}_\gamma \end{array}\right] \tag{8}$$

where $\mathcal{A}_d$ is defined by (3) and the other matrices in (8), $\hat{B}_\gamma, \hat{C}_{\gamma 1}, \hat{C}_{\gamma 2}$, and $\hat{D}_\gamma$, are given by several computations of matrix exponentials and factorizations [15] (see the Appendix).

Remark 1: It is well known that

$$\|\mathcal{G}_h\|_\infty \geq \|\hat{G}_\gamma[z]\|_\infty \tag{9}$$

holds for any $\gamma > 0$ (see [15, Proposition 4.2.3]). Also note that the equality $\|\mathcal{G}_h\|_\infty = \|\hat{G}_\gamma[z]\|_\infty$ holds for any $\gamma > 0$ under the assumption of $C_c e^{A_c t} B_c \equiv 0$.

Lemma 3 [12]: Consider a stable hybrid system $\mathcal{G}_h$ represented by (2). Let $\mathcal{C}_s(t)$ be

$$\mathcal{C}_s(t) := [\hat{C}_1(t) \quad \hat{C}_2(t)] \tag{10}$$

where $\hat{C}_1(t)$ and $\hat{C}_2(t)$ are defined in (40), and let $\mathcal{X}_d$ denote the solution of the following Lyapunov equation:

$$\mathcal{X}_d = \mathcal{A}_d \mathcal{X}_d \mathcal{A}_d^T + \begin{bmatrix} \int_0^\tau e^{A_c\xi} B_c B_c^T e^{A_c^T \xi}\, d\xi & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

Then, the H_2 norm of $\mathcal{G}_h$ defined in (5) is given by

$$\|\mathcal{G}_h\|_2^2 = \text{trace}\left\{ \frac{1}{\tau} \int_0^\tau \left(\int_0^t C_c e^{A_c\xi} B_c B_c^T e^{A_c^T\xi} C_c^T\, d\xi + \mathcal{C}_s(t) \mathcal{X}_d \mathcal{C}_s^T(t) \right) dt \right\}. \tag{12}$$

Direct but lengthy calculations based on the definition of the H_2 norm with hybrid state-space representation (2) lead to the following theorem, which connects the H_∞ norm and H_2 norm for sampled-data systems, and it will be used for deriving H_∞ and H_2-norm bounds in Sections III and IV.

Theorem 1: Consider a stable sampled-data system $\mathcal{G}_h$ expressed as (2). The H_2 norm of $\mathcal{G}_h$ defined in Definition 2 is given by

$$\|\mathcal{G}_h\|_2^2 = \frac{1}{\tau}\|\hat{G}_\infty[z]\|_2^2 + R_\tau(A_c, B_c, C_c) \tag{13}$$

where $\hat{G}_\infty[z]$ is an LTI discrete-time system given as the limit $(\gamma \to \infty)$ of $\hat{G}_\gamma[z]$ defined in Theorem 2, i.e.,

$$\hat{G}_\infty[z] := \lim_{\gamma\to\infty} \hat{G}_\gamma[z] \tag{14}$$

and

$$R_\tau(A_c, B_c, C_c) := \frac{1}{\tau}\text{trace}\left\{ C_c \int_0^\tau [W(t) - W(t) \times \{e^{A_c t} L(\tau) e^{A_c^T t}\}^+ W(t)]\, dt C_c^T \right\} \tag{15}$$

with

$$W(t) := \int_0^t e^{A_c\xi} B_c B_c^T e^{A_c^T \xi}\, d\xi$$
$$L(t) := \int_0^t e^{-A_c\xi} B_c B_c^T e^{-A_c^T\xi}\, d\xi. \tag{16}$$

Proof: The proof is done by the computation of the H_2 norm of $\hat{G}_\infty[z]$. Using the standard definition of the H_2 norm for discrete-time systems, the square of the H_2 norm of $\hat{G}_\infty[z]$ is expressible as

$$\|\hat{G}_\infty[z]\|_2^2 = \text{trace}\left\{ [\hat{C}_{\infty 1} \quad \hat{C}_{\infty 2}] \hat{\mathcal{X}}_d \begin{bmatrix} \hat{C}_{\infty 1}^T \\ \hat{C}_{\infty 2}^T \end{bmatrix} + \hat{D}_\infty \hat{D}_\infty^T \right\}$$

where $\hat{\mathcal{X}}_d$ is the solution of the following Lyapunov equation:

$$\hat{\mathcal{X}}_d = \mathcal{A}_d \hat{\mathcal{X}}_d \mathcal{A}_d^T + \begin{bmatrix} \hat{B}_\infty \hat{B}_\infty^T & 0 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

For simplicity, we assume that (A_c, B_c) is controllable, then we can see from elementary computations that $\hat{B}_\infty$ is given by

$$\hat{B}_\infty = -\int_0^\tau e^{A_c(\tau-\xi)} B_c B_c^T e^{-A_c^T\xi}\, d\xi \times \left(\int_0^\tau e^{-A_c t} B_c B_c^T e^{-A_c t}\, dt \right)^{-1/2}.$$

Hence, we have

$$\hat{B}_\infty \hat{B}_\infty^T = e^{A_c\tau} \int_0^\tau e^{-A_c\xi} B_c B_c^T e^{-A_c^T\xi}\, d\xi e^{A_c^T t} = \int_0^\tau e^{A_c\xi} B_c B_c^T e^{A_c^T\xi}\, d\xi.$$

This implies that the solutions of the Lyapunov equations (11) and (17) are coincident, i.e., $\hat{\mathcal{X}}_d = \mathcal{X}_d$.

Also we have the following relations:

$$\begin{aligned} \text{trace}&\left\{ [\hat{C}_{\infty 1} \quad \hat{C}_{\infty 2}] \hat{\mathcal{X}}_d \begin{bmatrix} \hat{C}_{\infty 1}^T \\ \hat{C}_{\infty 2}^T \end{bmatrix} \right\} \\ &= \text{trace}\left\{ \hat{\mathcal{X}}_d \int_0^\tau \begin{bmatrix} \hat{C}_1^T(t) \\ \hat{C}_2^T(t) \end{bmatrix} [\hat{C}_1(t) \quad \hat{C}_2(t)]\, dt \right\} \\ &= \text{trace}\left\{ \mathcal{X}_d \int_0^\tau \mathcal{C}_s^T(t) \mathcal{C}_s(t)\, dt \right\} \\ &= \text{trace}\left\{ \int_0^\tau \mathcal{C}_s(t) \mathcal{X}_d \mathcal{C}_s^T(t)\, dt \right\}. \end{aligned} \tag{18}$$

In order to compute $\text{trace}\{\hat{D}_\infty \hat{D}_\infty^T\}$, we note that

$$\begin{aligned} \hat{C}_{\infty 3}(t) &:= \lim_{\gamma\to\infty} \hat{C}_{\gamma 3}(t) \\ &= -C_c \int_0^t e^{A_c(t-\xi)} B_c B_c^T e^{-A_c^T\xi}\, d\xi L^{-1/2}(\tau) \\ &= -C_c e^{A_c t} L(t) L^{-1/2}(\tau). \end{aligned} \tag{19}$$

Hence, we have

$$\begin{aligned} \text{trace}&\{\hat{D}_\infty \hat{D}_\infty^T\} \\ &= \text{trace}\{\hat{D}_\infty^T \hat{D}_\infty\} \\ &= \text{trace}\left\{ \int_0^\tau \hat{C}_{\infty 3}^T(t) \hat{C}_{\infty 3}(t)\, dt \right\} \\ &= \text{trace}\left\{ \int_0^\tau \hat{C}_{\infty 3}(t) \hat{C}_{\infty 3}^T(t)\, dt \right\} \\ &= C_c \int_0^\tau e^{A_c t} L(t) L^{-1}(\tau) L(t) e^{A_c^T t}\, dt C_c^T \\ &= C_c \int_0^\tau W(t) \{e^{A_c t} L^{-1}(\tau) e^{A_c^T t}\}^{-1} W(t)\, dt C_c^T. \end{aligned}$$

On the other hand, the first part in the trace of $\tau\|\mathcal{G}_h\|_2^2$ is

$$\begin{aligned} \text{trace}&\left\{ \int_0^\tau \int_0^t C_c e^{A_c\xi} B_c B_c^T e^{A_c^T\xi} C_c^T\, d\xi\, dt \right\} \\ &= \text{trace}\left\{ C_c \int_0^\tau W(t)\, dt C_c^T \right\} \\ &= \text{trace}\left\{ \int_0^\tau \hat{C}_{\infty 3}^T(t) \hat{C}_{\infty 3}(t)\, dt \right\} + \tau \cdot R_\tau(A_c, B_c, C_c) \\ &= \text{trace}\{\hat{D}_\infty \hat{D}_\infty^T\} + \tau \cdot R_\tau(A_c, B_c, C_c) \end{aligned}$$

where $R_\tau(A_c, B_c, C_c)$ is defined in (15). This, together with (18), completes the proof. □

We can show that R_τ can be interpreted as the trace of the covariance of the average intersample estimation error knowing the influence of the disturbance input at the sampling instant. Also, it has the following properties.

Property 1:

$$R_\tau(A_c, B_c, C_c) \geq 0, \qquad \forall \tau \geq 0. \tag{20}$$

Property 2:

$$R_{\tau_1} \leq R_{\tau_2}, \qquad \forall 0 \leq \tau_1 \leq \tau_2. \tag{21}$$

Property 3:

$$\begin{aligned} R_\tau(A_c, B_c, C_c) =& R_1\tau + O(\tau^2) \\ R_1 :=& \frac{1}{6}\text{trace}\{C_cB_cB_c^TC_c^T\}. \end{aligned} \tag{22}$$

B. Fast Sampling and Lifting Approach

There is another approach called the *fast sampling and lifting approach* for the analysis and synthesis for sampled-data systems [16]. The idea is to approximate the continuous parts in the original sampled-data system by replacing them with their discrete-time hold-input approximation using fast sample and hold operators, and then we use a lifting operation to obtain a (single rate) time-invariant discrete-time system.

Let us denote the weighting function of the fast-sampled approximation to the sampled-data system denoted by $w_f(k, \ell)$, using fast sampling with sampling period τ/N.

Since we can see that

$$\int_{\ell\tau/N}^{(\ell+1)\tau/N} w(t,s)\,ds \simeq \frac{\tau}{N}w\left(t, \frac{\ell\tau}{N}\right)$$

holds for the weighting function $w(t, s)$ of the sampled-data system (2), the output of the sampler at time $k\tau/N$ due to an impulse input at time $\ell\tau/N$ can be approximated by $\frac{\tau}{N}w(\frac{k\tau}{N}, \frac{\ell\tau}{N})$. In other words, $w_f(k, \ell)$ can be approximated by

$$w_f(k,\ell) = \frac{\tau}{N}w\left(\frac{k\tau}{N}, \frac{\ell\tau}{N}\right). \tag{23}$$

Then, we have

$$\begin{aligned} &\int_{-\infty}^{k\tau/N} w\left(\frac{k\tau}{N}, s\right)w^T\left(\frac{k\tau}{N}, s\right)ds \\ &\simeq \sum_{\ell=-\infty}^{k} w\left(\frac{k\tau}{N}, \frac{\ell\tau}{N}\right)w\left(\frac{k\tau}{N}, \frac{\ell\tau}{N}\right)\cdot\frac{\tau}{N} \\ &= \frac{N}{\tau}\sum_{\ell=-\infty}^{k} w_f(k,\ell)w^T(k,\ell). \end{aligned}$$

This leads to

$$\begin{aligned} \|\mathcal{G}_h\|_2^2 &= \frac{1}{\tau}\int_0^\tau dt\int_{-\infty}^t w(t,s)w^T(t,s)\,ds \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N}\frac{\tau}{N}\int_{-\infty}^{k\tau/N} w\left(\frac{k\tau}{N}, s\right)w^T\left(\frac{k\tau}{N}, s\right)ds \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N}\frac{\tau}{N}\cdot\frac{N}{\tau}\sum_{\ell=-\infty}^{k} w_f(k,\ell)w_f^T(k,\ell) \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N}\sum_{\ell=-\infty}^{k} w_f(k,\ell)w_f^T(k,\ell). \end{aligned} \tag{24}$$

Let us now define another weighting function $w_s(i,j)$ as

$$\begin{bmatrix} w_f(iN, jN) & \cdots & w_f(iN, jN+N-1) \\ w_f(iN+1, jN) & \cdots & w_f(iN+1, jN+N-1) \\ \vdots & \vdots & \vdots \\ w_f(iN+N-1, jN) & \cdots & w_f(iN+N-1, jN+N-1) \end{bmatrix}.$$

We can see that the w_s is considered as a time-invariant "rearrangement" of w_f. Hence, we can define $w_s[k] := w_s(i, k+i) = w_s(j, k+j)$; $\forall i, j$ and easily obtain the following relation:

$$\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{k}\text{trace}\{w_f(k,\ell)w_f^T(k,\ell)\} = \sum_{k=0}^{N-1}\text{trace}\{w_s[k]w_s^T[k]\}.$$

This, together with (24), yields

$$\|\mathcal{G}_h\|_2^2 \simeq \frac{1}{\tau}\|w_s[k]\|_2^2 = \frac{1}{\tau}\|W_s[z]\|_2^2 \tag{25}$$

where $W_s[z]$ is the z-transformation of $w_s[k]$. The approximation becomes exact as $N \to \infty$.

Also, we can see from the similar investigation as in [16] that

$$\|\mathcal{G}_h\|_\infty \simeq \|w_f(k,\ell)\|_\infty = \|W_s[z]\|_\infty \tag{26}$$

holds for the H_∞ norm, although we here omit the details for the derivation.

III. H_∞-Norm Bounds

A. Discrete-Time System

We give an H_∞-norm bound for an MIMO discrete-time systems with rational transfer function in this section. The result is a generalization of that in [3] for the SISO case, and it will be used for the derivation of the H_∞-norm bound for sampled-data systems.

Consider a real rational, stable, and proper discrete-time transfer function with m inputs and p outputs expressed as

$$G_d[z] = B_0 + \sum_{i=1}^{n}\frac{B_i}{z - a_i}\,(B_i = B_j^* \text{ for } i,j \text{ satisfying } a_i = a_j^*) \tag{27}$$

where $|a_i| < 1$ and $a_i\,(i = 1 \sim n)$ are assumed to be distinct for simplicity.

Theorem 2: Consider $G_d[z]$ represented by (27). Define

$$\hat{\beta}_n := \begin{bmatrix} \frac{a_1a_1^*}{1-a_1a_1^*} & \cdots & \frac{a_1a_n^*}{1-a_1a_n^*} \\ \vdots & & \vdots \\ \frac{a_na_1^*}{1-a_na_1^*} & \cdots & \frac{a_na_n^*}{1-a_na_n^*} \end{bmatrix} \in \mathbf{C}^{n\times n} \tag{28}$$

$$\hat{C}_n(e^{j\omega}) := \left[\frac{a_1^*}{a_1^* - e^{-j\omega}}, \cdots, \frac{a_n^*}{a_n^* - e^{-j\omega}}\right] \in \mathbf{C}^{1\times n}. \tag{29}$$

Then, for any $B_i\,(i = 1 \sim n)$

$$\frac{1}{\min\{p,m\}}\|G_d[z]\|_2^2 \leq \|G_d[z]\|_\infty^2 \leq \hat{\alpha}_n\|G_d(z)\|_2^2 \tag{30}$$

holds, where

$$\hat{\alpha}_n := \|1 + \hat{C}_n(e^{j\omega})\hat{\beta}_n^{-1}\hat{C}_n^*(e^{j\omega})\|_\infty \tag{31}$$

(we omit the proof for the brevity).

Remark 2: Let $a_i = \sigma_i + j\omega_i$ with $|a_i| \leq \rho < 1$ for $i = 1 \sim n$. Then, with $\hat{\beta}_n$ and $\hat{C}_n(e^{j\omega})$ as defined in Theorem 2, we have

$$\begin{aligned} \hat{C}_n(e^{j\omega})\hat{\beta}_n^{-1}\hat{C}_n^*(e^{j\omega}) &= \sum_{i=1}^{n}\frac{1-|a_i|^2}{1+|a_i|^2 - 2\omega_i\sin\omega - 2\sigma_i\cos\omega} \\ &\leq \sum_{i=1}^{n}\frac{1+|a_i|}{1-|a_i|} \leq n\,\frac{1+\rho}{1-\rho}. \end{aligned} \tag{32}$$

Also note that a similar generalization is possible for the continuous-time case [10].

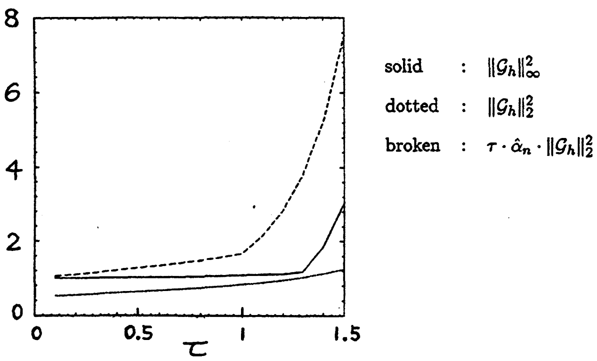

Fig. 2. Example 1.

B. Sampled-Data System

In this subsection, we will give H_∞-norm bounds for sampled-data systems, where we assume that we know all the eigenvalues, $a_i\ (i = 1 \sim n := n_c + n_d)$, of $\mathcal{A}_d$ defined in (3), which are supposed to be distinct and $|a_i| < 1$.

Theorem 3: Consider a stable hybrid system $\mathcal{G}_h$ represented by (2). Let $a_i\ (i = 1 \sim n := n_c + n_d)$ be eigenvalues of $\mathcal{A}_d$ defined in (3), which are assumed to be distinct. Then, we have

$$\|\mathcal{G}_h\|_\infty^2 \le \tau \cdot \hat{\alpha}_n \ \|\mathcal{G}_h\|_2^2 \tag{33}$$

where $\hat{\alpha}_n$ is defined in (31). Furthermore, the conclusion of Remark 2 also holds.

Proof: From (25) and (26) derived in Section II-B and (30) in Theorem 2, we obtain

$$\|\mathcal{G}_h\|_\infty^2 \simeq \|W_s(z)\|_\infty^2 \le \hat{\alpha}_n \ \|W_s(z)\|_2^2 \simeq \tau \cdot \hat{\alpha}_n \|\mathcal{G}_h\|_2^2.$$

Letting N approach infinity makes the approximate equalities exact, yielding (33). □

Theorem 3 implies that we can know an H_∞-norm bound of a given sampled-data system $\mathcal{G}_h$ under the information on the closed-loop poles (possibly inexact), the sampling period, and the H_2 norm. This reveals that H_2 synthesis with a restriction of the closed-loop pole region gives an H_∞-norm bound of the resultant system. It is also seen that (33) can be used as an initial upper bound of the bisection or γ-iteration procedure in the H_∞-norm computation, since we can compute the closed-loop poles and the H_2 norm without iteration. These points relate to our motivations m2) and m3) in Section 1.

Example 1: We will now show a simple numerical example. Consider a sampled-data feedback system with

$$P(s) = \begin{bmatrix} 1/s & 1/s \\ -1/s & -1/s \end{bmatrix}, \qquad K[z] = 1$$

in Fig. 1. The setting corresponds to a disturbance attenuation problem for a continuous-time plant $1/s$ controlled by a digital controller $K[z] = 1$, where the disturbance input $w(t)$ is added at the control input channel and the evaluated signal $z(t)$ is coincident with the plant output. Hence, we can readily see that the closed-loop system is stable iff $0 < \tau < 2$.

The solid, dotted, and broken curves in Fig. 2, respectively, represent $\|\mathcal{G}_h\|_\infty^2$, $\|\mathcal{G}_h\|_2^2$, and $\tau \cdot \hat{\alpha}_n \cdot \|\mathcal{G}_h\|_2^2$ against the plotted sampling period τ. We can see from the figure that (33) holds for any τ while the H_∞ norm is not bounded by the H_2 norm itself.

IV. H_2-Norm Bounds

If we do not restrict the bandwidth of a continuous-time system $G_c(s)$, we cannot derive any upper bound of the H_2 norm with respect to the H_∞ norm as mentioned in [3]. However, we can obtain H_2-norm upper bounds for the sampled-data system without restricting the bandwidth explicitly, since the hybrid system itself has a band-limiting feature due to the sampler. Hence, this section will discuss the H_2-norm bound for sampled-data systems based on the results in Section II. The result is given by using Theorem 1.

Theorem 4: Consider a stable hybrid system $\mathcal{G}_h$ represented by (2). Then, an H_2-norm bound of $\mathcal{G}_h$ is given by

$$\|\mathcal{G}_h\|_2^2 \le \frac{\nu_c}{\tau}\|\mathcal{G}_h\|_\infty^2 + R_\tau(A_c, B_c, C_c) \tag{34}$$

where

$$\nu_c := \operatorname{rank}\begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{n_c-1}B_c \end{bmatrix}. \tag{35}$$

Proof: From (13), (1), and (9), we have

$$\begin{aligned} \tau\|\mathcal{G}_h\|_2^2 &= \|\hat{G}_\infty[z]\|_2^2 + \tau \cdot R_\tau(A_c, B_c, C_c) \\ &\le \nu_c\|\hat{G}_\infty[z]\|_\infty^2 + \tau \cdot R_\tau(A_c, B_c, C_c) \\ &\le \nu_c \ \|\mathcal{G}_h\|_\infty^2 + \tau \cdot R_\tau(A_c, B_c, C_c) \end{aligned}$$

since the dimensions of state, input, and output of $\hat{G}_\infty[z]$ are $n_c + n_d$, ν_c, and $n_c + n_d + \nu_c$, respectively. □

Remark 3: If we assume that $R_\tau(A_c, B_c, C_c) = 0$, i.e., $C_1e^{A_pt}B_1 \equiv 0$, the factor ν_c can be improved by $\nu_k := \min\{m_k, p_k\}$, where m_k and p_k denote the dimensions of the input and output of the controller $K[z]$, respectively [10].

Theorem 4 implies that we can estimate the H_2-norm bound of a given hybrid system $\mathcal{G}_h$ if we know the sampling period, the H_∞ norm, and the continuous-time part A_c, B_c, C_c, i.e., no information on the discrete-time part is required for the estimation. Hence, the bound (34) is useful for the H_∞-control synthesis as well as the H_∞-norm computation.

Since any term except $\|\mathcal{G}_h\|_\infty$ in the right-hand side of (34) is independent of the controller to be designed, we can see that any γ-suboptimal H_∞-digital controller satisfying $\|\mathcal{G}_h\|_\infty < \gamma$ achieves the H_2 norm less than

$$\left(\frac{\nu_c}{\tau}\gamma^2 + R_\tau(A_c, B_c, C_c)\right)^{1/2}.$$

In this sense, the H_∞ norm minimization is indirectly effective for the H_2-norm reduction. This is a different feature in comparison with the continuous-time case. Also, from (37), we can get a lower bound of achievable H_∞ norm if we know the optimal H_2 norm $\gamma_{2\text{opt}} := \inf_{K[z]} \|\mathcal{G}_h\|_2$, which can be easily computed. The form is given as follows:

$$\gamma_{\infty\text{opt}}^2 := \inf_{K[z]} \|\mathcal{G}_h\|_\infty^2 \ge \frac{\tau}{\nu_c}\left\{\gamma_{2\text{opt}}^2 - R_\tau(A_c, B_c, C_c)\right\}. \tag{36}$$

For the computation, Theorem 4 gives an initial lower bound of the bisection procedure, since (34) can be rewritten as

$$\|\mathcal{G}_h\|_\infty^2 \ge \frac{\tau}{\nu_c}\left\{\|\mathcal{G}_h\|_2^2 - R_\tau(A_c, B_c, C_c)\right\} \tag{37}$$

where no iteration is required for the computation of the right-hand side.

The above investigations relate to our motivations m2) and m3) in Section I.

Example 2: We will consider the same numerical example as in Example 1. Fig. 3 illustrates the $\|\mathcal{G}_h\|_2^2$, $\frac{\nu_c}{\tau}\|\mathcal{G}_h\|_\infty^2 + R_\tau$, and $\frac{\nu_c}{\tau}\|\mathcal{G}_h\|_\infty^2$, where we can readily see from simple calculations that $R_\tau = \tau/6$. It is verified that (34) holds for any τ and that the term $R_\tau = \tau/6$ is necessary for assuring the bound.

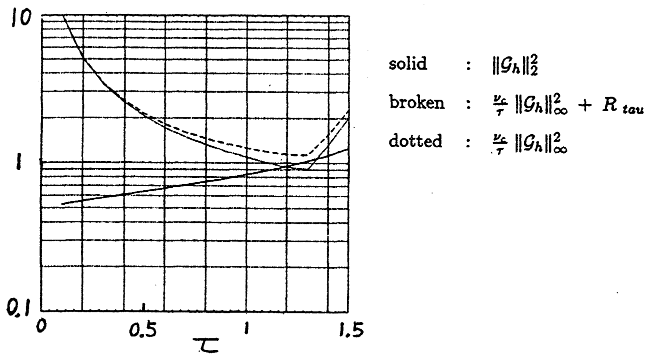

Fig. 3. Example 2.

V. Conclusion

The relationship between the H_∞ and H_2 norms has been investigated for MIMO sampled-data feedback control systems, where a continuous-time plant is controlled by a digital compensator with hold and sampler.

The derived bounds afford partial information about the closed-loop performances of sampled-data feedback control systems, and they are useful for H_∞-norm computation and optimization as remarked just before Examples 1 and 2.

Appendix

$\hat{B}_\gamma, \hat{C}_{\gamma 1}, \hat{C}_{\gamma 2}$, and $\hat{D}_\gamma$, are given by the following steps. Step 1 is related to a class of possibly worst case inputs. Step 2 gives the computation of the equivalent B matrix, $\hat{B}_\gamma$. The discretization of the output by averaging the intersampling output is executed in Step 3.

Step 1: Define the following matrix functions of $\gamma > 0$:

$$\hat{E} := \begin{bmatrix} A_c & -B_cB_c^T/\gamma \\ C_c^TC_c/\gamma & -A_c^T \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_1(t) \\ \hat{F}_2(t) \end{bmatrix} := \begin{bmatrix} A_{cs}(t) \\ C_c^TC_{cs}(t)/\gamma \end{bmatrix}$$

$$\begin{bmatrix} \hat{G}_1(t) \\ \hat{G}_2(t) \end{bmatrix} := \begin{bmatrix} A_{cd}(t) \\ C_c^TC_{cd}(t)/\gamma \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{\Phi}_{11}(t) & \hat{\Phi}_{12}(t) \\ \hat{\Phi}_{21}(t) & \hat{\Phi}_{22}(t) \end{bmatrix} := e^{\hat{E}t} + \int_0^t e^{\hat{E}(t-\xi)} \begin{bmatrix} \hat{F}_1(\xi) & 0 \\ \hat{F}_2(\xi) & 0 \end{bmatrix} d\xi$$

$$\begin{bmatrix} \hat{\Phi}_{13}(t) \\ \hat{\Phi}_{23}(t) \end{bmatrix} := \int_0^t e^{\hat{E}(t-\xi)} \begin{bmatrix} \hat{G}_1(\xi) \\ \hat{G}_2(\xi) \end{bmatrix} d\xi.$$

Also define

$$\hat{V}_\gamma(t) := -B_c^T \left[\hat{\Phi}_{21}(t) \quad \hat{\Phi}_{22}(t) \quad \hat{\Phi}_{23}(t)\right].$$

Then define a positive semidefinite matrix $\hat{W}_\gamma$ and the factor $\hat{Y}_\gamma$ of full row rank as

$$\hat{W}_\gamma := \int_0^\tau \hat{V}_\gamma^T(t)\hat{V}_\gamma(t)\,dt = \hat{Y}_\gamma^T\hat{Y}_\gamma.$$

Step 2: Let

$$\hat{B}_\gamma := \int_0^\tau e^{A_c(\tau-\xi)}B_c\hat{F}_\gamma(\xi)\,d\xi; \quad \hat{F}_\gamma(t) := \hat{V}_\gamma(t)\hat{Y}_\gamma^+ \tag{38}$$

where $^+$ denotes the pseudo inverse.

Step 3: Let

$$\left[\hat{C}_{\gamma 1} \quad \hat{C}_{\gamma 2} \quad \hat{D}_\gamma\right] = \hat{M}_\gamma^{1/2}$$

$$\hat{M}_\gamma := \int_0^\tau \begin{bmatrix} \hat{C}_1^T(t) \\ \hat{C}_2^T(t) \\ \hat{C}_{\gamma 3}^T(t) \end{bmatrix} \left[\hat{C}_1(t) \quad \hat{C}_2(t) \quad \hat{C}_{\gamma 3}(t)\right] dt \tag{39}$$

where $\hat{C}_1(t), \hat{C}_2(t)$, and $\hat{C}_{\gamma 3}(t)$ are given by

$$\begin{aligned} \hat{C}_1(t) &:= C_c\left(e^{A_ct} + \int_0^t e^{A_c(t-\xi)}A_{cs}(\xi)\,d\xi\right) + C_{cs}(t) \\ \hat{C}_2(t) &:= C_c\int_0^t e^{A_c(t-\xi)}A_{cd}(\xi)\,d\xi + C_{cd}(t) \\ \hat{C}_{\gamma 3}(t) &:= C_c\int_0^t e^{A_c(t-\xi)}B_c\hat{F}_\gamma(\xi)\,d\xi. \end{aligned} \tag{40}$$

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