Nonnegative Realization of a Linear System with Nonnegative Impulse Response

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***Abstract*— Let $H(z)$ be a rational transfer function, with associated nonnegative impulse response sequence. The paper considers the question: When does there exist a triple $A \in R^{N\times N}, b \in R^N, c \in R^N$ with all nonnegative entries and $H(z) = c'(zI - A)^{-1}b$? An essentially complete characterization is given of the $H(z)$ allowing such a realization, in terms of the location of the pole or poles of $H(z)$ with maximum modulus.**

I. Introduction

In this paper, we are concerned with the following question. Let $H(z)$ be a rational transfer function, with $H(z) = \sum_{k\geq 1} h_k z^{-k}$. When does there exist a nonnegative realization, i.e., a triple $A \in R_+^{n\times n}, b \in R_+^n, c \in R_+^n$ for which

$$H(z) = c'(zI - A)^{-1}b?$$

[Here $R_+^n, R_+^{n\times n}$ denote n-vectors and $n \times n$ matrices with all entries nonnegative]. Our main result provides a reasonably complete answer to the realization question, in terms of properties of the poles of $H(z)$. We will not consider the interesting and important questions of characterizing the minimal dimension of nonnegative realizations [which in general is of course not the same as the number of poles of $H(z)$], nor of characterizing in a helpful way the relationship between equivalent realizations.

There have been a number of contributions dealing with different aspects of this problem, see, e.g., [1]–[7].We will draw particularly from [3]. We were actually motivated to examine this problem in the course of examining some realization and approximation questions in the area of hidden Markov models [HMM's], which in some ways are a generalization [8]–[11]. It would appear that the concepts developed in this paper will find application also in HMM problems.

Aside from the possible application and the intrinsic system theoretic interest of the nonnegative realization problem, we should note that nonnegative realizations arise naturally in the modeling of linear compartmental systems, where physical constraints force nonnegativity, [1], [12]. Reference [3] adverts (with few details) to applications in engineering, economics and medicine. Reference [13] describes so-called charge routing networks, which are MOS structures for implementing digital filters where the nonnegativity constraints are a consequence of the physics. Further material appears in [14] and [15].

An outline of the paper is as follows. In the next section, by making comparatively minor modifications to [2] and [3], we describe how the existence question for nonnegative realizations transforms into a question on the existence of a polyhedral cone [this latter question at first glance being just as difficult]. Section III develops three helpful lemmas. The first is an easy restatement of the cone condition using impulse response data, the second is a trivial observation on realization of the transfer function $H_\alpha(z) = H(\alpha z)$ for α positive real, and the third is a result linking the maximum modulus of the poles of $H(z)$ with the maximum modulus eigenvalue of at least one positive realization of $H(z)$, assuming positive realizations exist. The main result is in Section IV.

Section V contains remarks on the continuous-time nonnegative realization problem, while Section VI contains concluding remarks.

Notice that if $h_k = 0$ for all $k > K$, the (discrete-time) nonnegative realization problem is a trivial one [Take A as a single $K \times K$ Jordan block with zero eigenvalues, $b = [h_1 \cdots h_{K-1}\, h_K]'$ and $c = [1\, 0 \cdots 0]'$.] So throughout this paper, we shall assume that $H(z)$ has poles other than just at the origin.

II. Realization in Terms of Polyhedral Cone

We begin with some definitions. Let X be a set of vectors in R^n. Cone (X), which we shall write as $\mathcal{X}$, is the set of all (finite) nonnegative linear combinations of X. The dual of $\mathcal{X}$, written $\mathcal{X}^*$, is defined by $\{y|x'y \geq 0 \forall x \in X\}$. The dual is a closed set for any $\mathcal{X}$, and $\mathcal{X} \subset \mathcal{X}^{**}$, with equality if and only if $\mathcal{X}$ is closed. The set X may have an infinite number of elements. We say it is finitely generated or polyhedral if there exists a finite set of n-vectors P for which $\mathcal{P} = \mathcal{X}$. If we think of P as a matrix, then the columns of P generate $\mathcal{P} =$ cone (P). It is trivial that if a cone $\mathcal{P}$ is finitely generated, so is $\mathcal{P}^*$.

We shall now state two theorems on nonnegative realizations. These theorems are minor restatements of results obtained in [2], [3], and [16].

Theorem 1: Let $H(z)$ be a rational transfer function, with minimal (i.e., controllable and observable) realization $\{F, g, h\}$; thus $H(z) = h'(zI - F)^{-1}g$. Let $\mathcal{R}$ denote the cone

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spanned by $g, Fg, F^2g, \ldots$. Then if $H(z)$ has a nonnegative realization, there exists a (finite) matrix P such that [with $\mathcal{P} = \text{cone}(P)$]

$$\mathcal{R} \subset \mathcal{P} \quad (1a)$$

$$F\mathcal{P} \subset \mathcal{P} \quad (1b)$$

$$h \in \mathcal{P}^* \quad (1c)$$

Proof: See Appendix A.

Remark: In [3], a more symmetric type of result is effectively established. Let $\mathcal{S} = \{x | h'F^i x \geq 0, i = 0, 1, \ldots\}$. Then the Theorem holds true if (2.1c) is replaced by

$$\mathcal{P} \subset \mathcal{S} \quad (1d)$$

In fact, (1d) is a simple consequence of (1b) and (1c). To establish (1d), notice that by (1c) for any $p \in \mathcal{P}, h'p \geq 0$. In the light of (1b), $F^i p \in \mathcal{P}$ also, so then $h'F^i p \geq 0$ for $i = 1, 2, \ldots$. For example, $\mathcal{P} \subset \mathcal{S}$. We will not use this alternative.

The second promised theorem is the converse of Theorem 2.1.

Theorem 2.2: Let $H(z)$ be a rational transfer function with minimal realization $\{F, g, h\}$. Let $\mathcal{R}$ denote Cone $[g, Fg, F^2g \ldots]$ and suppose that for some $n \times N$ matrix P, with $\mathcal{P} = cone(P)$ (1a), (1b), and (1c) hold. Then there exists a realization $\{A, b, c\}$ of $H(z)$ with $b, c \in R_+^N, A \in R_+^{N \times N}$.

Proof: See Appendix B

We commented earlier that in Theorem 2.1, the condition (1c), *viz* $h \in \mathcal{P}^*$, could be replaced by (1d) $\mathcal{P} \subset \mathcal{S}$. An easy consequence is that

$$F\mathcal{R} \subset \mathcal{R} \text{ and } \mathcal{F}\mathcal{S} \subset \mathcal{S}$$

In our assault on the realization problem, the first property we shall use is that $h_k \geq 0$ for all k. This is equivalent to the condition $\mathcal{R} \subset \mathcal{S}$. However, the tough part of the realization problem is to find an additional, easily checked, condition on $H(z)$ to ensure existence (and computability) of a **finitely generated** $\mathcal{P}$ lying between $\mathcal{R}$ and $\mathcal{S}$,

$$\mathcal{R} \subset \mathcal{P} \subset \mathcal{S}$$

and with the F-invariance property

$$F\mathcal{P} \subset \mathcal{P}$$

Without the finitely generated requirement, with $h_k \geq 0$ we could simply take $\mathcal{R} = \mathcal{P}$ or $\mathcal{P} = \mathcal{S}$. In general, $\mathcal{R}$ and $\mathcal{S}$ are not finitely generated, although in some cases they may have this property, see [16].

III. Reformulations and Lemmas

Theorems 2.1 and 2.2 employ an arbitrary minimal realization F, g, h. By specializing to a particular realization, we get a statement of the cone condition involving input-ouput quantities (Markov parameters) only. Suppose that

$$H(z) = \frac{p_1 z^{n-1} + \cdots + p_n}{z^n + q_1 z^{n-1} + \cdots + q_n} = \sum_{k \geq 1} h_k z^{-k} \quad (2)$$

where the polynomials $p_1 z^{n-1} + \cdots + p_n$ and $z^n + q_1 z^{n-1} + \cdots + q_n$ are coprime. It is standard (and easily checked) that a minimal realization for $H(z)$ is given by

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -q_n & -q_{n-1} & -q_{n-2} & \cdots & -q_1 \end{bmatrix}$$

$$g = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \quad h' = [1 \quad 0 \cdots 0] \quad (3)$$

It follows that

$$[g \quad Fg \quad F^2 g \cdots] = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ \vdots & \vdots & \vdots & \\ h_n & h_{n+1} & h_{n+2} & \cdots \end{bmatrix} \quad (4)$$

which is a truncated Hankel matrix, call it H_n.

We now obtain the following simple consequence of Theorems 2.1 and 2.2.

Theorem 3.1: Let $H(z)$ be the rational nth-order transfer function given in (2), with $h_k \geq 0$ for all k. Let F be as in (3) and let

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ \vdots & \vdots & \vdots & \\ h_n & h_{n+1} & h_{n+2} & \cdots \end{bmatrix} \quad (5)$$

Then $H(z)$ has a nonnegative realization if and only if for some N there exists an $n \times N$ matrix P such that

$$Cone(H_n) \subset \mathcal{P} \quad (6a)$$

$$F\mathcal{P} \subset \mathcal{P} \quad (6b)$$

The first row of P has nonnegative entries (6c)

In the sequel, we shall work with this formulation.

The second result, even simpler, allows us to position a pole of $H(z)$.

Lemma 3.1: Let $H(z)$ be a rational nth-order transfer function with nonnegative impulse response. Then $H(z)$ has a nonnegative realization if and only if $H_\alpha(z) = H(\alpha z)$ has a nonnegative realization for any positive α.

Proof: If $H(z) = c'(zI - A)^{-1}b$ where $A \in R_+^{N \times N}, b \in R_+^N, c \in R_+^N$, then $H_\alpha(z) = c'(zI - \alpha^{-1}A)^{-1}\alpha^{-1}b$ and $\{\alpha^{-1}A, \alpha^{-1}b, c\}$ defines a nonnegative realization for $H_\alpha(z)$. □

Evidently, if $H(z)$ has a positive real pole, we can assume, without any significant loss of generality in treating the positive realization problem, that the pole is located at $z = 1$.

The third result connects poles of $H(z)$ of maximum modulus with eigenvalues of the matrix A of a nonnegative realization, also of maximum modulus. To motivate it, we shall first make a simple calculation valid only in a restrictive situation in Lemma 3.2. The general result appears in Theorem 3.3.

Lemma 3.2: Let $H(z)$ be a rational nth-order transfer function with a nonnegative realization A, b, c in which A has all positive entries. Then the eigenvalue of A of maximum modulus which by the Perron-Frobenius theory [17]-[19] is *unique* and positive is also a pole of $H(z)$.

Proof: With A possessing all positive entries, it is standard that there exists a positive right and left eigenvector associated with the unique eigenvalue $\bar{\lambda}$ of maximum modulus (which is positive):

$$Av = \bar{\lambda}v$$
$$w'A = w'\bar{\lambda}$$

Now consider $h_{k+1} = c'A^k b$ for large k. It is standard that

$$\bar{\lambda}^{-k}c'A^k b \to c'[vw']b = c'vw'b$$

Because b, c are nonnegative and nonzero and v, w are positive vectors, $c'vw'b$ is positive. Hence $\bar{\lambda}$ must be a pole of $H(z)$.

Remark: Obviously, there can be no pole of $H(z)$ with modulus greater than $\bar{\lambda}$ since all poles of $H(z)$ must show up as eigenvalues of A. Of course, *certain* eigenvalues of A *may not* be poles of $H(z)$, because they are unobservable or uncontrollable. What we have just demonstrated is that if A is positive, the *maximal* eigenvalue of A must be observable and controllable in the realization $\{A, b, c\}$.

The task is now to improve on the result of Lemma 3.2 for the case of nonnegative as opposed to positive A. It turns out that there can be realizations with nonnegative A such that the maximum (real) eigenvalue of A is not a pole of $H(z)$. However, if this is so, we show we can find a lower dimension but still nonnegative realization without this eigenvalue. By repeating the argument we shall show:

Theorem 3.2: Let $H(z)$ be a rational n-th order transfer function with nonnegative impulse response and suppose it possesses a nonnegative realization $\{A, b, c\}$. Then it also possesses a nonnegative realization of $\{\bar{A}, \bar{b}, \bar{c}\}$ of lesser or equal dimension such that among the eigenvalues of $\bar{A}$ of maximum modulus, which necessarily contain a positive real eigenvalue $\bar{\lambda}, \bar{\lambda}$ is a pole of $H(z)$.

Proof: Let $\{A, b, c\}$ be a nonnegative realization of dimension N of $H(z)$. Let $\bar{\lambda}$ be the modulus of a maximum modulus eigenvalue of A. Then A has an eigenvalue at $\bar{\lambda}$ and the associated right and left eigenvalues v and w are nonnegative:

$$Av = \bar{\lambda}v \quad w'A = \bar{\lambda}w' \quad v \in R_+^N, \quad w \in R_+^N$$

We shall now show that either $c'v \neq 0$, i.e., this eigenvalue is observed, or unobservable states can be removed without destroying the nonnegativity of the realization.

Suppose then that $c'v = 0$. Without loss of generality, reorder the entries of the state vector so that

$$c' = (c_1' \; 0 \; 0)$$
$$v' = (0 \; 0 \; v_3')$$

with c_1 and v_3 positive.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Because $Av = \bar{\lambda}v$, the zeros in v force $A_{13} = 0, A_{23} = 0$. But now the zero blocks in c' and A mean that an unobservable part is displayed, and a lower dimension but still nonnegative realization is provided by

$$\left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_1' \;\; 0] \right\}$$

Similarly, if $w'A = \bar{\lambda}w'$ and $w'b = 0$, we can immediately eliminate certain uncontrollable blocks and still retain the nonnegativity of the realization.

Starting with an arbitrary nonnegative realization then, we can reduce it by eliminating uncontrollable and/or unobservable blocks, and retaining the nonnegativity, until the real eigenvalue of maximum modulus is controllable and observable, i.e., is a pole of $H(z)$. □

There is an important consequence of this result; *viz* a new necessary condition on $H(z)$ with nonegative impulse response for it to have a nonnegative realization.

Corollary 3.1: Let $H(z)$ be a rational transfer function with nonnegative impulse response. A necessary condition for $H(z)$ to have a nonnegative realization is that the poles of $H(z)$ of maximum modulus be a subset of those which are the allowed eigenvalues of maximum modulus of a nonnegative matrix.

As explained in for example [18], an $N \times N$ irreducible nonnegative A can have k eigenvalues of maximum modulus for any $k = 1, \cdots, N$. The eigenvalues are then located at $\omega_1 r, \omega_2 r, \cdots, \omega_k r$ where $\omega_1, \cdots, \omega_k$ are the distinct k-th roots of unity and r is the eigenvalue modulus. The eigenvalues of maximum modulus of an arbitrary (not necessarily irreducible) $N \times N$ nonnegative matrix will be the union of such sets.

Let us also note that in [3], Theorems 2 and 3 imply (actually for the continuous time version of the problem) that the maximal real pole of $H(z)$ must be dominant, which is a less powerful result.

Example: The nonnegative impulse response sequence $h_k = \left(\frac{1}{2}\right)^k \sin^2 k$ for which

$$H(z) = \frac{1}{2}\left[\frac{\frac{1}{2}}{z - \frac{1}{2}} - \frac{z\left(\frac{1}{2}\cos 2\right) - \frac{1}{4}}{z^2 - z\cos 2 + \frac{1}{4}}\right]$$

does not have a finite-dimensional nonnegative realization of any finite dimension. The poles of $H(z)$ are $1/2$, $(1/2)\exp(\pm 2j)$. If there were a nonnegative realization $\{A, b, c\}$, there would be one with $1/2$ as the maximum modulus for the eigenvalue of A, and then since the eigenvalues of A necessarily include the poles of $H(z)$, the requirement to include a pole at $(1/2)\exp(\pm 2j)$ is impossible.

A nonnegative matrix, as just mentioned, cannot have arbitrary eigenvalues for its eigenvalues of maximum modulus. Indeed, there are restrictions on the other eigenvalues also, [20]. However, these restrictions become less restrictive as the dimension of the matrix increases. This means that if an $H(z)$

has an acceptable pole pattern as far as its maximum modulus poles are concerned, there are no restrictions on the other poles in the sense that there will always exist a nonnegative matrix of sufficiently large dimension with eigenvalue set including all the poles of $H(z)$, irrespective of the pattern in which those not of maximum modulus lie. [Of course, whether b and c can also be found to realize $H(z)$ is a separate question].

IV. Realization Existence with Restrictions on Poles of Maximum Modulus

In the previous section we have explained that a necessary condition on $H(z)$ with nonnegative h_k to have a nonnegative realization is that the poles of maximum modulus obey a certain restriction. The main result of this paper is that nonnegative realizability is possible if the pole restriction is fulfilled *and* $\lim_{k\to\infty}\inf\bar{\lambda}^{-k}h_k > 0$, where $\bar{\lambda}$ is the magnitude of a maximum modulus pole of $H(z)$.

For pedagogical reasons, we shall first consider the restricted situation where $H(z)$ has a single pole of maximum modulus, which by Corollary 3.1 is necessarily real and positive under the restriction, if $H(z)$ is to have a nonegative realization.

By Lemma 3.1, there is no loss of generality in assuming that the pole is at $z = 1$. This means that as $k \to \infty, \lim_{k\to\infty} h_k$ exists (and is positive). For convenience, we scale h_k so that the limit is 1.

Under the assumption of existence of a single pole of maximum modulus at $z = 1$, with $\lim_{k\to\infty} h_k = 1$, we can consider the test contained in Theorem 3.1. Thus we seek an $n \times N$ matrix $\mathcal{P}$ such that

$$Cone\begin{pmatrix} h_1 & h_2 & \cdots \\ \vdots & \vdots & \\ h_n & h_{n+1} & \cdots \end{pmatrix} \subset \mathcal{P} \tag{7a}$$

$$F\mathcal{P} \subset \mathcal{P} \tag{7b}$$

and P has nonnegative entries in the first row, ie

$$P'(1,0,\ldots,0)' \in R^N_+ \tag{7c}$$

We shall establish

Theorem 4.1: Let $H(z) = \frac{\sum_{i=1}^n p_i z^{n-i}}{z^n+\sum_{i=1}^n q_i z^{n-i}}$ be a rational nth-order transfer function with nonnegative impulse response $(h_k, k = 0,1\ldots)$. Suppose $H(z)$ has one single pole of maximum modulus which is positive real. Then there exists for some N an $n \times N$ matrix $\mathcal{P}$ such that (7) holds, where

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -q_n & -q_{n-1} & -q_{n-2} & \cdots & -q_1 \end{bmatrix} \tag{8}$$

and accordingly $H(z)$ possesses a nonnegative realization.

Proof: Without loss of generality, suppose the maximum modulus pole is at $z = 1$, and $\lim_{k\to\infty} h_k = 1$.

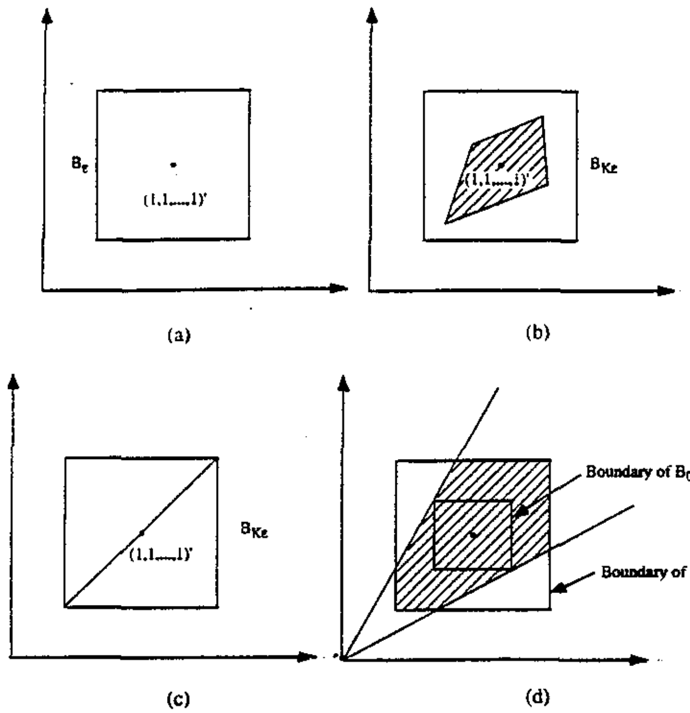

Fig. 1. Illustration of claims (a) – (d). (a) B_ϵ contains limiting u_m and all u_m for suitably large m. (b) Shaded figure shows general shape of $F^m B_\epsilon$ for any m, always fitting inside $B_{K\epsilon}$ for some K depending on F, and with origin excluded from $B_{K\epsilon}$. (c) Limiting points of $F^m x$ for all $x \in B_\epsilon, m \to \infty$ must lie on the diagonal line. (d) Shaded area denotes bound for $F^m B_\epsilon$ for all $m \geq M_2(\epsilon)$, and cone lies inside Cone (B_ϵ).

Let u_j denote $[h_j,\ldots,h_{j+n-1}]'$ and let H_{nN} denote the matrix $(u_1,\ldots,u_N)$. There are two possibilities to consider. Either the vector $[1\ldots1]'$ lies in the interior of the convex cone generated by $u_1, u_2, \ldots, u_M$ for some finite M, or it does not. If it does, since $u_p \to [1\ldots1]'$ as $p \to \infty$, there will be an N such that for all $p \geq N, u_p \in$ Cone $(u_1, u_2, \ldots, u_N)$, and then Cone H_n will be generated by $P = (u_1, u_2, \ldots, u_N)$. Further, since $Fu_i = u_{i+1}$ for all i [see e.g. (4)] $F\mathcal{P} \subset Cone(u_1, u_2, \ldots, u_{N+1}) \subset Cone H_n = \mathcal{P}$. Finally, the first row of P, being $[h_1, \ldots, h_N]$, is obviously nonnegative. Note that in this case, the controllability cone $\mathcal{R} = cone[g\ Fg\ F^2g\ldots]$ is itself finitely generated, which is not in general guaranteed.

Accordingly, suppose now that $[1\ldots1]'$ does not lie in the interior of the convex cone generated by $(u_1, u_2, \ldots, u_M)$ for any M.

For $1 > \epsilon > 0$, define a set $B_\epsilon \in R^n_+$ by $B_\epsilon = \{x | 1-\epsilon \leq x^i \leq 1+\epsilon\}$
and a set

$$B_{0\epsilon} = \{x | -\epsilon \leq x^i \leq \epsilon\}$$

Because F has one eigenvalue at 1 and all others in $|z| < 1$, it is obvious that $\bigcup_m F^m B_{0\epsilon}$ is a bounded set, so that

$$\bigcup_{m\geq 0} F^m B_{0\epsilon} \subset B_{0K\epsilon}$$

where $K \geq 1$ is a constant depending just on F. Choose ϵ so that $K\epsilon < 1$. Then the following properties hold [see Fig. 1].

Then

- i) There exists $M_1(\epsilon)$ such that for $m \geq M_1(\epsilon), u_m \in B_\epsilon$. [This follows because $u_m \to [1, 1 \dots 1]$, which is in the interior of B_ϵ].
- ii) With $K\epsilon < 1$, for all m all entries of $F^m B_\epsilon$ are positive (Think of $F^m B_\epsilon$ as $F^m B_{0\epsilon}$ translated to sit round the point $[1\ 1 \dots 1]'$. Since $F^m B_{0\epsilon} \subset B_{0K\epsilon}, F^m B_\epsilon$ sits in a square box $B_{K\epsilon}$ centered on $[1\ 1 \dots 1]'$ and this box does not contain the origin, because $K\epsilon < 1$.
- iii) For any $x \in B_\epsilon$, as $m \to 0, F^m x \to k(x)[1\ 1 \dots 1]$, where k is a positive scalar constant determined by x (This is a consequence of having all eigenvalues of F inside $|z| < 1$, save for an eigenvalue of 1, for which the associated eigenvector is checked to be $[1\ 1 \dots 1]'$).
- iv) There exists $M_2(\epsilon)$ such that for $m \geq M_2(\epsilon), Cone(F^m B_\epsilon) \subset Cone(B_\epsilon)$. [This is a consequence of ii) and iii)].

Now define

$$\mathcal{P} = Cone[u_1\ u_2 \dots u_{M_1}\quad B_\epsilon\quad FB_\epsilon \dots F^{M_2-1}B_\epsilon] \tag{4.3}$$

Obviously, $\mathcal{P}$ is finitely generated. By (i),

$$\begin{aligned} &Cone\begin{pmatrix} h_1 & h_2 & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \\ h_n & h_{n+1} & \cdots \end{pmatrix} \\ &= Cone\Big\{[u_1\ u_2 \dots u_{M_1}] \bigcup [u_{M_1+1}\ u_{M_1+2} \dots]\Big\} \\ &\subset Cone\Big\{[u_1\ u_2 \dots u_{M_1}] \bigcup B_\epsilon\Big\} \subset \mathcal{P} \end{aligned} \tag{4.4}$$

Next

$$\begin{aligned} &F\ Cone[u_1\ u_2 \dots u_{M_1}] \\ &= Cone[u_2 \dots u_{M_1+1}] \subset Cone[u_2 \dots u_{M_1}\ B_\epsilon] \subset \mathcal{P} \end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned} &FCone[B_\epsilon\ FB_\epsilon \dots F^{M_2-1}B_\epsilon] \\ &= Cone[FB_\epsilon \dots F^{M_2}B_\epsilon] \\ &\subset Cone[FB_\epsilon \dots F^{M_2-1}B_\epsilon\ B_\epsilon] \quad \text{by (iv)} \\ &\subset \mathcal{P} \end{aligned} \tag{4.6}$$

Hence from (4.3), (4.5) and (4.6), we see that

$$F\mathcal{P} \subset \mathcal{P} \tag{4.7}$$

Finally, all entries of $F^i B_\epsilon$ are positive for all i and so the vectors generating $\mathcal{P}$ all have nonnegative first entries. Since (7) has been verified, the result is established from (10) and (13). ☐

In the above theorem, we have started from the hypothesis that $H(z)$ has a single simple pole of maximum modulus. The theorem can be very easily extended to cope with the case of a single pole of maximum modulus with order greater than 1. Then we shall extend to the case where $H(z)$ has more than one distinct pole of maximum modulus; in this instance, we shall however impose the restrcition that the poles are simple, and that a certain restriction holds on the limiting ($k \to \infty$) behavior of h_k.

Corollary 4.1: Assume the same hypotheses as Theorem 4.1, save that the pole of $H(z)$ of maximum modulus is a multiple pole. Then the conclusion of the theorem remains valid.

Proof: Without loss of generality, supose the maximum modulus pole is al $z = 1$, and it has order p. Then $h_k = f(k) + o(1)$, where $f(k)$ is a polynomial in k of order $p - 1$. Since $h_k \geq 0$, the coefficient of k^{p-1} in $f(k)$ is necessarily positive. Using scaling if necessary, let us suppose it is 1. With u_j denoting $[h_j, \dots, h_{j+n-1}]'$, it is evident that $u_k/k^{p-1} \to [1, \dots, 1]$ as $k \to \infty$. The remainder of the proof of the Corollary is as for the theorem, apart from trivial variations which in some places replace u_k by u_k/k^{p-1}. ☐

We now turn to the more difficult problem arising when there are distinct maximum modulus poles.

Theorem 4.2: Let $H(z) = \frac{\sum_{i=1}^n p_i z^{n-i}}{z^n + \sum_{i=1}^n q_i z^{n-i}}$ be a rational nth-order transfer function with nonnegative impulse response $h_k, k = 0, 1 \dots$. Let $\bar{\lambda} \neq 0$ be the magnitude of the maximum modulus poles of $H(z)$ and suppose $\lim_{k\to\infty} \inf \bar{\lambda}^{-k} h_k > 0$. Suppose $H(z)$ has in addition to a simple pole at $\bar{\lambda}$ further simple poles of magnitude $\bar{\lambda}$ at angles corresponding to one or more roots of unity apart from 1, and all other poles lie in $|z| < \bar{\lambda}$. Then there exists for some N an $n \times N$ matrix P such that (7) holds and accordingly, $H(z)$ possesses a nonnegative realization.

Remark: The constraint $\lim_{k\to\infty} \inf \bar{\lambda}^{-k} h_k \neq 0$ does exclude a "small" class of systems with a nonnegative realization. Consider for example the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad c' = (\gamma \quad \beta \quad \alpha)$$

and c a nonnegative vector. Then $\bar{\lambda} = 1, h_1 = h_4 = \dots = \alpha, h_2 = h_5 = \dots = \beta, h_3 = h_6 = \dots = \gamma$ and taking $\alpha = 0$ gives $\lim_{k\to\infty} \inf \bar{\lambda}^{-k} h_k = 0$ while a nonnegative realization is available.

Proof: As before denote the successive columns of H_n by $u_1, u_2 \dots$ and without loss of generality assume $\bar{\lambda} = 1$. Suppose that poles of $H(z)$ occur at 1 and also at a nonzero number (but not necessarily all) of the pth roots of 1. Then it is not hard to check that there is a finite set of values $a_1, a_2 \dots, a_p$ such that, in the limit as $k \to \infty, h_k$ cycles through the values $a_1, a_2 \dots, a_p$ i.e.

$$\begin{bmatrix} h_{kp+1} \\ h_{kp+2} \\ \vdots \\ h_{kp+n} \end{bmatrix} \to \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

as $k \to \infty$. This means that with $b_1 = [a_1 \dots a_n]', b_2 = [a_2 \dots a_{n+1}]'$, etc., the columns of H_n obey

$$u_{kp+i} \to b_i$$

as $k \to \infty$. Because of the assumption $\lim_{k\to\infty} \inf h_k > 0$, the b_i are all positive vectors. Notice that because $Fu_i = u_{i+1}$ for all i, see (4), $Fb_i = b_{i+1}$ and $Fb_p = b_1$.

As in the proof of Theorem 4.1, if $b_1, b_2, \ldots, b_p$ all lie in the interior of the convex cone generated by $u_1, u_2, \ldots, u_M$ for some M, the proof is immediate, because $\mathcal{R}$ is finitely generated and $\mathcal{P} = \mathcal{R}$ suffices. So we shall suppose to the contrary.

Let us now observe that rank $[b_1 \ldots b_p]$ is precisely the number of (simple) unit circle poles of $H(z)$. For if $H(z) = K(z) + L(z)$ where $K(z)$ has only poles on $|z| = 1$ and $L(z)$ has only poles in $|z| < 1$, it is evident that the Hankel matrix with n rows and infinite number of columns associated with $K(z)$ is

$$[b_1\, b_2\, b_3 \ldots b_p \;\; b_1\, b_2 \ldots b_p \;\; b_1 \ldots]$$

(or a shifted version). A standard linear system result guarantees that the rank of this matrix is the number of poles of $K(z)$, i.e., the number of unit circle poles of $H(z)$, or equivalently, by minimality, the number of unit circle eigenvalues of F.

Now we shall try to mimic the proof of Theorem 4.1. Let $\Gamma = [\gamma_1, \ldots, \gamma_q]$ define a real basis for the space spanned by those eigenvectors and generalized eigenvectors of F corresponding to eigenvalues of modulus less than 1. Define a set of vectors $C_{\epsilon_1,\epsilon_2}$ by

$$C_{\epsilon_1,\epsilon_2} = \left\{ x | x = \sum_{i=1}^{p} \alpha_i b_i + \sum_{j=1}^{q} \beta_j \gamma_j, |\alpha_i| < \epsilon_1, |\beta_j| < \epsilon_2 \right\}$$

Notice that because rank $[b_1 \ldots b_p]$ is the number of unit circle eigenvalues of F, the set $C_{\epsilon_1,\epsilon_2}$ contains an n-dimensional box $B_{0\epsilon}$ for suitably small ϵ and is likewise contained in such a box for suitably large ϵ.

Recall that $Fb_i = b_{i+1}, Fb_p = b_1$. Also, $F^m \gamma_j \to 0$ as $m \to \infty$.

Hence for all m

$$F^m C_{\epsilon_1,\epsilon_2} \subset C_{\epsilon_1, K\epsilon_2}$$

for some $K > 1$ depending on F, and also

$$F^m C_{\epsilon_1,\epsilon_2} \to C_{\epsilon_1,0}$$

as $m \to \infty$; hence, there exists $M_2(\epsilon_2)$ so that for all $m \geq M_2(\epsilon_2)$

$$F^m C_{\epsilon_1,\epsilon_2} \subset C_{\epsilon_1,\epsilon_2}$$

Define also

$$B_\epsilon = \bigcup_{i=1}^{p} [b_i + C_{\epsilon,\epsilon}]$$

and require that ϵ is chosen sufficiently small that all vectors in $b_i + C_{\epsilon,K\epsilon} (i = 1, \ldots, p)$ have positive entries. [Note that the existence of ϵ relies on b_i being positive rather than just nonnegative; the positivity of b_i follows from the theorem hypothesis that $\lim_{k\to\infty} \inf h_k > 0$.] Now we consider parallels to the earlier claims i)–iv) made in proving Theorem 4.1.

- i) As before, there exists $M_1(\epsilon)$ because $u_{kp+1} \to b_i$ as $k \to \infty$, so that for all $m \geq M_1(\epsilon), u_m \in B_\epsilon$.
- ii) With ϵ chosen sufficiently small so that all vectors in $b_i + C_{\epsilon,K\epsilon}$ have positive entries, $F^m B_\epsilon$ for all m contain only positive vectors. [Use the fact that $Fb_i = b_{i+1}, Fb_p = b_1$, and $F^m C_{\epsilon,\epsilon} \subset C_{\epsilon,K\epsilon}$].
- iii) For any $y \in B_\epsilon$, there holds $F^m y \in B_\epsilon$ for $m \geq M_2 \epsilon$. [This follows because $m \geq M_2(\epsilon)$ yields $F^m C_{\epsilon,\epsilon} \subset C_{\epsilon,\epsilon}$].
- iv) For $m \geq M_2(\epsilon)$, $Cone(F^m B_\epsilon) \subset Cone(B_\epsilon)$. [This is a trivial consequence of iii)].

Now define

$$\mathcal{P} = Cone[u_1, u_2, \ldots u_M, \quad B_\epsilon, FB_\epsilon, \ldots F^{M_2-1} B_\epsilon]$$

The same argument now applies as in the proof of Theorem 4.1 to establish the result. □

Example: Consider the impulse response

$$\begin{aligned} h_k &= (\tfrac{1}{2})^k \left[1 + \cos\frac{2\pi k}{3} + \frac{1-\epsilon}{\sqrt{3}} \sin\frac{2\pi k}{3}\right] - \epsilon\left(\frac{1}{16}\right)^k \\ &= 2\left(\frac{1}{2}\right)^k - \epsilon\left(\frac{1}{16}\right)^k \qquad k = 0, 3, 6, \ldots \\ &= \left(1 - \frac{\epsilon}{2}\right)\left(\frac{1}{2}\right)^k - \epsilon\left(\frac{1}{16}\right)^k \quad k = 1, 4, 7, \ldots \\ &= \frac{\epsilon}{2}\left(\frac{1}{2}\right)^k - \epsilon\left(\frac{1}{16}\right)^k \quad k = 2, 5, 7, \ldots \end{aligned}$$

With ϵ small, this transfer function has a nonnegative realization since $h_k > 0$ for all k, the maximum modulus poles are at $\frac{1}{2}, \frac{1}{2}\exp j\frac{2\pi}{3}, \frac{1}{2}\exp j\frac{4\pi}{3}$ and $\lim_{k\to\infty} \inf \left(\frac{1}{2}\right)^{-k} h_k > 0$.

V. Continuous-Time Systems

In this section, we study rational transfer functions $H(s) = h'(sI - F)^{-1} g$ which are the Laplace transforms of nonnegative continuous-time impulse responses $h(t)$. Thus

$$h' e^{Ft} g \geq 0 \quad \text{for } t \geq 0 \tag{14}$$

Nonnegative realizations of $h(t)$ are those triples $\{A, b, c\}$ with $h(t) = c' e^{At} b$ where all entries of b and c and *off-diagonal* entries of A are nonnegative [3]. Note that the absence of restriction on the diagonal entries of A is standard and motivated in applications [1], [12], [14]. Also, $h(t) > 0$ for $t > 0$.

The main result is as follows.

Theorem 5.1: Let $H(z) = \frac{\sum_{i=1}^{n} p_i s^{n-i}}{s^n + \sum_{i=1}^{n} q_i s^{n-i}}$ be a rational n-th order transfer function corresponding to an impulse response $h(\cdot)$, with $h(t) \geq 0$ for all $t \geq 0$.

A necessary and sufficient condition for $H(s)$ to have a nonnegative realization is that:

- [(i)] there is a unique (possibly multiple) pole of $H(s)$ with maximal real part, and the pole is real
- [(ii)] with $H(s) = h(sI - F)^{-1} g$, there exists $\lambda > 0$ such that $H_{d\lambda}(z) = h(zI - F - \lambda I)^{-1} g$ has a nonnegative (discrete-time) impulse response.

Proof: Suppose first $H(s)$ has a nonnegative realization A, b, c. Then there exists $\lambda_0 \geq 0$ such that for all $\lambda \geq \lambda_0$, $\lambda I + A$ has all nonnegative entires. Then $H_{d\lambda}(z) = c(zI - A - \lambda I)^{-1}b$ is a discrete-time transfer function with nonnegative realization $\{\lambda I + A, b, c\}$. If $\{F, g, h\}$ is a minimal realization of $H(s)$, then $\{\lambda I + F, g, h\}$ is a minimal realization of $H_{d\lambda}(z)$. By corollary 3.1, there is a positive real maximum modulus pole of $H_{d\lambda}(z)$, call it z_0. Then $z_0 - \lambda$ is the unique pole of $H(s)$ with maximal real part, and this pole is real.

For the converse, choose λ_0 so that $h(zI - F - \lambda_0 I)^{-1}g$ has a nonnegative impulse response. It is easy to see then that $h(zI - F - \lambda I)^{-1}g$ has a nonnegative impulse response for an $\lambda \geq \lambda_0$.

The poles of $H_{d\lambda}(z)$ are those of $H(s)$ shifted rightwards by λ. By i), we can always find a suitably large λ to ensure that the maximum modulus pole of $H_{d\lambda}(z)$ is unique and real (though possibly not simple). By Corollary 4.1, $H_{d\lambda}(z)$ possesses a nonnegative realization, i.e., there exist nonnegative A_d, b_d and c_d such that

$$c_d(zI - A_d)^{-1}b_d = h(zI - F - \lambda I)^{-1}g$$

It follows that $(A_d - \lambda I, b_d, c_d)$ is a continuous-time nonnegative realization of $H(s)$.

Remark: It seems likely that if $h'e^{Ft}g > 0$ for all t, then there exists $\lambda > 0$ such that $h'(F + \lambda I)^k g > 0$ for $k = 0, 1, 2, \ldots$. If so, then ii) of the theorem could be dispensed with.

VI. Conclusion

This paper has revealed the key point that nonnegative realizability of a transfer function with nonnegative impulse response is essentially a property of the maximum modulus poles only. [We say essentially, since at most a finite number of values of the impulse response are allowed to be zero in our main result, which is a slight additional restriction]. A key aspect of the proof of this result was the preliminary result showing that nonnegative realizability implied the existence of a nonnegative realization with maximum eigenvalue modulus of the state transition matrix equal to the maximum pole modulus of the transfer function.

The difficult question of establishing the minimal degree of a nonnegative realization is much more bound up with all the poles. In fact, given an arbitrary integer N, it is possible to define a complex pole pair inside the unit circle (and thus a 3rd order system with nonnegative impulse response and one pole at 1), such that no nonnegative realization exists of dimension smaller than N. This is because there exist points inside the unit circle that cannot be an eigenvalue of any nonnegative matrix of size $M \times M$, $M \leq N$ when the maximum eigenvalue of that matrix is fixed at 1 [20].

Accordingly, we see questions of determining the minimal degree and relating minimal degree realizations as fundamentally different, and requiring more techniques than those used in this paper.

It also remains open to investigate the case of $H(z)$ with several poles of maximum modulus, where also $h_k \geq 0$ with $\lim_{k\to\infty} \inf h_k = 0$. Some results appear in [16].

An interesting question to consider is whether, given an $H(z)$ satisfying the pole constraint and the information that $h_k > 0$ for $k = 0, 1, \ldots, M$ say (with M finite), existence of a nonnegative realization can be inferred. Certainly, there does not exist an M depending just on the order of $H(z)$ - consider the sequence $h_k = 1 + \gamma\alpha^k \sin k\beta$ where $\alpha < 1, \gamma > 1$, which corresponds to a third-order $H(z)$. With $1 - \alpha$ and $\gamma - 1$ very small, the value of β may well determine whether $h_k > 0$ for all k. It is clear that, by choosing β small enough, an arbitrary large number of the initial h_k will be positive, even if eventually one or more can be negative.

Last, we note that an open problem remains for the case of continuous-time realization.

Appendix A
Proof of Theorem 2.1

The idea of the proof is to introduce matrices which transform a nonnegative realization A, b, c so that uncontrollable and unobservable parts can be cut out, and to construct $\mathcal{P}$ using these matrices. We begin by cutting out the uncontrollable part. Accordingly, with $\{A, b, c\}$ a nonnegative realization of $H(z)$, let $M = [M_1 \quad M_2]$ be a nonsingular matrix for which

$$A[M_1 \quad M_2] = [M_1 \quad M_2]\begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \qquad \text{(A.1a)}$$

$$b = [M_1 \quad M_2]\begin{bmatrix} e_1 \\ 0 \end{bmatrix} \qquad \text{(A.1b)}$$

$$c'[M_1 \quad M_2] = [f_1' \quad f_2'] \qquad \text{(A.1c)}$$

where (D_{11}, e_1) is completely controllable. Notice that $H(z) = f_1'(zI - D_{11})^{-1}e_1$. Also

$$AM_1 = M_1 D_{11} \quad b = M_1 e_1 \quad c'M_1 = f_1' \qquad \text{(A.2)}$$

Let

$$\mathcal{I} = [\text{Cone}(M_1')]^*. \qquad \text{(A.3)}$$

[This ensures that $\mathcal{I}$ is finitely generated.]

We claim that

$$\text{Cone}\,(e_1, D_{11}e_1, D_{11}^2 e_1, \ldots) \subset \mathcal{I} \qquad \text{(A.4a)}$$

$$D_{11}\mathcal{I} \subset \mathcal{I} \qquad \text{(A.4b)}$$

$$f_1 \subset \mathcal{I}^* \qquad \text{(A.4c)}$$

To see this, notice that

$$M_1 D_{11}^i e_1 = A^i b \in R_+^N$$

for all i, i.e. $D_{11}^i e_1 \in [Cone(M_1')]^*$. Equation (A.4a) is then immediate. Equation (A.4b) is straightforward to obtain from the first equation of (A.2). Finally, $f_1 = M_1'c$, i.e. $f_1 \in \text{Cone}\,(M_1') = \mathcal{I}^*$.

Next, we eliminate the unobservable parts from D_{11}, e_1, f_1: there exists a nonsingular matrix $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ such that

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} D_{11} = \begin{bmatrix} F & 0 \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \qquad \text{(A.5a)}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} e_1 = \begin{bmatrix} g \\ g_2 \end{bmatrix} \qquad \text{(A.5b)}$$

$$f_1' = [h' \quad 0]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{A.5c}$$

where $[F, h]$ is a completely observable pair and $[F, g]$ is a completely controllable pair. Of course, $H(z) = h'(zI - F)^{-1}g$ and

$$T_1 D_{11} = F T_1 \quad T_1 e_1 = g \quad f_1' = h' T_1. \tag{A.6}$$

Define

$$\mathcal{P} = \text{ Cone } (T_1 \mathcal{I}) \tag{A.7}$$

[which is finitely generated, since $\mathcal{I}$ is finitely generated]. We claim that the requirements of the theorem are fulfilled. From (A.6) we have

$$[g \; Fg \; F^2 g \ldots] = T_1 [e_1 \; D_{11} e_1 \; D_{11}^2 e_1 \ldots]$$

and then (2.1a) is an immediate consequence of (A.4a) and (A.7).

Next, using (A.7), (A.6), and (A.4c),

$$F\mathcal{P} = F \text{ Cone } (T_1 \mathcal{I}) = F T_1 \mathcal{I} = T_1 D_{11} \mathcal{I} \subset T_1 \mathcal{I} = \mathcal{P}.$$

Finally, since $f_1 \subset \mathcal{I}^*$, i.e. $T_1' h \subset \mathcal{I}^*$, i.e. $h \subset [Cone(T_1 \mathcal{I})]^* = \mathcal{P}^*$

APPENDIX B
PROOF OF THEOREM 2.2

From (1b), there follows

$$FP = PA$$

for some $A \in R_+^{N \times N}$. Since $g \in R$, from (1a) there follows

$$g = Pb$$

for some $b \in R_+^N$. Finally, from (1c) there follows

$$h'P = c'$$

for some $c \in R_+^N$. It is trivial to verify that $c' A^i b = h' F^i g$.

REFERENCES

[1] H. Maeda, S. Kodama, and F. Kajiya, "Compartmental system analysis: Realization of a class of linear systems with physical constraints," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 8-14, Jan. 1977.

[2] H. Maeda and S. Kodama, "Positive realization of difference equations," *IEEE Trans. Circuits Syst.*, vol. CAS-28, pp. 39-47, Jan. 1981.

[3] Y. Ohta, H. Maeda, and S. Kodama, "Reachability, observability and realizability of contiuous-time positive systems," *SIAM J. Contr. Optimization*, vol. 22, no. 2, 1984, pp. 171-180.

[4] J. W. Nieuwenhuis, "About nonnegative realizations," *Syste. Contr. Lett.*, vol. 1, no. 4, 1982, pp. 283-287.

[5] J. M. Van den Hof and J. H. van Schuppen, "Realizations of positive linear systems using polyhedral cones," submitted for publication 1994.

[6] H. Maeda and S Kodama, "Reachability, observability and realizability of linear systems with positive constraints," *Trans. IECE*, vol. 63, pp. 688-694, 1980.

[7] ——, "On realizability of compatmental systems," in *Proc ISCAS*, 1979, pp. 326-329.

[8] E. Picci, "On the internal structure of finite-state stochastic processes," in *Recent Developments in Variable Structure Systems*, R. Mohler and A. Ruberti, Eds., vol. 162 of *Lecture notes in Economics and Math Systems*. New York: Springer Verlag, 1978, pp. 288-304.

[9] S. W. Dharmadhikari, "Sufficient conditions for a stationary process to be a function of a finite Markov chain," *Annals Math. Statistics*, vol. 34, 1963, pp. 1033-1041.

[10] M. Fox and H. Rubin, "Functions of processes with Markovian states," *Annals Math. Statistics*, vol. 39, no. 3, 1968, pp 938-946.

[11] H. Ito, S. Amari, and Kobayashi, "Identifiability of Hidden Markov Information Sources and their minimum degrees of Freedom," *IEEE Trans. Inform. Theory*, vol. 38, pp. 324-333, Feb. 1992.

[12] J. A. Jacquez, *Compartmental Analysis in Biology and Medicine*. New York: Elsevier, 1972

[13] A. Gersho and B B Gopinath, "Charge-routing networks," *IEEE Trans Circuits Syst.*, vol. CAS-26, 1979, pp. 81-92.

[14] D. G. Luenberger, *Introduction to Dynamic Systems*. New York: Wiley, 1979.

[15] S. Rinaldi and L Farina, *Positive Linear Systems: Theory and Applications* (in Italian), Città Studi Edizioni, Milan, 1995.

[16] L. Farina and L. Benvenuti, "Positive realizations of linear systems," *Syst. Contr. Lett.*, to appear.

[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge: Cambridge Univ. Press, 1985.

[18] P. Lancaster and M. Tismenetsky, *The Theory of Matrices. Second Edition with Applications*. San Diego: Academic, 1985

[19] A. Berman and R. J. Plemmons, *Nonegative Matrices in the Mathematical Sciences*. New York: Academic, 1979.

[20] H. Minc, *Nonnegative Matrices*. New York: Wiley, 1988.

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