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Brief Paper

Solution Set Properties for Static Errors-in-variables Problems*

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Key Words—Linear systems; system identification; errors-in-variables; noisy inputs; symmetric modelling.

Abstract—This paper examines and refutes a conjecture to the effect that the solution set for a general (real) static errors-in-variables problem is a finite union of sets that are described by a finite number of linear inequalities. The conjecture is disproved by detailed examination of particular errors-in-variables problems with four variables. The solution set in this case is described by five surfaces, all intersecting in straight lines, but in general one of these surfaces is not flat. Copyright © 1996 Elsevier Science Ltd.

1. *Introduction*

In this paper we are concerned with the problem of finding linear (static) relations between the n components of a vector x of observations corrupted by noise. To be more precise, we consider the model

$$x = \hat{x} + \tilde{x}, \tag{1}$$

where the components of $\hat{x}$ are linearly dependent, i.e. they satisfy one or more relations of the form

$$w\hat{x} = 0, \quad w \in \mathbb{R}^{1\times n}, \ w \neq 0. \tag{2}$$

Here $\tilde{x}$ is interpreted as noise corrupting the 'true' variables $\hat{x}$. We shall assume throughout that the covariance matrices $\hat{\Sigma}$ and $\tilde{\Sigma}$ of $\hat{x}$ and $\tilde{x}$ respectively exist, that $\mathrm{E}\,\hat{x}\tilde{x}' = 0$ and that $\tilde{\Sigma}$ is diagonal. By our assumptions, the covariance matrix Σ of x may be written as

$$\Sigma = \hat{\Sigma} + \tilde{\Sigma}. \tag{3}$$

Usually for such a class of models, the term *static errors-in-variables models* is used. Because of the assumption that $\tilde{\Sigma}$ is diagonal, we speak of the Firsch case (see Frisch, 1934).

Note that $\hat{\Sigma}$ has to be singular, since, by assumption, the components of $\hat{x}$ satisfy a nontrivial relation (2). For a given $\hat{\Sigma}$, these relations are the nontrivial elements of the (left) kernel of $\hat{\Sigma}$. For given Σ, neither $\hat{\Sigma}$ nor its kernel are unique in this setting (see e.g. Deistler and Scherrer, 1992). Of course uniqueness may be achieved by additional assumptions, which may, however, lead to prejudiced results (see Kalman, 1982). In this paper we do not impose additional assumptions, and we shall describe the set of all linear relations compatible with Σ. Such compatible relations will be called *solutions*.

The cases $n = 2$ and $n = 3$ are well known. We here consider the case $n = 4$, which exhibits a number of complications. We give a certain description of the set of all solutions. The main result of this paper is a negative one, showing that the set of all solutions is not always a finite union of sets that are described by a finite number of inequalities. This disproves a conjecture in De Moor and Vandewalle (1986b), stating that the solution set is a collection of polyhedral sets in orthants.

2. *General properties*

In a first step we make certain notions precise. Throughout the paper it is assumed that Σ is an n-dimensional nonsingular covariance matrix, i.e. Σ is symmetric and positive-definite. The inverse of Σ is denoted by $S = \Sigma^{-1}$. A diagonal matrix $\tilde{\Sigma}$ is called *compatible* (with Σ) if $0 \leq \tilde{\Sigma} \leq \Sigma$ and $\mathrm{crk}\,(\Sigma - \tilde{\Sigma}) > 0$ holds. Here $\mathrm{crk}\,(R)$ denotes the corank of a (square) matrix R. The maximum corank of $\Sigma - \tilde{\Sigma}$, where $\tilde{\Sigma}$ is compatible, is denoted by $\mathrm{mc}\,(\Sigma)$. Note that $\mathrm{mc}\,(\Sigma) \geq 1$, since there always exist a compatible $\tilde{\Sigma}$. A vector $w \in \mathbb{R}^{1\times n}$, $w \neq 0$, is called a *solution* if there exists a compatible $\tilde{\Sigma}$ such that $w(\Sigma - \tilde{\Sigma}) = 0$. A vector $v \in \mathbb{R}^{1\times(n-1)}$ is called a *normalized solution* if $w = (v, 1)$ is a solution. The sets of all solutions and of all normalized solutions are denoted by $\mathscr{L} \subseteq \mathbb{R}^n$ and $\mathscr{N} \subseteq \mathbb{R}^{n-1}$ respectively.

Given any $w \in \mathbb{R}^{1\times n}$, we may define a unique diagonal matrix $\tilde{\Sigma}^w$, corresponding to w, by

$$\tilde{\sigma}^w_{ii} = \begin{cases} (w\Sigma)_i / w_i & \text{for } w_i \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Here r_{ij} is used for the (i, j)th element of a matrix R and c_i denotes the ith element of a vector c. Note that $w(\Sigma - \tilde{\Sigma}^w) = 0$ if w has no zero entries; however, in general, $w(\Sigma - \tilde{\Sigma}^w)$ is not necessarily equal to zero. The following lemma has a trivial proof.

Lemma 2.1. A vector w is a solution iff $w(\Sigma - \tilde{\Sigma}^w) = 0$ and $0 \leq \tilde{\Sigma}^w \leq \Sigma$. In addition, if w is a solution then, for all compatible $\tilde{\Sigma}$ satisfying $w(\Sigma - \tilde{\Sigma}) = 0$, the inequality $\tilde{\Sigma}^w \leq \tilde{\Sigma}$ holds.

We call a solution w an ith *elementary solution* if $w_i \neq 0$ holds and all diagonal elements of $\tilde{\Sigma}^w$ are zero, except for the element in the ith position. These solutions correspond to the regression of the ith component of x onto the remaining components. In this case all noise is added to the ith component of x and the corresponding noise covariance is given by $\tilde{\Sigma}^i = \mathrm{diag}\,(0, \ldots, 0, \tilde{\sigma}^i_{ii}, 0, \ldots, 0)$. Note that s_i,

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which denotes the ith row of $S=\Sigma^{-1}$, is an ith elementary solution, since $s_i\Sigma=(0,\dots,0,1,0,\dots,0)=s_i\,\mathrm{diag}\,(0,\dots,0,s_{ii}^{-1},0,\dots,0)$. A vector $e_i\in\mathbb{R}^{1\times(n-1)}$ is called the ith *normalized elementary solution* if $(e_i,1)$ is an ith elementary solution.

The following theorem (see e.g. Klaman, 1982) gives a complete picture, both for the characterization of the case $\mathrm{mc}\,(\Sigma)=1$ in terms of Σ, and for the characterization of the solution set in this case. See also Fig. 1 for a simple three-dimensional example.

Proposition 2.2.

(i) $\mathrm{mc}\,(\Sigma)=1$ iff there exists a diagonal matrix T with diagonal entries ±1, such that all entries of $(T\Sigma T)^{-1}$ are strictly positive.

(ii) In this case the set of normalized solutions is the convex hull generated by the n normalized elementary solutions e_i.

The case $\mathrm{mc}\,(\Sigma)=n-1$ is treated in Anderson and Deistler (1993). For this situation, both a characterization of the case $\mathrm{mc}\,(\Sigma)=n-1$ in terms of Σ and a complete description of the solution set are available. Here we only give some results for the case $n=3$. See also Fig. 2 for a plot of the normalized solution set.

Lemma 2.3. For the three-dimensional case $n=3$, we have the following

(i) $\mathrm{mc}\,(\Sigma)=2$ iff S is not diagonal and $s_{12}s_{23}s_{31}\le 0$.

(ii) Let us assume that S (and thus also Σ) has at most two zero entries. The following equation defines a plane in $\mathbb{R}^3$:

$$w_1\sigma_{12}\sigma_{13}+w_2\sigma_{21}\sigma_{23}+w_3\sigma_{31}\sigma_{32}=0. \qquad (5)$$

(a) For $\mathrm{mc}\,(\Sigma)=1$, the above plane contains no solutions.

(b) For $\mathrm{mc}\,(\Sigma)=2$, all entries of Σ must be nonzero, and the above relation is equivalent to $w(\Sigma-\tilde\Sigma^0)=0$, where

$$\tilde\Sigma^0=\mathrm{diag}\left(\sigma_{11}-\frac{\sigma_{21}\sigma_{13}}{\sigma_{23}},\ \sigma_{22}-\frac{\sigma_{12}\sigma_{23}}{\sigma_{13}},\ \sigma_{33}-\frac{\sigma_{13}\sigma_{32}}{\sigma_{12}}\right). \qquad (6)$$

$\tilde\Sigma^0$ is the only compatible matrix with $\mathrm{crk}\,(\Sigma-\tilde\Sigma)=2$. Thus this plane (except for the zero point) is the set of all solutions w where either w has a zero entry or $\mathrm{crk}\,(\Sigma-\tilde\Sigma^w)=2$ (in which case $\tilde\Sigma^w=\tilde\Sigma^0$).

The above results together with the ideas in De Moor and Vandewalle (1986a, b) give rise to the following conjecture for Σ of arbitrary dimension.

Conjecture. The normalized solution set is a union of a finite number of subsets, where each of these subsets is described by a (finite) set of linear inequalities. In every orthant these subsets are convex.

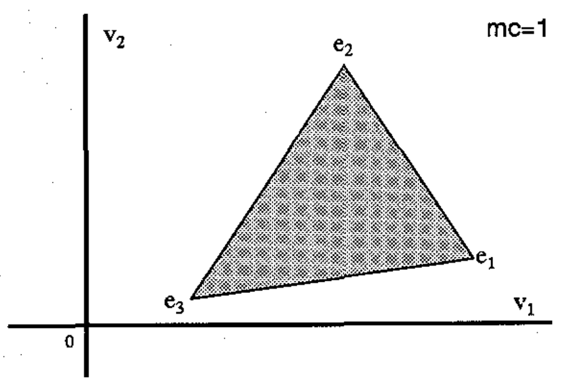

Fig. 1. Plot of the normalized solution set $\mathcal{N}$ for the case $n=3$, $\mathrm{mc}\,(\Sigma)=1$.

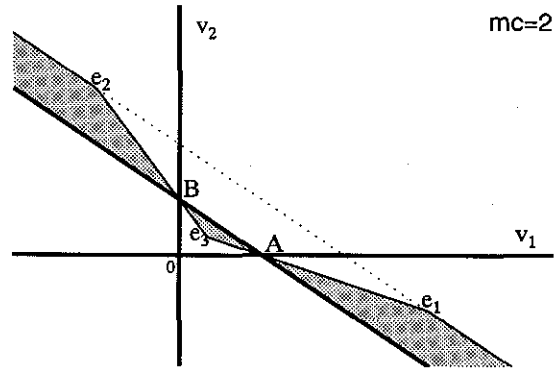

Fig. 2. Plot of the normalized solution set $\mathcal{N}$ for the case $n=3$, $\mathrm{mc}\,(\Sigma)=2$. The (doubly infinite) line passing through the points A and B is the intersection of the plane (5) with the plane $w_3=1$. Thus points on this line correspond to the unique $\tilde\Sigma^0$ with $\mathrm{crk}\,(\Sigma-\tilde\Sigma^0)=2$.

In this paper the above conjecture will be disproved by a particular example, namely the case $n=4$, $\mathrm{mc}\,(\Sigma)=2$.

The next porposition gives some results about the topological properties of the solution set. For a subset $\mathcal{A}\subseteq\mathbb{R}^n$, we denote its closure by $\bar{\mathcal{A}}$ and its interior by $\mathcal{A}^\circ$. The boundary of $\mathcal{A}$ is defined as $\partial\mathcal{A}=\bar{\mathcal{A}}\backslash\mathcal{A}^\circ$.

Proposition 2.4.

(i) $\bar{\mathcal{L}}=\mathcal{L}\cup\{0\}$.

(ii) If $w\in\mathcal{L}$, $\tilde\Sigma^w>0$ and $\mathrm{crk}\,(\Sigma-\tilde\Sigma^w)=1$ then $w\in\mathcal{L}^\circ$.

(iii) If $w\in\mathcal{L}$ has no zero entries, $\tilde\Sigma^w$ is singular and $\mathrm{crk}\,(\Sigma-\tilde\Sigma^w)=1$ then $w\in\partial\mathcal{L}\cap\bar{\mathcal{L}}^\circ$.

The same statments hold for the normalized solution set $\mathcal{N}$, except for (i), which has to be replaced by $\bar{\mathcal{N}}=\mathcal{N}$.

Proof. (i) Consider a convergent sequence $w^k\in\mathcal{L}$, with $w^k\to w^0\neq0$. Let $\tilde\Sigma^k$ denote the noise covariance corresponding to w^k. Since $0\le\tilde\Sigma^k\le\Sigma$, there exists a convergent subsequence of $(\tilde\Sigma^k)$ with limiting point, $\tilde\Sigma^0$ say. Then clearly $0\le\tilde\Sigma^0\le\Sigma$ and $w^0(\Sigma-\tilde\Sigma^0)=0$, and thus w^0 is a solution, i.e. $w^0\in\mathcal{L}$.

(ii) By the assumption $\tilde\Sigma^w>0$, w has no zero component, and thus the mapping $\bar w\to\tilde\Sigma^{\bar w}$ is continuous in a neighborhood of w. This implies that $\tilde\Sigma^{\bar w}>0$, $\Sigma-\tilde\Sigma^{\bar w}\ge0$, and thus $\bar w\in\mathcal{L}$ in a suitably chosen neighborhood of w. The semipositivity of $\Sigma-\tilde\Sigma^{\bar w}$ follows from the fact that $\Sigma-\tilde\Sigma^w$ has $n-1$ strictly positive eigenvalues, and therefore the same holds true for all $\Sigma-\tilde\Sigma^{\bar w}$, where $\tilde\Sigma^w-\tilde\Sigma^{\bar w}$ is small enough.

(iii) Let $\tilde\sigma_{ii}^w=0$. We consider a small perturbation of w of the form $w+\epsilon s_i$, where s_i denotes the ith row of $S=\Sigma^{-1}$. From the equation $(w+\epsilon s_i)\Sigma=(w+\epsilon s_i)\tilde{\tilde\Sigma}$ (where $\tilde{\tilde\Sigma}$ corresponds to the perturbed vector $(w+\epsilon s_i)$), we obtain

$$\epsilon=(w_i+\epsilon s_{ii})\tilde{\tilde\sigma}_{ii},$$

$$0=(w_j+\epsilon s_{ij})\tilde{\tilde\sigma}_{jj}\quad\text{for } j\neq i \text{ and } \tilde\sigma_{jj}^w=0,$$

$$w_j\tilde\sigma_{jj}^w=(w_j+\epsilon s_{ij})\tilde{\tilde\sigma}_{jj}\quad\text{for } j\neq i \text{ and } \tilde\sigma_{jj}^w>0.$$

By the same reasoning as above, it follows that $\tilde{\tilde\Sigma}\le\Sigma$ and $\mathrm{crk}\,(\Sigma-\tilde{\tilde\Sigma})=1$ for small ϵ. From the first of the above equations, it follows that we may find an arbitrarily small ϵ (which may be negative) such that $\tilde{\tilde\sigma}_{ii}<0$. Thus w is contained in the boundary of $\mathcal{L}$.

On the other hand, we can find an ϵ such that $\tilde{\tilde\sigma}_{ii}>0$, $\tilde{\tilde\sigma}_{jj}=0$ for all $j\neq i$ such that $\tilde\sigma_{jj}^w=0$, and $\tilde{\tilde\sigma}_{jj}>0$ for all $j\neq i$ such that $\tilde\sigma_{jj}^w>0$. Proceeding in this manner, one can construct a $\bar w\in\mathcal{L}$ such that $\tilde\Sigma^{\bar w}>0$ and $\mathrm{crk}\,(\Sigma-\tilde\Sigma^{\bar w})=1$, and thus, by (ii), $\bar w\in\mathcal{L}^\circ$.

The corresponding results for $\mathcal{N}$ may be proved in a completely analogous way. □

Note that point (ii) of the proposition above in particular implies that for any point of the boundary of $\mathcal{L}$, either $\mathrm{crk}(\Sigma - \tilde{\Sigma}^w) > 1$ holds or w or $(w\Sigma)$ contain a zero element.

3. *The case* $n = 4$

Throughout this section, we shall consider this case $n = 4$. It is assumed that Σ has full rank and that all entries of the last row and column of $S = \Sigma^{-1}$ are nonzero. Thus by a coordinate transformation $x \to Tx$, $\Sigma \to T\Sigma T$, $S \to T^{-1}ST^{-1}$ with $T = (1/\sqrt{s_{44}})\,\mathrm{diag}(s_{14}, \ldots, s_{44})$, all entries of the last row and column of S can be scaled to one. The normalized solutions are then transformed by $v \to v[s_{44}\,\mathrm{diag}(s_{14}, s_{24}, s_{34})^{-1}]$.

Using an obvious partitioning, we shall write Σ, $\tilde{\Sigma}^w$ and $S = \Sigma^{-1}$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{14} \\ \Sigma_{41} & \sigma_{44} \end{pmatrix}, \quad \tilde{\Sigma}^w = \begin{pmatrix} \tilde{\Sigma}_{11}^w & 0 \\ 0 & \tilde{\sigma}_{44}^w \end{pmatrix}, \quad S = \begin{pmatrix} E & \mathbf{1}' \\ \mathbf{1} & 1 \end{pmatrix}$$

where $e_4 = \mathbf{1} = (1, 1, 1)$. The rows of E are denoted by e_i, $i = 1, 2, 3$. Note that e_i is the ith normalized elementary solution. To avoid 'unnecessary' complications, we shall in addition assume, that all super diagonal elements of E are distinct. (For a discussion of nongeneric cases, see Scherrer (1991).)

Let us consider a preliminary calculation. It can easily be verified that for $\alpha \neq 1$,

$$[\Sigma - \mathrm{diag}(0, 0, 0, \alpha)]^{-1} = \frac{1}{1-\alpha}\begin{pmatrix} E(\alpha) & \mathbf{1}' \\ \mathbf{1} & 1 \end{pmatrix},$$
$$E(\alpha) = \alpha \mathbf{1}\mathbf{1}' + (1-\alpha)E. \tag{7}$$

The rows of $E(\alpha)$ are denoted by $e_1(\alpha), e_2(\alpha)$ and $e_3(\alpha)$ respectively. Let $w = (v, 1)$; then $\tilde{\sigma}_{44}^w = \alpha$ is equivalent to

$$v\Sigma_{14} + (\sigma_{44} - \alpha) = 0 \tag{8}$$

The above relation defines an affine plane, termed the α plane below. This plane is parallel to the plane spanned by e_1, e_2, e_3, and it contains the points $e_1(\alpha), e_2(\alpha), e_3(\alpha)$ (see Fig. 3). Thus $\tilde{\sigma}_{44}^w = \alpha$ holds iff v is contained in the corresponding α plane. Using this remark, one can prove that for a fixed α, $0 \leq \alpha < 1$, the problem of finding the corresponding solutions is essentially equivalent to a three-dimensional EV problem. To be more specific, we have the following lemma.

Lemma 3.1. Let v be contained in a fixed but arbitrary α plane.

(i) If v is a normalized solution then $0 \leq \alpha \leq 1$.

(ii) If $\alpha = 1$ then v is a normalized solution iff $v = e_4$ (in this case $\tilde{\Sigma}_{11}^w = 0$).

(iii) For $0 \leq \alpha < 1$, v is a normalized solution iff $v[E(\alpha)^{-1} - (1-\alpha)^{-1}\tilde{\Sigma}_{11}^w] = 0$ and $0 \leq (1-\alpha)^{-1}\tilde{\Sigma}_{11}^w \leq E(\alpha)^{-1}$ (i.e. if v is a solution of the three-dimensional problem $E(\alpha)^{-1}$).

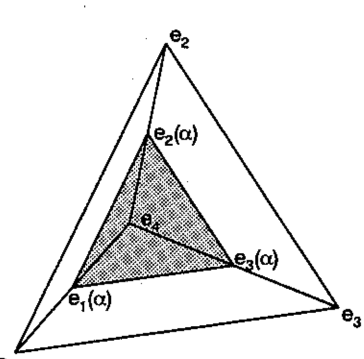

Fig. 3. Plot of the four elementary solutions e_i and of the points $e_i(\alpha)$ for the case $n = 4$.

Proof. (i) Clearly for a solution $w = (v, 1)$, $0 \leq \mathrm{diag}(0, 0, 0, \alpha) \leq \Sigma$ must hold, which, by (7), implies $0 \leq \alpha \leq 1$.

(ii) For $\tilde{\sigma}_{44}^w = \alpha = 1$, we have

$$(e_4, 1)(\Sigma - \tilde{\Sigma}^w)(e_4, 1)' = \underbrace{(e_4, 1)[\Sigma - \mathrm{diag}(0, 0, 0, 1)](e_4, 1)'}_{=0} - (\tilde{\sigma}_{11}^w)^2 - (\tilde{\sigma}_{22}^w)^2 - (\tilde{\sigma}_{33}^w)^2,$$

and thus in this case w is a solution if and only if $w = (e_4, 1)$.

(iii) For $0 \leq \alpha < 1$, first note that, by (7), $\Sigma_{11} - \Sigma_{14}(\sigma_{44} - \alpha)^{-1}\Sigma_{41} = (1-\alpha)E(\alpha)^{-1}$. Furthermore, $0 \leq \tilde{\Sigma}^w \leq \Sigma$ is equivalent to $0 \leq \tilde{\Sigma}_{11}^w \leq (1-\alpha)E(\alpha)^{-1}$, and $w(\Sigma - \tilde{\Sigma}^w) = 0$ is equivalent to $v[(1-\alpha)E(\alpha)^{-1} - \tilde{\Sigma}_{11}^w] = 0$. □

In the following reasoning, we define the e_i plane as the plane which is spanned by the three points $e_j, j \neq i$, and the v_i plane as the coordinate plane $v_i = 0$. For $\alpha \neq 1$, let $\sigma_{ij}(\alpha)$ denote the entries of $E(\alpha)^{-1}$. We define the α line as the intersection of the α plane (8) with the plane given by

$$v_1\sigma_{12}(\alpha)\sigma_{13}(\alpha) + v_2\sigma_{21}(\alpha)\sigma_{23}(\alpha) + v_3\sigma_{31}(\alpha)\sigma_{32}(\alpha) = 0. \tag{9}$$

Then, by the results of Lemmas 2.3 and 3.1, we have the following.

(i) If $e_{12}(\alpha)e_{23}(\alpha)e_{31}(\alpha) > 0$ then this α line contains no normalized solution.

(ii) If $e_{12}(\alpha)e_{23}(\alpha)e_{32}(\alpha) \leq 0$ then this α line is contained in $\mathcal{N}$. In addition, this α-line is the set of all normalized solutions v in the α plane, where either $v_i = w_i = 0$ or $\mathrm{crk}(\Sigma - \tilde{\Sigma}^w) = 2$ for the corresponding solution $w = (v, 1)$.

Note that we may also define such an α line for $\alpha = 1$, by replacing $\sigma_{ij}(\alpha)$ in (9) by $(1-\alpha)\sigma_{ij}(\alpha) = \sigma_{ij} - (\sigma_{44} - \alpha)^{-1}\sigma_{i4}\sigma_{4j}$ and then allowing α to approach 1. The set of all α lines generates a (smooth) ruled surface. We shall call this surface the crk-2 surface (see Fig. 4, which will be described in norm detail below). Let $u_i(\alpha)$ denote the intersection of the α line with the v_i plane. Since all above-diagonal elements of E are distinct, these intersection points exist and are unique. By simple algebra, one can show that

$$\det(u_1(0)' - u_2(1)', u_2(0)' - u_2(1)', u_1(1)' - u_2(1)') = \frac{\det(S)(e_{23} - e_{13})}{(e_{13} - e_{12})(e_{23} - e_{12})} \neq 0.$$

Thus these four points $u_1(0)$, $u_1(1)$, $u_2(0)$ and $u_2(1)$ are not contained in a plane, which implies that the crk-2 surface is not flat.

For a solution $w = (v, 1)$, $\tilde{\sigma}_{ii}^w = 0$ implies that $\gamma_i = 0$, where $\gamma = w\Sigma$. Since $w = \gamma S$, we see that in this case v is an element of the e_i plane. This remark and (i) and (ii) above, together with the result of Proposition 2.4, give the following Lemma.

Lemma 3.2. The boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is a subset of the union of the e_i planes together with the crk-2 surface.

The four e_j planes and the crk-2 surface partition $\mathbb{R}^3$. Thus we may represent $\mathbb{R}^3$ as the union of the crk-2 surface, the four e_i planes and a finite number of open and connected subsets $\mathcal{A}_i$. Then, by Proposition 2.4 and the above lemma, the normalized solution set can be constructed as follows. For each $\mathcal{A}_i$, we pick an arbitrary element, v^i say. If v^i is a normalized solution then $\bar{\mathcal{A}}_i \subseteq \mathcal{N}$. If v^i is not a normalized solution then $\mathcal{A}_i$ contains no normalized solutions. The normalized solution set $\mathcal{N}$ then is the union of the 'compatible' $\bar{\mathcal{A}}_i$s together with the α lines, for which $e_{12}(\alpha)e_{23}(\alpha)e_{31}(\alpha) \leq 0$.

To make a figure, we analyze the intersections of the solution set with the α planes. It is easy to see that the 'character' of this intersection changes iff the α line crosses a point $e_i(\alpha)$—in other words, iff one of the elements of the

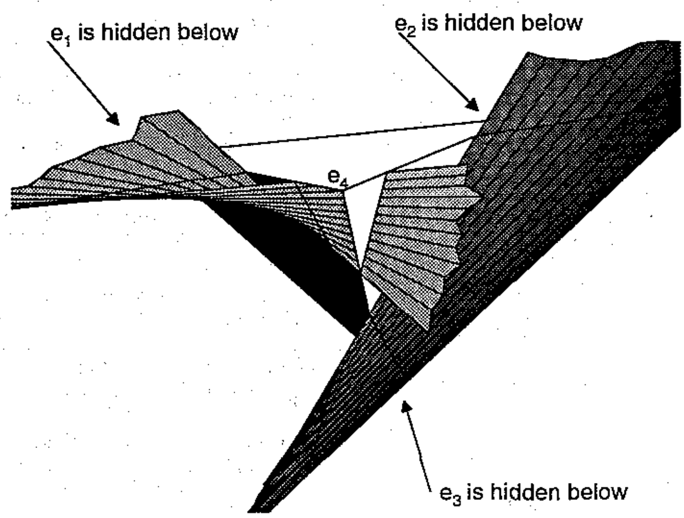

Fig. 4. The part of the crk-2 surface contained in the normalized solution set for the case $e_{12}<e_{13}<e_{23}\leq 0$ (and thus $0\leq\alpha_{23}<\alpha_{13}<\alpha_{12}<1$) is shown. This is a view orthogonal to the plane spanned by e_1, e_2, e_3, such that e_4 lies above this plane. To improve the '3D feeling' of this picture, the triangles (e_1, e_2, e_3 (black)), (e_1, e_2, e_4 (white)) and (e_2, e_3, e_4 (white)) are also plotted.

matrix $E(\alpha)$ becomes zero. The (i, j)th entry of $E(\alpha)$ is zero iff $\alpha=\alpha_{ij}=e_{ij}/(e_{ij}-1)$. Note that $0\leq\alpha_{ij}<1$ is equivalent to $e_{ij}\leq 0$. By a reordering of the variables, we can always achieve $e_{12}<e_{12}<e_{23}$. Now there are essentially four cases, depending on the number of negative entries of E (which is equal to the number of α_{ij}s contained in the interval $[0, 1]$).

In the case $e_{12}<e_{13}<e_{23}\leq 0$ we have $0\leq\alpha_{23}<\alpha_{13}<\alpha_{12}<1$. Note that, by Lemma 2.3, $E(\alpha)^{-1}$ allows a corank ≥ 2 decomposition iff $e_{12}(\alpha)e_{13}(\alpha(e_{23}(\alpha)\leq 0$, which in this case occurs for $\alpha\in[\alpha_{13}, \alpha_{12}]$ and $\alpha\in[0, \alpha_{23}]$. Thus essentially four different types of intersections of the solution set with an α plane may occur, as depicted in Fig. 5. Although these intersections correspond to a three-dimensional EV problem, the cases for $\alpha<\alpha_{12}$ look quite different than in the plots 1 and 2. The reason is that the plots 1 and 2 show the intersection of the solution set with the 'normalizing plane' $v_3=1$, whereas here we show the intersection with the corresponding α plane.

Figure 4 shows a plot of the part of the crk-2 surface contained in $\mathcal{N}$. Note that an α line is contained in $\mathcal{N}$ iff α lies in one of the two intervals $[\alpha_{13}, \alpha_{12}]$ and $[0, \alpha_{23}]$. Thus the part of the crk-2 surface contained in $\mathcal{N}$ splits up into two disjoint parts: the light-shaded part corresponding to $\alpha\in[\alpha_{13}, \alpha_{12}]$, and the dark-shaded part corresponding to $\alpha\in[0, \alpha_{23}]$. This last part lies outside the tetrahedron spanned by the elementary solutions, whereas the first part crosses this tetrahedron. The crk-2 surface intersects the e_i planes in straight lines, which are the intersection of the e_i planes with the corresponding coordinate plane $v_i=0$.

In the case $e_{12}<e_{13}\leq 0<e_{23}$ we have $0\leq\alpha_{13}<\alpha_{12}<1$. Thus only intersections of the type corresponding to cases (a), (b) and (c) in Fig. 5 occur.

In the case $e_{12}\leq 0<e_{13}<e_{23}$ we have $0\leq\alpha_{12}<1$. Thus only intersections of the type corresponding to cases (a) and (b) in Fig. 5 occur.

In the case $0<e_{12}<e_{13}<e_{23}$ only intersections of the type

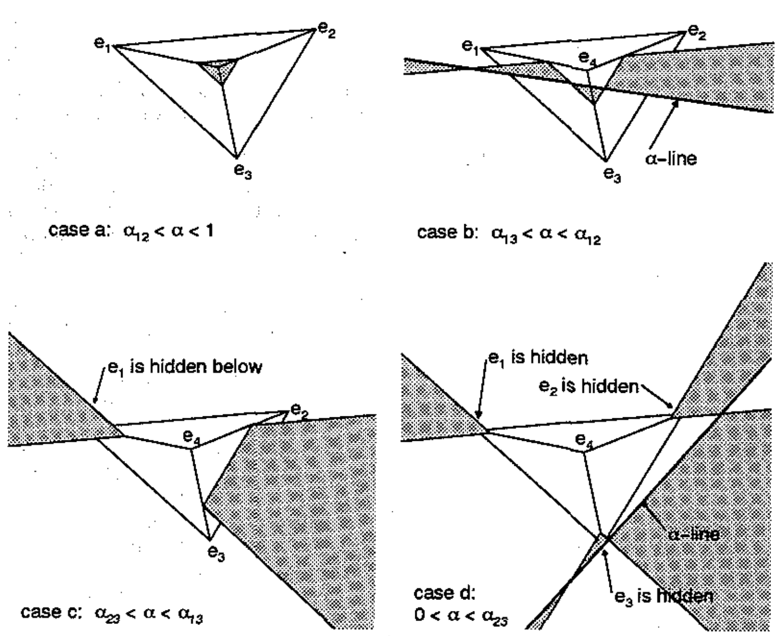

Fig. 5. The intersection of the normalized solution set $\mathcal{N}$ with four α planes for the case $e_{12}<e_{13}<e_{23}\leq 0$ (and thus $0\leq\alpha_{23}<\alpha_{13}<\alpha_{12}<1$) is shown. The intersection is shaded gray. In addition, the lines connecting the four elementary solutions are plotted. (The α line is not shown for cases (a) and (c), since no part of it is contained in the normalized solution set.)

corresponding to case (a) in Fig. 5 occur. This is of course the case mc $(\Sigma) = 1$, as described in Proposition 2.2.

4. *Conclusions*

The main goal of this paper was to examine a conjecture to the effect that the solution set in a general errors-in-variables problem comprised a union of convex polyhedral sets. The conjecture was disproved by examining the four-dimensional case. It has been shown that in the case mc $(\Sigma) = 2$, generically, the normalized solution set is a union of subsets, where each of this subsets is bounded by the planes spanned by the elementary solutions and by a ruled (nonflat) surface.

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References

Anderson, B. D. O. and M. Deistler (1993). Identification of dynamic systems from nosiy data: the single factor case. *Math. Control, Sig. Syst.*, **6**, 61–65.

Deistler, M. and W. Scherrer (1992). Identification of linear systems from nosiy data. In D. Brillinger, P. Caines, J. Geweke, E. Parzen, M. Rosenblatt, and M. S. Taqqu (Eds), *New Directions in Time-Series Analysis*, Part II, pp. 21–42. Springer-Verlag, Berlin.

De Moor, B. and J. Vandewalle (1986a). A geometrical approach to the maximal corank problem in the analysis of linear systems. In *Proc. 25th IEEE Conf. on Decision and Control*, Athens, Greece, pp. 1990–1995.

De Moor, B. and J. Vandewalle (1986b). The uniqueness versus the nonuniqueness principle in the identification of linear relations from noisy data. In *Proc. 25th IEEE Conf. on Decision and Control*, Athens, Greece, pp. 1663–1665.

Frisch, R. (1934). Statistical confluence analysis by means of complete regression systems. Publication No. 5, University of Oslo, Economic Institute.

Kalman, R. E. (1982). System identification from noisy data. In A. Bednarek and L. Cesari (Eds), *Dynamical Systems II*, pp. 331–342. Academic Press, New York.

Scherrer, W. (1991). Strukturtheorie von linearen dynamischen Fehler-in-den-Variablen Modellen mit diagonalem Fehlerspektrum. PhD thesis, Technical University of Vienna.