The Square Root of Linear Time Varying Systems with Applications

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Abstract—**This paper considers the extension of a number of passive multiplier theory based results, previously known only for linear time invariant scalar systems, to linear time varying (LTV) multivariable settings. The extensions obtained here have important applications to the stability of both adaptive systems and linear systems in general. We demonstrate in this paper that at the heart of the extensions carried out here lies the result that if a stable multivariable, linear time varying system is stable under all scalar constant, positive feedback gains, then it has a well defined square root. The existence of this square root is demonstrated through a constructive Newton-Raphson based algorithm. The various extensions provided here though different in form from their linear time invariant scalar counterparts, do recover these as special cases.**

I. Introduction and Problem Motivation

THIS PAPER is concerned with finding time-varying multivariable generalizations of some multiplier theory results involving strictly positive real (SPR) functions.

The following is a well-known result in linear systems theory [1]. Consider an asymptotically stable linear time invariant (LTI), single input single output (SISO) system with a strictly proper transfer function $H(s)$. Then the system in Fig. 1 is asymptotically stable for all

$$0 \leq k \leq 1 \tag{1.1}$$

if, and only if, there exists a SPR scalar operator $Z(s)$, such that

$$Z(s)(1 + H(s)) \tag{1.2}$$

is SPR. The concept of a SPR operator is defined as follows.

Definition 1.1: A real, square matrix transfer function $Z(s)$ is positive real (PR) if:

1) $Z(s)$ is analytic in the right half plane; and
2) for all $Re[s] \geq 0$,

$$Z(s) + Z^H(s) \geq 0$$

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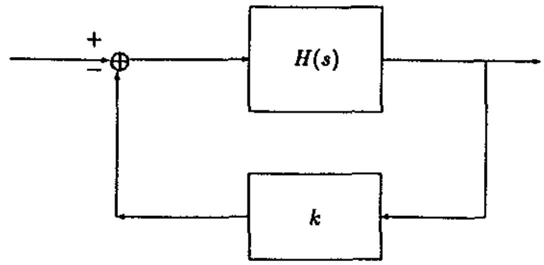

Fig. 1. A closed-loop configuration.

where the superscript H denotes the Hermitian transpose.

We say $Z(s)$ is SPR if for some $\alpha > 0, Z(s - \alpha)$ is PR.

We note that strictly speaking this result as presented in [1] allows for the presence of simple purely imaginary poles in both $H(s)$ and the closed loop above. In such a case all appearances of SPR operators in the statement of the result of [1], must be replaced by PR operators. For the purposes of this paper, however, only the case concerning asymptotic stability is relevant, and imaginary axis poles are precluded.

From this result spring a number of other important results of which two are cited below: The first states that two scalar polynomials of equal degree $p_1(s)$ and $p_2(s)$, have the property that $p_1(s) + kp_2(s)$ is Hurwitz (i.e., has roots in the open left half plane) for all k as in (1.1) iff there exists an asymptotically stable minimum phase $G(s)$, such that $G(s)(p_1(s) + kp_2(s))$ is SPR for all k as in (1.1); in turn, this holds iff there exists an asymptotically stable minimum phase $G(s)$ such that $G(s)p_1(s)$ and $G(s)(p_1(s) + p_2(s))$ are SPR. As will be evident in a later section of this paper, this has an important application in certain adaptive systems problems involving a single unknown parameter. An alternative way of viewing this result is that every convex combination of two monic polynomials with the same degree is Hurwitz iff there exists a *single* stable minimum phase operator whose product with every such convex combination is SPR (see [2] which in fact considers the more general case of convex combinations of more than two polynomials).

The second result concerns the stability of a class of linear time varying (LTV) systems. Specifically, suppose that the configuration in Fig. 1 is stable with *a degree of stability* α for all k as in (1.1). Now consider the LTV systems obtained in Fig. 1, when the feedback gain $k(t)$ is allowed to be time varying while obeying

$$0 < k(t) < 1. \tag{1.3}$$

Then it has been shown in [3] and [4] that the closed loop retains stability whenever, there exist T and $\delta \in (0,\alpha)$ for which

$$\sup_{t\geq 0} \frac{1}{T}\int_t^{t+T}\left[\frac{d}{d\tau}\ln\frac{k(\tau)}{1-k(\tau)}\right]^+ d\tau < 2(\alpha-\delta) \quad (1.4)$$

where

$$[a]^+ = \begin{cases} a; & a\geq 0\\ 0; & a<0.\end{cases}$$

See [5] for an association between the result of [1] and that of [3] and [4], using tools that include the Popov-Kalman-Yakubovic (PKY) Lemma.

The question addressed in this paper is to what extent do these results extend to systems that are LTV or for that matter multiple input multiple output (MIMO) LTI? The ability to answer this question depends critically on the existence of the square root of certain LTV systems. This can be understood by noting that the result of [1] can itself be viewed in the following terms. The stability of the closed loop of Fig. 1 for all $k \in [0,1]$ is equivalent to the transfer function $1 + H(s)$ having a phase function that lies in $(-\pi,\pi)$, [6]. Accordingly, there is a well-defined square root of $1+H(s)$. Call this square root $G(s)$. Clearly this possibly nonrational $G(s)$ has a phase that lies in $(-\pi/2,\pi/2)$ and is consequently SPR, as also is its inverse $G^{-1}(s)$ and indeed $(1+H(s))/G(s)$. Then a rational $Z(s)$ chosen as an arbitrarily good approximant of the inverse of this square root will be SPR, as indeed will be the product in (1.2).

Having dispensed with some preliminaries in Section II, the first question we ask concerns square, LTV, strictly causal continuous operators H (observe this class obviously includes square MIMO, LTI operators) such that both H and $[I+kH]^{-1}$ are stable for all k as in (1.1). Observe, that this corresponds to a stable closed loop of the form of Fig. 1, with the LTV operator H occupying the position of $H(s)$. Then, using a Newton-Raphson technique, we demonstrate in Section III, that in such a case $I + H$ does indeed have a square root. Of course H is presumed to be the operator relating inputs and outputs of a strictly causal system. Stability corresponds to the boundedness of the operator given suitable input and output norms.

Section IV provides the analog of the result of [1]. Specifically it shows that given a finite dimensional L_2 stable $H, [I + kH]^{-1}$ is L_2 stable iff there exist L_2 stable operators X_1 and X_2 such that both $X_1[I + H]X_2$ and X_1X_2 are strictly passive. Section V derives the analog of the first consequence of [1] mentioned in the foregoing. Specifically, it shows that given two polynomial matrices $A_1(s)$ and $A_2(s)$, $\det(A_1(s) + kA_2(s))$ is Hurwitz for all scalar $k \in [0,1]$ iff there exist proper rational matrices $Z_1(s)$ and $Z_2(s)$ such that $Z_1(s)(A_1(s) + kA_2(s))Z_2(s)$ is SPR for all $k \in [0,1]$. This thus resolves an open problem presented in [7]. Section V also discusses the application of this latter result to certain types of adaptive identification algorithms involving MIMO, LTI systems.

Section VI derives the analog of the [3], [4] result. As in [3] and [4] this result assumes that the closed loop in Fig. 1 is stable, with a degree of stability α, for all *constant* scalar feedback $k \in[0, 1]$, but with H, possibly LTV. Under this assumption it shows that when the feedback gain is time varying but obeys the logarithmic bound on the rate of time variation given in (1.4), then there exists a stable, stable invertible operator W, such that with the forward loop replaced by WHW^{-1}, the closed loop retains stability. Here W is determined completely from H. Thus as in [3] and [4], this results explores the connection between stability under constant scalar feedback and stability under scalar *time varying* feedback.

These last three results assume that H is finite dimensional, i.e., has a finite dimensional state variable description. As scalar LTI operators commute, each of the results in Sections IV through VI, though different from their SISO, LTI counterparts, capture these as special cases. Section VII is the conclusion.

II. Preliminaries

In this section we make precise the general framework of this paper by presenting some definitions and assumptions. All systems in this paper will be represented by square, LTV, real, continuous operators mapping L_2 to L_2. Consider such an operator G. Then G^a will denote the *Adjoint* of G, i.e., if G has impulse response $g(t,\tau)$ then G^a has the impulse response $g'(\tau,t)$. For an input signal $x(t), Gx$ will denote the corresponding output, i.e., if $g(t,\tau)$ is the impulse response of G then

$$[Gx](t) = \int_{-\infty}^{\infty} g(t,\tau)x(\tau)\, d\tau. \quad (2.1)$$

This operator is *causal* if $g(t,\tau) = 0\ \forall t<\tau$. In this case the upper limit in the integral of (2.1) can be replaced by t. The *inner product* between two signals $x(t)$ and $y(t)$ will be

$$\langle x,y\rangle = \int_{-\infty}^{\infty} x'(t)y(t)\, dt \quad (2.2)$$

and the *norm* of a signal $x(t)$ will denote the L_2-norm

$$\|x\| = \sqrt{\langle x,x\rangle}. \quad (2.3)$$

The norm of G will be the induced L_2 operator norm

$$\|G\| = \sup_{\|x\|=1}\|Gx\|. \quad (2.4)$$

In the sequel we will use the terms bounded and stable interchangeably to signify operators that have a finite norm. Moreover, the operator G^n for a positive integer n will designate the combined operator obtained by a cascade of n operators G. A bounded operator R: $L_2 \to L_2$ will be called the inverse of G if $GR = RG = I$. In such a case we denote $R = G^{-1}$ and note that the existence of G^{-1} automatically signifies its stability. Further, G will be called symmetric or self-adjoint if

$$G^a = G. \quad (2.5)$$

Every symmetric operator G can in turn be expressed as

$$G = G_c + G_{ac} \quad (2.6)$$

where G_c is causal and called the causal part of $G; G_{ac}$ is anticausal and called the *anticausal part* of G; and together they obey

$$G_c^a = G_{ac}. \tag{2.7}$$

If a term such as αI appears in G then it will be shared equally between G_c and G_{ac}; i.e., each of G_c and G_{ac} will get $0.5\alpha I$.

While the results of Section III rely only on an operator based description of the underlying systems, those of the subsequent sections do require that some of these operators have a State Variable Description. In either case certain stability and existence assumptions are made. The first is on the operator based description of the open loop system H mentioned in the Introduction.

Assumption 2.1: The operator H: $L_2 \to L_2$ is causal and $[I + kH]^{-1}$: $L_2 \to L_2$ exists and is causal for all $k \in [0, 1]$. Further, there exist numbers M_1 and M_2 such that

$$\|[I + kH]^{-1}\| \leq M_1 \quad \forall k \in [0, 1] \tag{2.8}$$

and

$$\|H\| \leq M_2. \tag{2.9}$$

Moreover, the impulse response $h(t, \tau)$ of H is finite for all finite t and τ.

Remark 2.1: The boundedness assumption on $h(t, \tau)$ precludes the presence of impulse functions in $h(t, \tau)$, much like the strict properness assumption in the LTI case. In actual fact all the results derived here remain valid, if one permits the presence of terms like $\delta(t - \tau)$ in $h(t, \tau)$, as long as $[I + kH]^{-1}$ exists and is causal $\forall k \in [0, 1]$. This boundedness assumption is made as it considerably simplifies the notation in Section VI.

To provide the assumption on the state variable realization (SVR) of H we introduce the following notation. For a given continuous square matrix function $A(t)$ we designate

$$A_\alpha(t) = \alpha I + A(t). \tag{2.10}$$

We will be concerned with the notion of degree of stability of operators such as H. To this end we introduce H_α having SVR

$$\{A_\alpha(t), B(t), C(t)\} \tag{2.11}$$

where each of $A_\alpha(t), B(t), C(t)$ is a bounded, continuous function of time. We also make the following definition.

Definition 2.1: The matrix $A(t)$ is exponentially asymptotically stable with degree of stability $\alpha > 0$ (α-eas) if for the LTV system

$$\dot{x}(t) = A(t)x(t) \tag{2.12}$$

$\exists c, \gamma > 0$ such that for all $x(t_0)$ and $t \geq t_0$,

$$\|x(t)\|e^{\alpha(t-t_0)} \leq c\|x(t_0)\|e^{-\gamma(t-t_0)}. \tag{2.13}$$

If $\alpha = 0$, we simply say that $A(t)$ is eas. Further, we will call a system with SVR, $\{A(t), B(t), C(t), D(t)\}$, all matrices bounded and continuous, α-eas (respectively, eas) if $A(t)$ is α-eas (respectively, eas).

Then we have the following assumption.

Assumption 2.2: The system H_α has an SVR

$$\{A_\alpha(t), B(t), C(t)\}$$

such that $[A_\alpha(t), B(t)]$ is uniformly completely controllable (u.c.c.), [8], $[A_\alpha(t), C(t)]$ is uniformly completely observable (u.c.o.), [8], and both the systems H_α and

$$\{A_\alpha(t) - kB(t)C'(t), B(t), -kC(t), I\}$$

are eas, for all $k \in [0, 1]$.

Remark 2.2: It should be noted that any H_α that satisfies Assumption 2.2 also satisfies Assumption 2.1 as $\{A_\alpha(t) - kB(t)C'(t), B(t), -kC(t), I\}$ is precisely the SVR of $[I + kH_\alpha]^{-1}$. Further under the given uco and ucc conditions, BIBO stability, as exemplified by (2.8) and (2.9), is equivalent to eas, [8]

Remark 2.3: Observe that a system having an SVR is necessarily causal. Since in this paper eas as a property has been defined in terms of SVR's, any statement to the effect that a given system is eas, will implicitly indicate the causality of that system.

We now introduce the concept of a strictly positive or passive operator and note that for LTI systems SPR is equivalent to strict positivity.

Defintion 2.2: An operator P: $L_2 \to L_2$ is called strictly positive ($P \geq \epsilon I > 0$) if for all x in L_2

$$\langle x, Px \rangle \geq \epsilon \langle x, x \rangle. \tag{2.14}$$

In Section IV we will need the concept of *Spectrum* of a LTV operator.

Defintion 2.3: The resolvent set $\rho(H)$ of an operator H: $L_2 \to L_2$ is the set of all complex numbers λ such that $[\lambda I - H]^{-1}$: $L_2 \to L_2$ exists. The complement of all $\rho(H)$ in the complex plane is called the spectrum of H and is denoted $\sigma(H)$.

Remark 2.4: The set $\sigma(H)$ is a bounded closed set, [9]. Further, [10] its elements vary continuously with continuous variations in H.

In the sequel, an open set Ω in the complex plane will be called an *open neighborhood* of $\sigma(H)$ if $\sigma(H) \subset \Omega$. The class of functions $\mathcal{F}(H)$ denotes the family of complex valued functions f that are analytic on some open neighborhood of $\sigma(H)$.

III. Existence of the Square Root

The principal contribution of this section is

1) to demonstrate that subject to Assumption 2.1, $I + H$ has a square root and
2) to give an algorithm for constructing this square root.

In the sequel, we say that G: $L_2 \to L_2$ is the square root of $I + F$ with F: $L_2 \to L_2$ if

$$G^2 = I + F. \tag{3.1}$$

To compute the square root we propose a Newton-Raphson algorithm obtained as follows. Suppose the current estimate of the square root, should of course the square root exist, is G_i and that the true square root is $G_i + \Delta G$. Then

$$[G_i + \Delta G]^2 = I + F$$

when neglecting the second order term, and assuming that G_i is invertible and that G_i and ΔG commute, one obtains

$$\Delta G = \tfrac{1}{2}[(I + F)G_i^{-1} - G_i]$$

and

$$G_i + \Delta G = \tfrac{1}{2}[(I + F)G_i^{-1} + G_i].$$

Indeed the Newton-Raphson algorithm we propose computes the successive iterates via

$$G_{i+1} = \tfrac{1}{2}[(I + F)G_i^{-1} + G_i]. \quad (3.2)$$

When initiated with $G_0 = I$, it will be shown that the successive G_i are rational in F, that G_i and ΔG commute, and that under suitable assumptions on F, G_i is invertible for all i.

By equating G_i to G_{i+1} in (3.2) one obtains the following Lemma, which shows that should the iterations in (3.2) converge, they do so to the square root of $I + F$.

Lemma 3.1: Suppose in (3.2)

$$G_i = G_{i+1}. \quad (3.3)$$

Then

$$G_i^2 = I + F. \quad (3.4)$$

In the sequel the convergent point of (3.2), should it exist, with

$$G_0 = G^*$$

will be denoted $NR(I + F, G^*)$.

Unfortunately the global convergence of (3.2) is difficult to demonstrate. On the other hand as we now show, it is possible to determine a number ϵ such that whenever

$$\|F\| \leq \epsilon \quad (3.5)$$

(3.2) converges uniformly whenever it is initiated by $G_0 = I$, i.e., $NR(I + F, I)$ exists. This Theorem is proved in Section III-B.

Theorem 3.1: Let F: $L_2 \rightarrow L_2$ be bounded and causal, $[I + kF]^{-1}$ exist and be causal for all $k \in [0, 1]$ and let $G_i, i = 0, 1, \cdots$, be the sequence of operators defined by (3.2) and $G_0 = 1$. Then, there exists an $\epsilon > 0$ such that whenever (3.5) holds, so do the following for all $k \in [0, 1]$: i) there exists bounded $G(kF) = \lim_{i\rightarrow\infty} G_i(kF)$: $L_2 \rightarrow L_2$; ii) $G(kF)^{-1}$: $L_2 \rightarrow L_2$ exists and both $G(kF)$ and $G(kF)^{-1}$ are causal; iii) $G(kF)$ and $G(kF)^{-1}$ commute with any operator that commutes with F; iv) $G(0) = I$; v) $G(kF)$ varies continuously with k; and vi) $G^2(kF) = I + kF$.

Remark 3.1: The proof of this Theorem shows that the convergence rate is faster than exponential.

To circumvent the apparent difficulty inherent in the restriction (3.5), we will adopt a nested Newton-Raphson strategy for determining the square root of $I + H$. Specifically, we will select suitably small δ_1 and δ_2, so that [see (2.8) and (2.9)]

$$\delta_1 M_2 \leq \min\{\epsilon, 1\} \quad (3.6)$$

and

$$\delta_2 M_1 M_2 \leq \min\{\epsilon, 1\}. \quad (3.7)$$

Further, choose $(1 - \delta_1)/\delta_2$ as an *integer* [N.B. this can always be done without violating (3.6) and (3.7)], and define N

$$N = \frac{1 - \delta_1}{\delta_2}. \quad (3.8)$$

Note

$$\delta_1 + N\delta_2 = 1. \quad (3.9)$$

Then the nested approach first divides the interval [0, 1] into intervals $[0, \delta_1], [\delta_1, \delta_1 + \delta_2], [\delta_1 + \delta_2, \delta_1 + 2\delta_2]$, etc. up to and including $[\delta_1 + (N - 1)\delta_2, 1]$ [see (3.9)]. It then uses (3.2) to compute the square root of $[I + \delta_1 H]$. Because of (2.9), (3.6) and the definition of ϵ, this is possible. It then uses the square root of $[I + \delta_1 H]$ to compute the square root of $[I + (\delta_1 + \delta_2)H]$, etc. until eventually the square root of $[I + H]$ has been obtained. More precisely the nested algorithm proceeds as follows.

1) If $N = 0$, i.e., $\delta_1 = 1$, then the definition of ϵ and (2.9) and (3.6) assure that $NR(I + H, I)$ exists. Thus, from here onwards assume $N > 0$.
2) Find

$$U_0 = NR(I + \delta_1 H, I). \quad (3.10)$$

3) For all $1 \leq m \leq N$, determine, should it exist

$$V_m = NR(I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}, I) \quad (3.11)$$

and

$$U_m = U_{m-1} V_m. \quad (3.12)$$

Remark 3.2: As will become evident in the sequel, for each $m \geq 0, U_m$ represents the square root of $[I + (\delta_1 + m\delta_2)H]$, and V_m the square root of $[I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}]$. Moreover, V_m exists because of (3.7) and (3.6) which together will be shown to force

$$\|\delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}\| \leq \epsilon. \quad (3.13)$$

Notice also, (3.9), that U_N is the square root of $[I + H]$.

Then the following Theorem proved in Section III-C demonstrates the convergence of the nested Newton-Raphson algorithm and hence the existence of the required square root.

Theorem 3.2: Consider the nested Newton-Raphson algorithm, i.e., (3.5)–(3.12), with $NR(I+F, I)$ the convergent point (should it exist) of (3.2) initiated with $G_0 = I$. Suppose Assumption 2.1 holds as do (3.6), (3.7) and (3.9) and that ϵ obeys the restrictions set out in Theorem 3.1. Then the bounded operators U_N: $L_2 \to L_2$ and U_N^{-1}: $L_2 \to L_2$ exist and

$$U_N^2 = I + H.$$

It is clear from (3.10), (3.11), and Theorem 3.1 that each of the operators U_0 and $V_m, m = 1, \cdots, N$ are obtained as the limit point of *uniformly convergent* sequences. Thus, by running each procedure implicit in their determination for a finite but arbitrarily large number of iterations, one can obtain a $\hat{X}$ such that with X the square root of $I+H$, $\|\hat{X} - X\|$ is arbitrarily small. Since the convergence rate is greater than exponential one can expect the number of iterations needed to secure an acceptable tolerance to be small. Also note that not only are X and $\hat{X}$ causal stable, but they also have inverses that are causal stable.

In Section IV we will be concerned with the continuity of the square root $X(kH)$ of $I + kH$ as k varies continuously in [0, 1]. To obtain $X(kH)$ for $k < 1$ one can introduce an obvious modification of the nested Newton-Raphson Algorithm under study. Specifically, define integer μ_1 and real $0 \le \mu_2 < \delta_2$ such that for some integer $0 \le m < N$,

$$k = \delta_1 + m\mu_1 + \mu_2.$$

Select

$$V_{m+1}(\mu_2) = NR(I + \mu_2 U_m^{-1} H U_m^{-1}, I).$$

Then

$$X(kH) = U_m V_{m+1}(\mu_2).$$

Remark 3.3: Indeed, from iv) and v) of Theorem 3.1 one can readily deduce that $X(0) = I$ and $X(kH)$ varies continuously with k.

In Section VI, in our search for the analog of the result of [3], we are concerned with the square roots of systems H having an SVR and comparing them with the square root of H_α. Observe that while in this case X may not have a SVR, its approximant $\hat{X}$ will (see Lemma 3.4). In order to ensure that the SVR of the approximate square root of H is related to that of H_α in a manner that facilitates future analysis we first present the following Lemma.

Lemma 3.2: Suppose for some $\alpha > 0$, H_α has SVR $\{A_\alpha(t), B(t), C(t)\}$, [see (2.10)] and H has SVR $\{A(t), B(t), C(t)\}$. Then

$$\|H_\alpha\| < M_2$$

implies that

$$\|H\| < M_2.$$

Proof: See the Appendix. ■

In a similar vein $\|[I + kH_\alpha]^{-1}\| < M_1$ implies $\|[I + kH]^{-1}\| < M_1$. Thus, define M_i according to $\|[I + kH_\alpha]^{-1}\| < M_1$ and $\|H_\alpha\| < M_2$ and select the parameters δ_i in (3.6, 3.7) with these M_i. Then one can operate successfully the Nested Newton-Raphson iterations for both $I+H$ and $I + H_\alpha$ using this same pair of δ_i. What is more, if H has degree of eas α then so does $\hat{X}$. Moreover, as H_α is eas and one carries the nested Newton-Raphson iterations for the same number of times with respect to both H and H_α, and if $\hat{X}$ so determined for H has SVR $\{A_x(t), B_x(t), C_x(t), I\}$, then the corresponding approximate square root of H_α will have SVR

$$\begin{aligned}\{\alpha I + A_x(t), B_x(t), C_x(t), I\} \\ = \{A_{x\alpha}(t), B_x(t), C_x(t), I\}.\end{aligned} \tag{3.14}$$

This is easy to see for a LTI operator with transfer function $H(s)$ as

$$I + H(s) = X^2(s)$$

implies that

$$I + H(s - \alpha) = X^2(s - \alpha).$$

For LTV H this can be deduced from the definition of the Nested Newton-Raphson algorithm, Lemma 3.4, the rules for combining state variable realizations and the fact that any δ_i, N that can be used with H also apply to H_α. Thus, one can legitimately, associate with H_α, the corresponding approximate square root $\hat{X}_\alpha$. Note also that the SVR in question can be chosen to be uco and ucc (see Assumption 2.2). From here onwards to simplify the notation we will refer to $\hat{X}_\alpha$ by X_α.

Observe also that the SVR corresponding to the inverse of this approximate square root, i.e.

$$\{\alpha I + A_x(t) - B_x(t)C_x(t), B_x(t), -C_x(t), I\}$$

will be eas.

The rest of this section is divided as follows. In Section III-A we will provide certain general relations concerning the iterations in (3.2). Sections III-B and Section III-C will, respectively, prove Theorems 3.1 and 3.2.

A. Some Properties of the Newton-Raphson Algorithm

In this subsection we will derive some important expressions for the G_i in (3.2) generated by

$$G_0 = I. \tag{3.15}$$

In doing so, on occasions we will write $G_i(F)$ instead of G_i, while sometimes this argument will be dropped. The first result is a trivial consequence of the fact that for a linear operator F and scalar β_i and any integer n_i (these can be negative)

$$\begin{aligned}(\beta_1 I + \beta_2 F)^{n_1}(\beta_3 I + \beta_4 F)^{n_2} \\ = (\beta_3 I + \beta_4 F)^{n_2}(\beta_1 I + \beta_2 F)^{n_1}.\end{aligned} \tag{3.16}$$

Lemma 3.3: Suppose F: $L_2 \to L_2$ is a linear operator, and P(F) is a polynomial in F, i.e., for suitable scalar constants p_i and integer n

$$P(F) = \sum_{i=0}^{n} p_i F^i.$$

Suppose that with scalar β_i

$$P(x) = \prod_{1=1}^{n} (1 + \beta_i x).$$

Then

$$P(F) = \prod_{1=1}^{n} (I + \beta_i F).$$

This brings us to the following expression for the $G_i(F)$.

Lemma 3.4: Let F: $L_2 \to L_2$ be bounded and causal and suppose that for all $k \in [0,1]$ $[I + kF]^{-1}$ exists and is causal. Suppose also that G_i is defined for $i = 0, 1, \cdots$ by $G_0 = I$ and (3.2). Then bounded $G_m(F)$: $L_2 \to L_2$ and $G_m^{-1}(F)$: $L_2 \to L_2$ exist, and are causal. Further, with

$$\hat{N}(m) = 2^{m-1} - 1$$

for every integer $m > 0$, there exist real scalar $\beta_i^{(m)}, i = 0, 1, \cdots, \hat{N}(m)$ and $\alpha_i^{(m)}, i = 1, \cdots \hat{N}(m)$ obeying

$$0 < \beta_0^{(m)} < \alpha_1^{(m)} < \beta_1^{(m)} < \alpha_2^{(m)} < \cdots < \alpha_{\hat{N}(m)}^{(m)} < \beta_{\hat{N}(m)}^{(m)} < 1 \tag{3.17}$$

such that

$$G_m(F) = (I + \beta_0^{(m)} F) \prod_{i=1}^{\hat{N}(m)} [(I + \beta_i^{(m)} F)(I + \alpha_i^{(m)} F)^{-1}] \tag{3.18}$$

$\forall\ 0 \le n \le \hat{N}(m)$

$$\alpha_{2n+1}^{(m+1)} = \beta_n^{(m)} \tag{3.19}$$

and $\forall\ 0 < n \le \hat{N}(m)$

$$\alpha_{2n}^{(m+1)} = \alpha_n^{(m)}. \tag{3.20}$$

Proof: See the Appendix. ■

Remark 3.4: Observe that for all $m > 0$, under the hypothesis of Lemma 3.4 G_i, G_j, and their inverses commute with each other and with any operator that commutes with F.

The next Lemma demonstrates a further relation between the constants appearing in (3.17).

Lemma 3.5: Under the hypothesis of Lemma 3.4, $\forall m \ge 1$,

$$\beta_0^{(m)} + \sum_{i=1}^{\hat{N}(m)} [\beta_i^{(m)} - \alpha_i^{(m)}] = \tfrac{1}{2}. \tag{3.21}$$

Proof: See the Appendix. ■

Define now

$$\Delta_i = G_i - G_{i-1}. \tag{3.22}$$

Then the following Lemma holds.

Lemma 3.6: Under the hypothesis of Lemma 3.4 and with Δ_i as in (3.22), we have that

$$I + F - G_{i+1}^2 = -\Delta_{i+1}^2 \tag{3.23}$$

and

$$\Delta_{i+2} = -\tfrac{1}{2} G_{i+1}^{-1} \Delta_{i+1}^2. \tag{3.24}$$

Proof: In view of Remark 3.4 and (3.2)

$$\begin{aligned} I + F - G_{i+1}^2 &= I + F - \tfrac{1}{4}[(I+F)G_i^{-1} + G_i]^2 && (3.25)\\ &= I + F - \tfrac{1}{4} G_i^{-1}[(I+F) \\ &\quad + G_i^2] G_i^{-1}[(I+F) + G_i^2] && (3.26)\\ &= -\tfrac{1}{4}[(I+F)G_i^{-1} - G_i]^2. && (3.27) \end{aligned}$$

Also from (3.2)

$$\Delta_{i+1} = \tfrac{1}{2}[(I+F)G_i^{-1} - G_i].$$

Thus, (3.23) follows. Further

$$\begin{aligned} \Delta_{i+2} &= \tfrac{1}{2}[(I+F)G_{i+1}^{-1} - G_{i+1}] && (3.28)\\ &= \tfrac{1}{2} G_{i+1}^{-1}[(I+F) - G_{i+1}^2] && (3.29)\\ &= -\tfrac{1}{2} G_{i+1}^{-1} \Delta_{i+1}^2. && (3.30) \end{aligned}$$

■

Observe that the repeated application of (3.24) helps establish the following expression. For all $n > 1$

$$\Delta_n = -(\tfrac{1}{2})^{2^{n-1}-1} \Delta_1^{2^{n-1}} \prod_{j=1}^{n-1} (G_{n-j})^{-2^{j-1}}. \tag{3.31}$$

B. Proof of Convergence Under Small Perturbation

In this section we prove Theorem 3.1.

Notice, (3.5) implies that $\|kF\| < \epsilon$ for all $k \in [0,1]$. Thus for the purposes of proving i)–iii) and vi), it suffices to work with $k = 1$. According to [10], if one chooses $\epsilon < 1$, then from (3.5), for all $k \in [0,1]$

$$\|[I + kF]^{-1}\| \le \frac{1}{1 - \epsilon k}.$$

Thus, for this choice of ϵ the hypothesis of Lemmas 3.4 and 3.5 hold. Hence, with

$$\delta_i^{(m)} = \alpha_i^{(m)} - \beta_{i-1}^{(m)}$$

from (3.21) we have that

$$\begin{aligned} \sum_{i=1}^{\hat{N}(m)} \delta_i^{(m)} &= \beta_{\hat{N}(m)} - \tfrac{1}{2} && (3.32)\\ &< \tfrac{1}{2}. && (3.33) \end{aligned}$$

From (3.18), we find that

$$G_m^{-1} = (I + \beta_{\hat{N}(m)}^{(m)} F)^{-1} \cdot \prod_{i=1}^{\hat{N}(m)} [I + \delta_i^{(m)} F(I + \beta_{i-1}^{(m)} F)^{-1}]. \tag{3.34}$$

and

$$G_m = (I + \beta_{\hat{N}(m)}^{(m)} F) \prod_{i=1}^{\hat{N}(m)} [I - \delta_i^{(m)} F(I + \alpha_{i-1}^{(m)} F)^{-1}]. \quad (3.35)$$

Hence, with $\epsilon < 1$, from (3.17)

$$\|FG_m^{-1}\| \leq \frac{\epsilon}{1-\epsilon} \prod_{i=1}^{\hat{N}(m)} \left[1 + \frac{\delta_i^{(m)}\epsilon}{1-\epsilon}\right] \quad (3.36)$$

$$\leq \frac{\epsilon}{1-\epsilon} e^{\epsilon/2(1-\epsilon)} \quad (3.37)$$

$$= \epsilon_1 \quad (3.38)$$

where the last inequality follows from (3.32) and [11]. Similarly, using (3.34)

$$\|G_m^{-1}\| \leq \frac{1}{1-\epsilon} e^{\epsilon/2(1-\epsilon)}$$

and using (3.35)

$$\|G_m\| \leq (1+\epsilon) e^{\epsilon/2(1-\epsilon)}.$$

Thus, proving i) will automatically also prove ii). Further, from Lemma 3.3, iii) will also follow. Now for small enough ϵ,

$$\epsilon_1 < 4.$$

Also observe, that because of (3.15) and (3.2)

$$\Delta_1 = \tfrac{1}{2} F.$$

Thus, from (3.31), and the fact that

$$\sum_{j=1}^{n-1} 2^{j-1} = 2^{n-1} - 1$$

we have that

$$\Delta_n = -\tfrac{1}{2} F \prod_{j=1}^{n-1} (\tfrac{1}{4} F G_{n-j}^{-1})^{2^{j-1}}.$$

Hence, from the definition of ϵ_1, one obtains

$$\|\Delta_m\| \leq \frac{\epsilon}{2}\left(\frac{\epsilon_1}{4}\right)^{2^m - 1} \quad (3.39)$$

when, for every $\epsilon_2 > 0$ there exists m, such that for all $m > n, \|G_n - G_m\| < \epsilon_2$. Hence i) follows as does vi) from Lemma 3.1. The fact that iv) is true is a trivial consequence of (3.18). Finally v) follows by examining (3.18) and employing techniques similar to those above, which yield the existence of $f(k_1, k_2)$, independent of m and having the property that $\lim_{k_1 \to k_2} f(k_1, k_2) = 0$, such that for all $k_1, k_2 \in [0, 1]$

$$\|G_m(k_1 F) - G_m(k_2 F)\| \leq f(k_1, k_2).$$

C Proof of Convergence of the Nested Algorithm

In this section we prove Theorem 3.2.

We will prove the Theorem using an inductive argument. To this end the following Lemma helps sustain the induction.

Lemma 3.7: With the conditions of Theorem 3.2 in force suppose that for some $0 < m \leq N$ the following hold.

1) The bounded operators U_{m-1} and its inverse exist; and both commute with any operator that commutes with H.

2)

$$U_{m-1}^2 = [I + (\delta_1 + (m-1)\delta_2)H]. \quad (3.40)$$

Then the bounded operators U_m and its inverse exist; both commute with any operator that commutes with H; and

$$U_m^2 = [I + (\delta_1 + m\delta_2)H]. \quad (3.41)$$

Proof: First observe that because of item 1 of the Lemma statement

$$\begin{aligned}
&\|\delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}\| \\
&\quad = \|\delta_2 H U_{m-1}^{-2}\| \\
&\quad = \|\delta_2 H [I + (\delta_1 + (m-1)\delta_2)H]^{-1}\| \\
&\quad \leq \delta_2 M_1 M_2 \\
&\quad \leq \epsilon.
\end{aligned} \quad (3.42)$$

where the last two inequalities arise consequent to (2.8), (2.9), (3.7), (3.6), (3.9). Thus, from Theorem 3.1 and (3.11), V_m and its inverse exist and commute with any operator that commutes with $U_{m-1}^{-1} H U_{m-1}^{-1}$. Then because of the commutativity and invertibility hypothesized in item 1 of the Lemma statement, $U_{m-1}^{-1} H U_{m-1}^{-1}$ commutes with any operator that commutes with H. Hence, V_m and its inverse also commute with any operator that commutes with H. Thus, from (3.12) U_m^{-1} exists and together with U_m satisfies the required commutativity property.

Because of (3.11)

$$V_m^2 = I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}. \quad (3.43)$$

Further, the commutativity property established on V_m assures that V_m commutes with both U_{m-1} and H.

Thus, from (3.12), (3.40), and (3.43) we obtain

$$\begin{aligned}
&[I + (\delta_1 + m\delta_2)H] \\
&\quad = [I + (\delta_1 + (m-1)\delta_2)H] + \delta_2 H \\
&\quad = U_{m-1}[I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}] U_{m-1} \\
&\quad = U_{m-1} V_m^2 U_{m-1} \\
&\quad = (U_{m-1} V_m)^2 \\
&\quad = U_m^2.
\end{aligned} \quad (3.44)$$

This completes the proof. ∎

Theorem 3.2 then follows readily from Lemma 3.7, and the fact that (2.9) and (3.6) ensure that U_0 in (3.10), together with its inverse, exists and commutes with any operator that commutes with H.

IV. Existence of Passive Multipliers

Having demonstrated the existence of the square root of $I + H$, we now generalize the result of [1] and its first implication discussed in the Introduction. Instead of focussing on PR type properties, we will consider SPR type (or strict passivity)

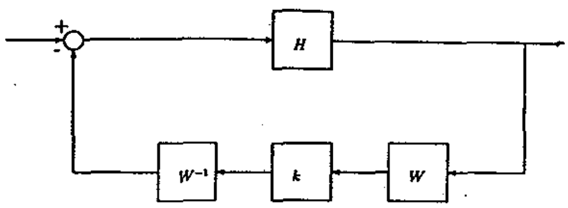

Fig. 2. A closed-loop under scalar time invariant feedback.

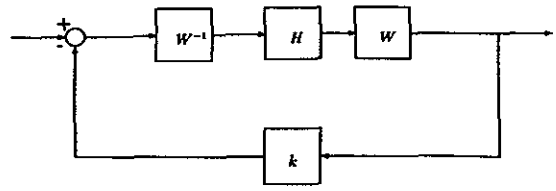

Fig. 3. Closed-loop with stability equivalent to the stability of Fig. 1.

properties. This is simply a matter of minor technicality in an attempt to avoid having to deal with singular situations.

In the spirit of [1] the principal result we derive takes the following form: Under Assumption 2.2 there exist operators, $X_{1\alpha}$ and $X_{2\alpha}$, both eas and having eas inverses, for which:

1) $X_{1\alpha}X_{2\alpha}$ is strictly positive.
2) $X_{1\alpha}[I+H_\alpha]X_{2\alpha}$ is strictly positive.

More precisely, we have the following Theorem that captures one direction of the result of [1].

Theorem 4.1: Under Assumption 2.2, there exist eas and eas invertible operators X_α^{-1} and W such that

$$[W(I+H_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I+H_\alpha)X_\alpha^{-1}W^{-1} > 0. \quad (4.1)$$

and

$$[WX_\alpha^{-1}W^{-1}]^a + [WX_\alpha^{-1}W^{-1}] > 0. \quad (4.2)$$

Further X_α has SVR of the form (3.14), with all matrices in (3.14) continuous, and W has an SVR of the form

$$\{\alpha I + A_w(t), B_w(t), C_w(t), I\} \quad (4.3)$$

again with all matrices continuous.

Proof: See Section IV-A.

Remark 4.1: As an alternative to the interpretation provided at the beginning of this section, this theorem also says that there is a causal operator $WX_\alpha^{-1}W^{-1}$ that is strictly positive (i.e., (4.2) holds) and such that the product of this operator with $W(I+H_\alpha)W^{-1}$ is also positive (i.e., (4.1) holds). This follows by noting that W is stable invertible and that

$$[W(I+H_\alpha)W^{-1}][WX_\alpha^{-1}W^{-1}] = W(I+H_\alpha)X_\alpha^{-1}W^{-1}.$$

Notice that Assumption 2.1 holds with H replaced by WHW^{-1}. In particular the fact that k is a scalar constant ensures that k commutes with both W and W^{-1}. Consequently, the stability and stable invertibility of W ensures that the stability of the closed loop in Fig. 1 is equivalent to that in Fig. 2. This is in turn equivalent to the stability of the closed loop in Fig. 3. So the difference with the original time-invariant scalar result lies in the introduction of W. Because in the time-invariant scalar case the various operators commute W drops out of the picture. This difference reappears in the next section when we generalize the result of [3].

Before discussing the second direction of the [1] result we turn now to the following Corollary.

Corollary 4.1: Under Assumption 2.2, there exist eas and eas invertible operators X_α^{-1} and W, with SVR of the form above, such that for all $k \in [0,1]$

$$[W(I+kH_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I+kH_\alpha)X_\alpha^{-1}W^{-1} > 0.$$

Proof: Follows from the fact that the above equation holds for $k=0$ and $k=1$ and the fact that Positivity is a convex property.

Observe eas Strictly Positive operators have an inverse that is eas (a fact easily proved from the PKY Lemma). Thus as long as H_α is eas and one can find eas and eas invertible operators W, X_α such that (4.1) and (4.2) hold, then the operator $(I+kH_\alpha)^{-1}$ must be eas for all $k \in [0,1]$. Thus the analog of the reverse direction of the [1] result also holds.

A. Proof of Theorem 4.1

The proof we give is constructive, in that it provides algorithms for determining both X_α^{-1} and W. In view of the results of Section III, the starting point of our development here will be that there exists an X which is causal stable and has a causal stable inverse such that

$$X^2 = I + H_\alpha. \quad (4.4)$$

Hence

$$X = [I+H_\alpha]X^{-1} = X^{-1}[I+H_\alpha]. \quad (4.5)$$

We need to concern ourselves with the Spectrum of $[I+H_\alpha]$, in particular the fact that it avoids the negative real axis.

Lemma 4.1: Under Assumption 2.2

$$\sigma(I+H_\alpha) \cap (-\infty, 0] = \{\phi\}.$$

Proof: Suppose the result were false. Then for some $\lambda \in (-\infty, 0]$

$$(\lambda - 1)I - H_\alpha$$

is not invertible. As for such a $\lambda, 1-\lambda > 1$, choosing

$$\mu = \frac{1}{1-\lambda}$$

$\exists \mu \in (0,1]$ such that

$$I - \mu H_\alpha$$

is not invertible. This violates Assumption 2.1 and hence, (see Remark 2.2), Assumption 2.2. ∎

This in turn has implication for the spectrum of X, specifically that it is in the open right half plane.

Lemma 4.2: Suppose H_α satisfies Assumption 2.2 and X is obtained as the convergent point of the Nested Newton-Raphson Algorithm of Section III. Then, there exists an $\epsilon > 0$ such that for every $\lambda \in \sigma(X)$

$$\Re(\lambda) > \epsilon. \quad (4.6)$$

where $\Re$ denotes the real part.

Proof: According to [9] if $f \in \mathcal{F}(X)$ (see the end of Section II) then

$$\sigma(f(X)) = \{f(\lambda) | \lambda \in \sigma(X)\}.$$

Observe, from Lemma 4.1 and the fact that $\sigma(X^2)$ is a bounded closed set that $\sigma(X^2)$ avoids an open neighborhood of the negative real axis. At this point for convenience we will use the notation $X(kH_\alpha), k \in [0,1]$ to denote the square root of $I + kH_\alpha$ as provided by the Nested Newton Raphson iterations of the previous section. As the square root function is analytic in the complement of the negative real axis, the spectrum of $X(kH_\alpha)$ (see Remark 3.3 and the discussion preceding it) cannot intersect a neighborhood of the imaginary axis for all $k \in [0,1]$. Now suppose it did intersect the left half plane. Consider $X(kH_\alpha)$ as k varies from 0 to 1. Clearly, from Remark 3.3, at $k = 0$ its spectral set comprises the single point 1. Also from Remarks 3.3 and 2.4, the spectrum of $X(kH_\alpha)$ varies continuously with k. Then for the spectrum of $X(H_\alpha)$ to intersect the left half plane for some $k \in (0,1)$, the spectrum of $X(kH_\alpha)$ must intersect the imaginary axis. Hence the result follows from contradiction. ∎

It is easy to see that X^{-1} also obeys (4.6). In light of the discussion in Section III, and the closed and bounded nature of $\sigma(X)$ it follows that an X_α obtained by utilizing a sufficient number of iterations in the underlying nested Newton-Raphson algorithm, will have the property that both $[I + H_\alpha]X_\alpha^{-1}$ and X_α have spectra confined to the open right half plane ORHP). In other words the spectral confinement property is essentially *robust*. Of course X_α is stable and stable invertible and has SVR as in (3.14).

Unlike SISO, LTI operators, a general linear, even MIMO, LTI operator, need not be strictly positive even if its spectra lie entirely in ORHP. Herein lies the need for finding a combination of left and right multipliers. To convert this spectral confinement to a strict positivity requirement we note, [9], that the open right half plane spectral confinement of the two operators mentioned in the foregoing suffices for the existence of a *symmetric* operator $P = P^a$, such that

$$X^a P + PX > 0 \tag{4.7}$$

and, in view of (4.5)

$$[(I + H_\alpha)X^{-1}]^a P + P[(I + H_\alpha)X^{-1}] > 0. \tag{4.8}$$

Observe also that as X is causal, stable invertible, by post and premultiplying (4.7) by X^{-1} and $[X^a]^{-1}$, respectively, we obtain

$$[X^a]^{-1}P + PX^{-1} > 0 \tag{4.9}$$

in other words both $P[(I + H_\alpha)X^{-1}]$ and PX^{-1} are strictly positive.

Further, due to the robustness of the strict positivity property, we have that for some X_α causal, stable invertible, and with SVR as in (3.14)

$$[(I + H_\alpha)X_\alpha^{-1}]^a P + P[(I + H_\alpha)X_\alpha^{-1}] > 0 \tag{4.10}$$

and

$$[X_\alpha^a]^{-1}P + PX_\alpha^{-1} > 0. \tag{4.11}$$

Here X_α is obtained by carrying each stage of the nested Newton-Raphson algorithm through a sufficient number of iterations.

A difficulty with the above is that P, being self adjoint, is noncausal. To circumvent this difficulty we first provide an algorithm for computing P through an obvious analog of the Cayley transform.

Lemma 4.3: Adopt the hypothesis of Lemma 4.2, and let X_α be an approximation to the operator X such that (4.8, 4.9) hold for a symmetric P. Define

$$\Gamma_\alpha = [(I + H_\alpha)X_\alpha^{-1}] \tag{4.12}$$

Then $\Gamma_\alpha + I$ has an inverse (by defintion bounded) and with

$$\Omega_\alpha = [\Gamma_\alpha - I](\Gamma_\alpha + I)^{-1} \tag{4.13}$$

there holds for all $\lambda \in \sigma(\Omega_\alpha)$

$$|\lambda| < 1. \tag{4.14}$$

Proof: Because of (4.8) Γ_α has a spectrum in the open right half plane. Hence $\Gamma_\alpha + I$ has an inverse. Then (4.14) follows from the fact that $\lambda \in \sigma(\Gamma_\alpha)$ implies $\Re(\lambda) > \epsilon$ and standard arguments surrounding the Cayley transform.

We now use this transformation to derive a constructive procedure for obtaining P.

Lemma 4.4: With Ω_α as above, P is given by the following uniformly convergent series

$$P = \sum_{i=0}^{\infty} [\Omega_\alpha^a]^i \Omega_\alpha^i.$$

Further for every $n \geq 1$, the causal part of P_n below can be realized by some eas SVR of the form $\{\alpha I + A_{pn}(t), B_{pn}(t), C_{pn}(t), D_{pn}\}$ with all matrices continuous

$$P_n = \sum_{i=0}^{n} [\Omega_\alpha^a]^i \Omega_\alpha^i. \tag{4.15}$$

Proof: Observe that in light of the fact that H_α and X_α have continuous state variable realizations, so also, because of (4.12) and (4.13), must for all finite i, Ω_α^i. Thus, [10] Ω_α^i can be represented by a kernel of finite rank, i.e., its spectrum comprises a finite number of points. Thus again from [10] one can show that the fact that the spectrum of Ω_α is in the unit disc, implies that there exists $0 < \lambda < 1$ such that for all finite $i, \|[\Omega_\alpha^a]^i \Omega_\alpha^i\| < \lambda^i$. Thus the series above is uniformly convergent. The fact that the causal part of P_n has an SVR of the form postulated follows from (4.12) and (4.13) and the fact that there are only a finite number of terms in the defining summation (4.15) and the SVR's for H_α and X_α^{-1}.

Now observe that with Ω_α as in (4.13)

$$P\Gamma_\alpha + \Gamma_\alpha^a P > 0 \tag{4.16}$$
$$\Leftrightarrow -P\Gamma_\alpha - \Gamma_\alpha^a P < 0 \tag{4.17}$$
$$\Leftrightarrow [\Gamma_\alpha^a - I]P[\Gamma_\alpha - I] - [\Gamma_\alpha^a + I]P[\Gamma_\alpha + I] < 0 \tag{4.18}$$
$$\Leftrightarrow [\Gamma_\alpha^a + I]^{-1}[\Gamma_\alpha^a - I]P[\Gamma_\alpha - I][\Gamma_\alpha + I]^{-1} - P < 0 \tag{4.19}$$
$$\Leftrightarrow \Omega_\alpha^a P \Omega_\alpha - P < 0. \tag{4.20}$$

Thus, to solve for (4.10), it suffices to solve the equation

$$\Omega_\alpha^a P \Omega_\alpha - P = -I \tag{4.21}$$

which in turn can be shown to have the postulated solution by direct substitution. ∎

Observe also that the uniform convergence of the power series realizing P and the robustness of the strict positivity property together assure that for sufficiently large n, P_n obeys both

$$[(I+H_\alpha)X_\alpha^{-1}]^a P_n + P_n[(I+H_\alpha)X_\alpha^{-1}] > 0 \tag{4.22}$$

and

$$[X_\alpha^a]^{-1}P_n + P_n X_\alpha^{-1} > 0. \tag{4.23}$$

Of course P_n is also symmetric. It follows from [13] that P_n has a spectral factorization of the form

$$P_n = W^a W \tag{4.24}$$

and that W can be chosen to have the same A and C matrices as the causal part of P_n. Thus in fact W is eas and has a SVR of the form (4.3). with all matrices continuous, [13]. Further, from [14], W^{-1} can be chosen to be eas as well (see [15], for a Newton-Raphson based algorithm for computing W). Then the proof of Theorem 4.1 follows from (4.24) and the multiplication of (4.22) and (4.23) by $[W^a]^{-1}$ from the left and W^{-1} from the right.

V. Solution to a Problem Posed in [7]

Motivated by adaptive systems problems, [7] had posed the following question: Suppose the following set of square Matrix Polynomials:

$$\{A_1(s) + kA_2(s) | k \in [0,1]\} \tag{5.1}$$

has all its members Hurwitz (i.e., the determinant is Hurwitz). Does there exist a single LTI operator $Z(s)$ such that all members of the set

$$\{[A_1(s) + kA_2(s)]Z(s) | k \in [0,1]\}$$

are SPR. The next Theorem shows that such construction of SPR products is possible provided one allows multiplication from both sides.

Theorem 5.1: Suppose $A_1(s)$ and $A_2(s)$ are two square matrix polynomials, and $A_1^{-1}A_2$ is strictly proper. Then all members of the set (5.1) are Hurwitz iff there exist, square, stable minimum phase matrix transfer functions $Z_1(s)$ and $Z_2(s)$ with the former strictly proper and the latter biproper, such that

$$\{Z_1(s)[A_1(s) + kA_2(s)]Z_2(s) | k \in [0,1]\}$$

are biproper and SPR.

Proof: For sufficiency, note i) that the inverse of an SPR matrix is SPR, and thus stable and ii) the stability of Z_i prevents unstable zeros of $A_1 + kA_2$ from being cancelled in forming the product $Z_1[A_1 + kA_2]Z_2$. For necessity, note that the operator corresponding to the biproper transfer function

$$I + k[A_1(s)]^{-1}A_2(s) \tag{5.2}$$

satisfies Assumption 2.2. Thus, from Corollary 4.1, there exist biproper (as can be seen from their SVR), operators $W(s)$, and $X(s)$, each eas and having eas inverse, such that

$$W(s)[I + k[A_1(s)]^{-1}A_2(s)]X^{-1}(s)W^{-1}(s)$$

is SPR for all $k \in [0,1]$. Then, choosing $Z_1(s) = W(s)[A_1(s)]^{-1}$ and $Z_2(s) = X(s)W^{-1}(s)$, the result follows.

The main application of this result is in output error adaptive identification [16]. Consider the identification of the proper MIMO plant

$$[A_1(s) + kA_2(s)]Y(s) = [B_1(s) + kB_2(s)]U(s) \tag{5.3}$$

with k a scalar unknown parameter and $u(t)$ and $y(t)$, the input and output of the plant. To identify the plant generally, one performs state variable filtering to avoid explicit differentiation of the various signals. This requires rewriting of the model as

$$Z_1(s)[A_1(s) + kA_2(s)]Y(s) = Z_1(s)[B_1(s) + kB_2(s)]U(s) \tag{5.4}$$

such that $Z_1(s)[A_1(s) + kA_2(s)]$ is biproper. Then, for exponential convergence of the underlying identification algorithm, one requires that $Z_1(s)[A_1(s) + kA_2(s)]$ be SPR. This can be seen readily from the result of [17] which treats the SISO case. As k is unknown the underlying SPR condition is difficult to ensure. However, suppose *a priori* bounds are available for k. In fact without sacrificing generality, assume that $k \in [0,1]$. Then as long as $[A_1(s) + kA_2(s)]$ is Hurwitz for all $k \in [0,1]$, one can choose square, stable, minimum phase matrix transfer functions $Z_1(s)$ and $Z_2(s)$ such that the requirements of Theorem 5.1 are satisfied. Then, noting that $Z_2(s)$ is biproper, one can rexpress the plant as

$$Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)\overline{Y}(s) = Z_1(s)[B_1(s) + kB_2(s)]U(s)$$

where

$$\overline{Y}(s) = Z_2^{-1}(s)Y(s)$$

acts as the converted output. Observe it can be constructed from $Y(s)$ without any explicit differentiation. Further, as $Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)$ is SPR, the output error identification algorithm for this redefined system will be exponentially convergent.

VI. Generalization of the Freedman Zames Result

In this section we generalize the second consequence of the result of [1] namely that of [3]. To this end the principal result to be derived is as follows.

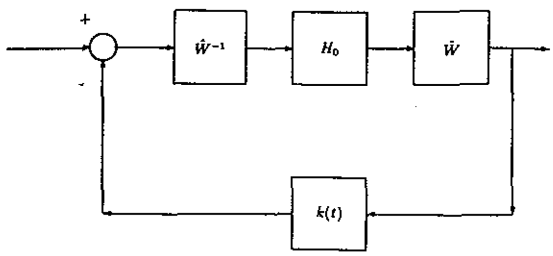

Fig. 4. Closed-loop under time varying feedback.

Theorem 6.1: Suppose Assumption 2.2 holds. Then there exists an eas and eas invertible operator $\hat{W}$ (independent of k) such that, the operator $[I + k(t)\hat{W}H_0\hat{W}^{-1}]^{-1}$ is eas provided (1.3) and (1.4) hold. Here $H_0 = H_\alpha$, with $\alpha = 0$.

Before proving this result we first discuss its implication. Essentially, it states that provided the closed loop of Fig. 1 (with $H(s)$ replaced by H_0) is α-eas for all time invariant feedback gains in the open interval [0,1], then under a logarithic time variation bound as in [3], by suitable pre and post filtering of H_0, the closed loop in Fig. 4 is also stable. Observe, if $H(s)$ is scalar LTI, then the underlying commutativity recovers the result of [3]. Moreover, the fact that the pre and post filters $\hat{W}^{-1}$ and $\hat{W}$ are independent of the particular trajectory that the time-varying feedback gain follows simplifies their selection.

In the remainder of this section we will prove this result. We first appeal to the time-varying version of the Positive Real Lemma, namely that an eas LTV system with SVR

$$\{\overline{A}(t), \overline{B}(t), \overline{C}(t), I\}$$

is strictly positive iff there exists a uniformly positive definite symmetric matrix $\Pi(t)$ such that [18]

$$\begin{bmatrix} -\dot{\Pi}(t) - (\overline{A}'(t)\Pi(t) + \Pi(t)\overline{A}(t)) & (\Pi(t)\overline{B}(t) - \overline{C}(t)) \\ (\Pi(t)\overline{B}(t) - \overline{C}(t))' & 2I \end{bmatrix} \tag{6.1}$$

is uniformly positive definite. Notice, as long as $\overline{A}(t)$ is eas, no restriction is placed on the controllability and or observability of the strictly passive system (see Lemma 7 of [18]).

Now let W and X_α be the operators whose existence has been referred to in Corollary 4.1 and consider the eas LTV system:

$$W(I + kH_\alpha)X_\alpha^{-1}W^{-1} = WX_\alpha^{-1}W^{-1} + kWH_\alpha X_\alpha^{-1}W^{-1} \tag{6.2}$$

the commutation being made possible by the fact that k is a scalar constant. Under Assumption 2.2, in view of Corollary 4.1, this system is strictly positive for all $k \in [0,1]$. With W having SVR $[A_w(t), B_w(t), C_w(t), I]$ and X_α having SVR $[A_x(t), B_x(t), C_x(t), I]$, using elementary rules of combining SVR's (see, e.g., [19]), the system of (6.2) can be verified as having the SVR $\{\overline{A}_\alpha(t), \overline{B}(t), \overline{C}(k,t), I\}$ where $\overline{A}_\alpha(t)$ is given by (6.3) shown at the bottom of the page

$$\overline{B}(t) = [B_w'(t), B_x'(t), B'(t), 0, B_w'(t)]' \tag{6.4}$$

and

$$\overline{C}(k,t) = [-C_w(t), -C_x(t), kC(t), kC_w(t), C_w(t)]. \tag{6.5}$$

Now since this system is strictly positive for $k = 0$ and $k = 1$, we have that there exist uniformly positive definite symmetric matrices $\hat{\Pi}_k(t), k = 0$ and $k = 1$ such that, [18], (see (6.6) at the bottom of the page). Using the convexity of the positive definiteness property we then have that with

$$\Pi_0(t) = \hat{\Pi}_0(t)$$

and

$$\Pi_1(t) = \hat{\Pi}_1(t) - \hat{\Pi}_0(t)$$

there exists a

$$\Pi(k,t) = \Pi_0(t) + k\Pi_1(t) \tag{6.7}$$

$$\begin{bmatrix} \alpha I + A_w(t) - B_w(t)C_w(t) & 0 & 0 & 0 & 0 \\ -B_x(t)C_w(t) & \alpha I + A_x(t) - B_x(t)C_x(t) & 0 & 0 & 0 \\ -B(t)C_w(t) & -BC_x(t) & \alpha I + A(t) & 0 & 0 \\ 0 & 0 & B_w(t)C(t) & \alpha I + A_w(t) & 0 \\ -B_w(t)C_w(t) & -B_w(t)C_x(t) & 0 & 0 & \alpha I + A_w(t) \end{bmatrix} \tag{6.3}$$

$$\begin{bmatrix} -\dot{\hat{\Pi}}_k(t) - (\overline{A}_\alpha'(t)\hat{\Pi}_k(t) + \hat{\Pi}_k(t)\overline{A}_\alpha(t)) & (\hat{\Pi}_k(t)\overline{B}(t) - \overline{C}'(k,t)) \\ (\hat{\Pi}_k(t)\overline{B}(t) - \overline{C}'(k,t))' & 2I \end{bmatrix} > 0. \tag{6.6}$$

$$\begin{bmatrix} -\dot{\Pi}(k,t) - (\overline{A}_\alpha'(t)\Pi(k,t) + \Pi(k,t)\overline{A}_\alpha(t)), & (\Pi(k,t)\overline{B}(t) - \overline{C}'(k,t)) \\ (\Pi(k,t)\overline{B}(t) - \overline{C}'(k,t))' & 2I \end{bmatrix}. \tag{6.8}$$

that is uniformly positive definite symmetric for all $k \in [0,1]$ such that the matrix in (6.8) shown at the bottom of the previous page is uniformly positive definite. Then pre- and post-multiplying both sides of (6.8) by T and T' with

$$T = \begin{bmatrix} I & \overline{C}(k,t) \\ 0 & I \end{bmatrix}$$

we have that the matrix in (6.9) at the bottom of the page is uniformly positive definite. In particular, this implies that for all $k \in [0,1]$

$$\dot{\Pi}(k,t) + (\overline{A}_\alpha(t) - \overline{B}(t)\overline{C}(k,t))'\Pi(k,t) + \Pi(k,t)(\overline{A}_\alpha(t) - \overline{B}(t)\overline{C}(k,t)) > 0 \quad (6.10)$$

with $\Pi(k,t)$ as in (6.7).

Now, using techniques developed in [5] we have the following Lemma the proof of which is in the Appendix.

Lemma 6.1: Suppose the matrix $\Psi_\alpha(k,t)$ obeys, for all constant $k \in [0,1]$

$$\dot{\Pi}(k,t) + \Psi'_\alpha(k,t)\Pi(k,t) + \Pi(k,t)\Psi_\alpha(k,t) > 0 \quad (6.11)$$

with $\Pi(k,t)$ obeying (6.7) and uniformly positive definite for all constant $k \in [0,1]$. Then with $k(t)$ time varying and obeying (1.3) and (1.4), the matrix $\Psi_0(k(t),t)$ is eas.

Now, applying Lemma 6.1, to (6.10), one obtains that $\overline{A}_0(t) - \overline{B}(t)\overline{C}(k(t),t)$ (note $k(t)$ is now time varying and no α figures in the matrix) is eas, i.e., the matrix is eas in (6.12) at the bottom of the page. Because of the block upper triangular structure of this matrix, it follows that in fact $\hat{A}(t)$ at the bottom of the page, is eas. Thus, as all matrices in question are bounded, the system with the following SVR

$$\{\hat{A}(t), \hat{B}(t), \hat{C}(t), I\}$$

with

$$\hat{B}(t) = [B'(t), 0, B'_w(t)]'$$

and

$$\hat{C}(t) = -[k(t)C(t), k(t)C_w(t), C_w(t)]$$

is also eas.

It is readily verified that this system is in fact

$$[\hat{W} + k(t)\hat{W}H_0]^{-1} \quad (6.13)$$

with $\hat{W}$ having the SVR

$$\{A_w(t), B_w(t), C_w(t), I\}$$

which from the results of the previous section is eas and has eas inverse. Thus, by pre and post-multiplying (6.13) by $\hat{W}$ and its inverse, respectively, we have that

$$[I + k(t)\hat{W}H_0\hat{W}^{-1}]^{-1}$$

is eas, thereby proving the result of Theorem 6.1.

VII. Conclusion

In this paper we have a generalized a number of results concerning passive multiplier theory to MIMO LTV systems. Between them these results cover important issues spanning a number of areas. Though the focus here is on continuous time systems extensions to discrete time systems is straight forward. To what extent generalizations to nonlinear time varying systems problems are possible remains however an open problem.

Appendix

We first prove Lemma 3.4.

Proof of Lemma 3.4: Observe the existence, stability and causality of the operators in question will readily follow from the hypothesis that $[I + kF]^{-1}$ exists for all $k \in [0,1]$ as long as (3.18) holds. We will use induction to prove (3.17)–(3.20). Clearly the result holds for $m = 1$. Suppose it holds for some

$$\begin{bmatrix} -\dot{\Pi}(k,t) - (\overline{A}_\alpha(t) - \overline{B}(t)\overline{C}(k,t))'\Pi(k,t) - \Pi(k,t)(\overline{A}_\alpha(t) - \overline{B}(t)\overline{C}(k,t)) & (\Pi(k,t)\overline{B}(t) + \overline{C}'(k,t)) \\ (\Pi(k,t)\overline{B}(t) + \overline{C}'(k,t))' & 2I \end{bmatrix}. \quad (6.9)$$

$$\begin{bmatrix} A_w(t) & B_w(t)C_x(t) & -k(t)B_w(t)C(t) & -k(t)B_w(t)C_w(t) & -B_w(t)C_w(t) \\ 0 & A_x(t) & -k(t)B_x(t)C(t) & -k(t)B_x(t)C_w(t) & -B_x(t)C_w(t) \\ 0 & 0 & A(t) - k(t)B(t)C(t) & -k(t)B(t)C_w(t) & -B(t)C_w(t) \\ 0 & 0 & B_w(t)C(t) & A_w(t) & 0 \\ 0 & 0 & -k(t)B_w(t)C(t) & -k(t)B_w(t)C_w(t) & A_w(t) - B_w(t)C_w(t) \end{bmatrix} \quad (6.12)$$

$$\hat{A}(t) = \begin{bmatrix} A(t) - k(t)B(t)C(t) & -k(t)B(t)C_w(t) & -B(t)C_w(t) \\ B_w(t)C(t) & A_w(t) & 0 \\ -k(t)B_w(t)C(t) & -k(t)B_w(t)C_w(t) & A_w(t) - B_w(t)C_w(t) \end{bmatrix}$$

$m > 1$. In view of Lemma 3.3 we will work with $G_i(x)$ for scalar x. Then from (3.2)

$$G_{m+1}(x) = \frac{1}{2}\left[\frac{(1+x)\prod_{i=1}^{\hat{N}(m)}(1+\alpha_i^{(m)}x)}{(1+\beta_0^{(m)}x)\prod_{i=1}^{\hat{N}(m)}(1+\beta_i^{(m)}x)} + \frac{(1+\beta_0^{(m)}x)\prod_{i=1}^{\hat{N}(m)}(1+\beta_i^{(m)}x)}{\prod_{i=1}^{\hat{N}(m)}(1+\alpha_i^{(m)}x)}\right]. \quad \text{(A.1)}$$

Clearly,

$$G_{m+1}(x) = \frac{R(x)}{(1+\beta_0^{(m)}x)\prod_{i=1}^{\hat{N}(m)}[(1+\beta_i^{(m)}x)(1+\alpha_i^{(m)}x)]}$$

with $R(x)$ a suitable polynomial. Thus (3.18–3.20) follow. It remains to be established that (3.17) holds with m replaced by $m+1$.

Given that in (A.1) $G_{m+1}(0) = 1$, and $G_{m+1}(x) > 0$ for all $x \geq -1$, the result will follow using the theory of Cauchy indexes (see [20]) as long as all the residues in a partial fraction expansion of $G_{m+1}(x)$ have the same sign. This can be seen readily from an examination of (A.1), combined with the use of (3.17).

Proof of Lemma 3.5: We will use induction to prove (3.21). Clearly the result holds for $m = 1$. Suppose it holds for some $m > 1$, i.e., for this m (3.21) holds. With x a scalar, observe that for arbitrary n,

$$\beta_0^{(n)} + \sum_{i=1}^{N(n)}[\beta_i^{(n)} - \alpha_i^{(n)}] = G_n'(0) \quad \text{(A.2)}$$

with $G_n'(x)$ denoting the derivative of $G_n(x)$. Now,

$$G_{m+1}(x) = \frac{1}{2}\left[\frac{1+x}{G_m(x)} + G_m(x)\right].$$

Hence

$$G_{m+1}'(x) = \frac{1}{2}\left[\frac{1}{G_m(x)} - \frac{(1+x)G_m'(x)}{G_m^2(x)} + G_m'(x)\right].$$

Thus, because of (A.2) and the induction hypothesis,

$$G_{m+1}'(0) = \tfrac{1}{2}[1 - G_m'(0) + G_m'(0)] \quad \text{(A.3)}$$
$$= \tfrac{1}{2}. \quad \text{(A.4)}$$

Thus, the result follows from (A.2).

Proof of Lemma 3.2: One can readily deduce from the result of [12] that $\|H\| < M_2$ iff there exists a symmetric positive semidefinite matrix $P_1(t)$ such that

$$-\dot{P}_1(t) = A'(t)P_1(t) + P_1(t)A(t) + \frac{1}{M_2^2}P_1(t)B(t)B'(t)P_1(t) + C'(t)C(t)$$

and $A(t) + B(t)B'(t)P_1(t)/M_2^2$ is eas. Thus $\|H_\alpha\| < M_2$ iff there exists a positive semidefinite symmetric matrix $P(t)$ such that

$$-\dot{P}(t) = A'(t)P(t) + P(t)A(t) + \frac{1}{M_2^2}P(t)B(t)B'(t)P(t) + C'(t)C(t) + 2\alpha P(t)$$

with $\alpha I + A(t) + B(t)B'(t)/M_2^2 P(t)$ eas. Then it readily follows that the system with SVR $\{A(t), B(t), C_1(t)\}$ has norm smaller than M_2, where

$$C_1(t) = [C'(t), \sqrt{2\alpha P(t)}]'.$$

Thus $\|H\| < M_2$.

Proof of Lemma 6.1: First observe, that the eas of $\Psi_0(k(t), t)$, concerns the eas of the equation

$$\dot{x}(t) = \Psi_0(k(t), t)x(t).$$

Then with

$$\lambda = \frac{k}{1-k}$$

i.e.,

$$k = \frac{\lambda}{1+\lambda} \quad \text{(A.5)}$$

and

$$\tilde{\Psi}_0(\lambda(t), t) = \Psi_0(k(t), t)|_{k=(\lambda/1+\lambda)}$$

this is equivalent to the eas of

$$\dot{x}(t) = \tilde{\Psi}_0(\lambda(t), t)x(t). \quad \text{(A.6)}$$

Observe that the assumed conditions on $k(t)$ translate to

$$0 < \lambda(t) < \infty \quad \text{(A.7)}$$

and with T and δ as in (1.4)

$$\sup_{t\geq 0}\frac{1}{T}\int_t^{t+T}\left[\frac{d}{d\tau}\ln\,\lambda(\tau)\right]^+ d\tau < 2(\alpha - \delta) \quad \text{(A.8)}$$

Define also

$$\tilde{\Pi}(\lambda, t) = (\lambda + 1)\Pi(k(t), t)|_{k=(\lambda/1+\lambda)}.$$

Observe that

$$\tilde{\Pi}(\lambda, t) = \tilde{\Pi}_0(t) + \lambda\tilde{\Pi}_1(t)$$

where

$$\tilde{\Pi}_0 = \Pi_0 > 0$$

and

$$\tilde{\Pi}_1 = \Pi_0 + \Pi_1 > 0.$$

In both cases positive definiteness is uniform. For all constant $\lambda \in [0,\infty)$, (6.11) becomes:

$$\dot{\tilde{\Pi}}(\lambda,t)+\Psi_0'(\lambda,t)\tilde{\Pi}(\lambda,t)+\tilde{\Pi}(\lambda,t)\Psi_0(\lambda,t)<-2\alpha\tilde{\Pi}(\lambda,t). \tag{A.9}$$

With $\lambda(t)$ time varying and obeying (A.7) and (A.8) one obtains

$$\begin{aligned}
&\dot{\tilde{\Pi}}(\lambda(t),t)+\Psi_0'(\lambda(t),t)\tilde{\Pi}(\lambda(t),t)+\tilde{\Pi}(\lambda(t),t)\Psi_0(\lambda(t),t)\\
&\quad<-2\alpha\tilde{\Pi}(\lambda(t),t)+\dot{\lambda}(t)\tilde{\Pi}_1(t)\\
&\quad\le -2\alpha\tilde{\Pi}(\lambda(t),t)+\left[\frac{\dot{\lambda}(t)}{\lambda(t)}\right]^+\lambda(t)\tilde{\Pi}_1(t)\\
&\quad\le -2\alpha\tilde{\Pi}(\lambda(t),t)+\left[\frac{d}{dt}\ln\,\lambda(t)\right]^+\lambda(t)\tilde{\Pi}_1(t)\\
&\quad\le -2\alpha\tilde{\Pi}(\lambda(t),t)+\left[\frac{d}{dt}\ln\,\lambda(t)\right]^+\tilde{\Pi}(\lambda(t),t).
\end{aligned}$$

Thus, with

$$L(x(t),t)=x'(t)\tilde{\Pi}(\lambda(t),t)x(t)$$

using (A.8) we have that, along the trajectories of (A.6),

$$L(x(t+T),t+T)<e^{-2\delta T}L(x(t),t).$$

Then a standard set of arguments (see, e.g., [5]), proves the result.

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