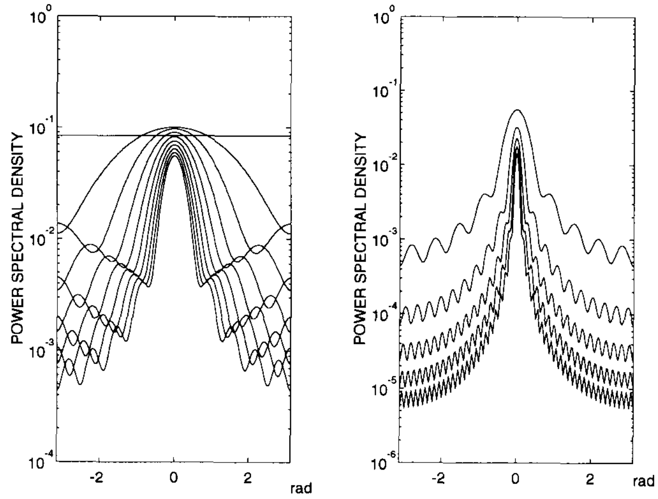

(a) (b)

Fig. 3. Autocovariance power spectral density function $S(e^{j\omega})$ of the dilated and eroded iid source uniformly distributed over $(0, 1)$ for various values of L: (a) Straight line corresponds to $L = 1$, and the rest to $L = 2$ to $L = 10$ from the top to the bottom at $\omega = 0$; (b) from the top ($L = 10$) to the bottom ($L = 50$) at intervals of 10.

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An Efficient Calculation of the Moments of Matched and Mismatched Hidden Markov Models

Mehmet Karan, Brian D. O. Anderson, and Robert C. Williamson

Abstract— **Strelt analysed the classification of an unknown hidden Markov model (HMM) using a set of prescribed HMM's. He proposed a suboptimal test statistic that can be approximated by certain moments for this classification. In this correspondence, the algorithm given by Streit to derive these moments is reformulated in a matrix algebra setting that gives a better insight into the algorithm. Also, an asymptotic analysis of the algorithm is derived using the reformulation.**

I. INTRODUCTION

Hidden Markov models (HMM's) have been popular in several areas, especially with applications in speech recognition [1]. Streit, in [2], has analyzed the problem of classifying an HMM among a set of HMM's by observing a finite output sequence generated by the HMM. He proposed a suboptimal classifier that can be approximated by the moments of the output sequence probabilities of an HMM with respect to another HMM. Let $O(t)$ for $t = 1, 2, \cdots$ be the output at time t of an HMM λ_j that can take M discrete values and let O_T be a T-tuple output sequence $(O(1), O(2), \cdots, O(T))$. If λ_i is another HMM whose output $O(t)$ at time t can take the same M discrete-values, then the moments of the output sequence probabilities of the HMM λ_j with respect to λ_i are defined in [2] as

$$M_{ji}(k, T) = E\{P_j(O_T)^k\}. \tag{1}$$

Here, P_j is the probability measure defined by the parameters of the HMM λ_j and the expectation is calculated using the probability measure P_i defined by the parameters of the HMM λ_i. Note that the amount of computation required for the direct calculation of (1) grows exponentially as T increases. This computational difficulty is overcome in [2] by calculating the moments $M_{ji}(k, T)$ using a recursive algorithm. However, the formulation of this algorithm does not give very much insight and its asymptotic behavior is not established. In this correspondence, we reformulate the algorithm in a matrix algebra framework that allows us to carry out the asymptotic analysis of the algorithm. This allows us to derive an approximate formula for $M_{ji}(k, T)$ that we show by example is surprisingly accurate, even for moderate T.

Another motivation for the calculation of the moments in (1) is a conjecture in [2] that states that the entropies and the relative entropies of HMM's can be calculated asymptotically from the moments $M_{ji}(k, T)$. This conjecture makes use of a central limit theorem type of another conjecture. However, one has to calculate the moments of $P_j(O_T)^{1/T}$, not $P_j(O_T)$, to calculate the entropy and relative entropy rates of HMM's, as is shown in [3] by specializing the conjecture to Bernoulli processes. Unfortunately, calculating the moments of $P_j(O_T)^{1/T}$ does not seem to be easy.

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II. Moments of Output Sequence Probabilities of HMM's

A discrete state, discrete output HMM is defined using a state set $\mathcal{S} = \{1, 2, \cdots, N\}$ and an output set $\mathcal{O} = \{1, 2, \cdots, M\}$ where the output at time t of the HMM, $O(t) \in \mathcal{O}$, is either a deterministic or a probabilistic function of the state of the HMM at time t, $X(t) \in \mathcal{S}$. An HMM can be defined in terms of certain real parameters $\lambda = (A, B, \Pi)$. Here, $A = [a_{ij}]_{N \times N}$ is the state transition probability matrix defined by

$$a_{ij} = \Pr\{X(t+1) = j \mid X(t) = i\} \qquad i, j = 1, \cdots, N \tag{2}$$

and $B = [b_{kj}]_{M \times N}$ is the output probability matrix where

$$b_{kj} = \Pr\{O(t) = k \mid X(t) = j\} \qquad k = 1, \cdots, M \quad \text{and} \quad j = 1, \cdots, N \tag{3}$$

and $\Pi = [\pi_1, \pi_2, \cdots, \pi_N]$ is the initial state probability vector defined by $\pi_i = \Pr\{X(0) = i\}$, $i = 1, \cdots, N$. If the HMM is stationary and A satisfies well known conditions ensuring the existence of a unique steady state probability vector for the states, one can postulate that Π is this vector, and then Π in λ is redundant.

If the parameters of an HMM are known, the probability of a consecutive output sequence $O_T = (O(1), \cdots, O(T))$ of length T, is found using the probability measure $P(\cdot)$ defined by

$$P(O_T) = \Pi \mathcal{B}(O(1)) A \mathcal{B}(O(2)) \cdots A \mathcal{B}(O(T)) \underline{\mathbf{1}}_N \tag{4}$$

where $\underline{\mathbf{1}}_N$ is an N-dimensional column vector of ones and $\mathcal{B}(O(t))$ for $O(t) = 1, \cdots, M$ is the diagonal matrix obtained from the matrix B as

$$\mathcal{B}(O(t)) = \mathbf{diag}\{b_{O(t),1}, b_{O(t),2}, \cdots, b_{O(t),N}\}. \tag{5}$$

Given two HMM's $\lambda_i = (A^{(i)}, B^{(i)}, \Pi^{(i)})$ with N_i states and $\lambda_j = (A^{(j)}, B^{(j)}, \Pi^{(j)})$ with N_j states and both with the same number M of outputs, the moments of the output sequence probabilities of λ_j with respect to λ_i is defined by

$$\begin{aligned} M_{ji}(k, T) &= E\{P_j(O_T)^k\} \\ &= \sum_{O_T} P_i(O_T) P_j(O_T)^k \end{aligned} \tag{6}$$

where O_T is an output sequence of length T. Here, k is a positive integer and P_i and P_j are the probability measures defined by the parameters of the HMM's λ_i and λ_j, respectively. In [2], the computational difficulty in evaluating $M_{ji}(k, T)$ was solved by deriving an algorithm for the calculation of $M_{ji}(k, T)$. Now, we will reformulate this algorithm in a matrix algebra framework.

As is done in [2], let us define a quantity $R(k, T)$ by

$$R(k, T) = \sum_{O_T} \prod_{\nu=0}^{k} P_\nu(O_T) \tag{7}$$

where $P_\nu(\cdot)$ is the probability measure defined by the parameters of the HMM $\lambda_\nu = (A^{(\nu)}, B^{(\nu)}, \Pi^{(\nu)})$ with N_ν states and M outputs where $\nu = 0, 1, \cdots, k$. Note that when λ_0 is λ_i and λ_ν is λ_j for $\nu = 1, \cdots, k$, then $R(k, T)$ is equal to the moment $M_{ji}(k, T)$.

It is possible to express the quantity $R(k, T)$ in (7) as

$$\begin{aligned} R(k, T) &= \sum_{O_T} \left\{ \prod_{\nu=0}^{k} \left\{ \sum_{X_T^{(\nu)}} P_\nu(O_T \mid X_T^{(\nu)}) P_\nu(X_T^{(\nu)}) \right\} \right\} \\ &= \sum_{X_T^{(0)}} \cdots \sum_{X_T^{(k)}} \left\{ \sum_{O_T} \prod_{\nu=0}^{k} P_\nu(O_T \mid X_T^{(\nu)}) \right\} \\ &\quad \times \left\{ \prod_{\nu=0}^{k} P_\nu(X_T^{(\nu)}) \right\} \\ &= \sum_{X^{(0)}(T)} \cdots \sum_{X^{(k)}(T)} \theta_T(X^{(0)}(T), \cdots, X^{(k)}(T)) \end{aligned} \tag{8}$$

where $X^{(0)}(T), \cdots, X^{(k)}(T)$ are states at time T of the HMM's $\lambda_0, \cdots, \lambda_k$, respectively, and $\theta_T(X^{(0)}(t), \cdots, X^{(k)}(t))$ is determined by the following recursive equation:

$$\begin{aligned} &\theta_t(X^{(0)}(t), \cdots, X^{(k)}(t)) \\ &= \left\{ \sum_{O(t)} \prod_{\nu=0}^{k} P_\nu(O(t) \mid X^{(\nu)}(t)) \right\} \sum_{X^{(0)}(t-1)} \cdots \sum_{X^{(k)}(t-1)} \\ &\quad \times \left[\left\{ \prod_{\nu=0}^{k} P_\nu(X^{(\nu)}(t) \mid X^{(\nu)}(t-1)) \right\} \right. \\ &\quad \left. \times\ \theta_{t-1}(X^{(0)}(t-1), \cdots, X^{(k)}(t-1)) \right] \end{aligned} \tag{9}$$

with the initial condition

$$\begin{aligned} &\theta_1(X^{(0)}(1), \cdots, X^{(k)}(1)) \\ &= \left\{ \sum_{O(1)} \prod_{\nu=0}^{k} P_\nu(O(1) \mid X^{(\nu)}(1)) \right\} \left\{ \prod_{\nu=0}^{k} P_\nu(X^{(\nu)}(1)) \right\}. \end{aligned} \tag{10}$$

Now, define a column vector Θ_t whose entries are $\theta_t(X^{(0)}(t), \cdots, X^{(k)}(t))$ in lexicographic order. For example, when $k = 1$, $N_0 = 2$ and $N_1 = 3$, the vector Θ_t becomes

$$\Theta_t = [\theta_t(1,1), \theta_t(1,2), \theta_t(1,3), \theta_t(2,1), \theta_t(2,2), \theta_t(2,3)]' \tag{11}$$

where $(\cdot)'$ denotes the transpose of $(\cdot)$. After some matrix manipulation, it is possible to show that the equation in (9) can be written in a matrix algebra setting as

$$\Theta_t = \bar{F} \Theta_{t-1} \tag{12}$$

where the matrix $\bar{F}$ is obtained from

$$\bar{F} = C \bar{A}'. \tag{13}$$

In (13), C is a diagonal matrix whose diagonal entries $\Gamma(r_0, \cdots, r_k)$ ($r_\nu = 1, \cdots, N_\nu$ where $\nu = 1, \cdots, k$) that are also ordered lexicographically and are defined by

$$\begin{aligned} \Gamma(r_0, \cdots, r_k) &= \sum_{O(t)} \left\{ \prod_{\nu=0}^{k} P_\nu(O(t) \mid X^{(\nu)}(t) = r_\nu) \right\} \\ &= \sum_{l=1}^{M} \left\{ \prod_{\nu=0}^{k} b_{l, r_\nu}^{(\nu)} \right\}. \end{aligned} \tag{14}$$

The matrix $\bar{A}$ in (13) is defined by

$$\bar{A} = A^{(0)} \otimes A^{(1)} \otimes \cdots \otimes A^{(k)} \tag{15}$$

where $\otimes$ denotes the Kronecker product [4]. Similarly, the initial condition in (10) can be rewritten in this framework as

$$\Theta_1 = C(\Pi^{(0)} \otimes \Pi^{(1)} \otimes \cdots \otimes \Pi^{(k)})'. \tag{16}$$

Using the relation between $R(k,T)$ and the moments $M_{ji}(k,T)$, we conclude that the integer moments $M_{ji}(k,T)$ of output sequence probabilities of HMM λ_j with respect to λ_i can be calculated by summing the entries of a column vector Θ_T which can be obtained from the recursive equation in (12) by setting λ_0 to λ_i, and λ_ν for $\nu = 1,\cdots,k$ to λ_j. Hence, the recursive algorithm given in [2] to calculate the moments $M_{ji}(k,T)$ can be reformulated as in the following theorem[1]:

Theorem 1: The moments $M_{ji}(k,T)$ are given by the output of an autonomous linear system of order $\bar{N} = N_i N_j^k$

$$\Theta_t = F\Theta_{t-1}, \quad (17)$$

$$M_{ji}(k,t) = h\Theta_t \quad (18)$$

where the system matrix $F = C(A^{(i)} \otimes (A^{(j)})^{[k]})'$ and the output vector $h = \underline{1}'_{\bar{N}}$. The initial condition for the above linear system is given by

$$\Theta_1 = C(\Pi^{(i)} \otimes (\Pi^{(j)})^{[k]}). \quad (19)$$

Here, $A^{[k]}$ denotes the k-fold Kronecker product of the matrix A by itself, and similarly, $\Pi^{[k]}$ denotes the k-fold Kronecker product of the vector Π by itself.

III. A Simple Calculation of Integer Moments of Output Sequence Probabilities

The asymptotic behavior of the autonomous system given in Theorem 1 can be analyzed using nonnegative matrix theory since the matrix F is nonnegative. If the matrix F is a primitive matrix, i.e. it is irreducible and has only one eigenvalue of maximum modulus, then the Perron–Frobenius theorem [4] implies that

$$\lim_{t\to\infty}[\rho(F)^{-1}F]^t = uv' \quad (20)$$

where $\rho(F)$ is the eigenvalue of F that has the maximum modulus and real, and u and v' are the right and left eigenvectors of F corresponding to this eigenvalue such that $v'u$ is equal to one. The Perron–Frobenius theorem also states that $\rho(F)$ is positive and the entries of u and v are positive as well.

Let $\bar{\rho}(F)$ be an eigenvalue of F that has the *second* largest modulus, then an approximate value for the calculation of the finite-time moments of HMM's can be obtained from the fact (see [4]) that

$$\left\|\left[\frac{F}{\rho(F)}\right]^t - uv'\right\|_\infty \le K\beta^t \qquad t = 1,2,3,\cdots \quad (21)$$

where $\beta = |\bar{\rho}(F)|/\rho(F)$ which is less than one and K is independent of time and its value depends on the matrix F.

These results lead us to the following theorem.

Theorem 2: If the system matrix F in (17) is primitive, the moments $M_{ji}(k,T)$ can be approximated as

$$M_{ji}(k,T) \approx \rho(F)^T \underline{1}'_{\bar{N}} uv'\Theta_1 \quad (22)$$

where $\rho(F)$ is the maximum eigenvalue of F, and u, v' are the right and left eigenvectors of F corresponding to this eigenvalue such that $v'u = 1$. The approximation error decreases as in (21). The vector Θ_1 is given in (19).

A consequence of Theorem 2 is that the integer moments decrease exponentially and the rate of decrease is given by the maximum eigenvalue of the matrix F.

Although the above theorem gives a simple way of calculating the asymptotic moments of HMM's, it is not obvious how to test whether the matrix F is primitive[2] by examining the parameters of the HMM's *a priori*. Of course, it is possible to check whether the matrix F is irreducible or not by finding the maximum eigenvalue of F with its multiplicity. However, it is desirable to simplify the primitivity condition test on F to some conditions on the parameters of the HMM's. This is possible if the matrix F is restricted to be a positive matrix. Note that when F is a positive matrix, it is also primitive.

Now, if the output probability matrices $B^{(i)}$ and $B^{(j)}$ are positive, the diagonal entries of the matrix C are positive as well. If the state transition probability matrices $A^{(i)}$ and $A^{(j)}$ are positive matrices, then the matrix $\bar{A}$ is a positive matrix. Hence, the matrix F that is a product of C and the transpose of $\bar{A}$, is positive. Consequently, if the matrices $A^{(i)}$, $A^{(j)}$ and $B^{(i)}$ and $B^{(j)}$ are positive matrices, then the moments $M_{ji}(k,T)$ can be calculated by the approximate formula in (22).

Note that when the dimension of the matrix F is quite large, then the computational cost of the approximate formula in (22) for the calculation of the moments can be higher than that for the recursive formula since one has to calculate the eigenvalues and eigenvectors of a large matrix. However, if the eigenvalues and eigenvectors of the matrix F are calculated once, then they can be reused for the calculation of all of the moments $M_{ji}(k,T)$ for all large T since the logarithms of the moments $M_{ji}(k,T)$ vary linearly with T.

TABLE I

The Exact and Approximate Values of the Integer Moments of HMM's Whose Parameters Are Given in the Example. These Moments Were Calculated Using the Formulae in Theorem 1 and 2. Here, $\rho(F)$ is the Maximum Modulus Eigenvalue of F and $\bar{\rho}(F)$ is the Eigenvalue of F That Has the Second Largest Absolute Value

		$\log_{10}(M_{ji}(k,T))$			
		T = 5		T = 50	
k	$\lvert\bar{\rho}(F)\rvert/\rho(F)$	Theorem 1	Theorem 2	Theorem 1	Theorem 2
1	0.35	-1.9643354	-1.9646047	-19.23120967	-19.23120976
2	0.25	-3.418649035	-3.418728318	-33.56021539	-33.56021541

IV. Example

The following example illustrates the usage of the approximate formula given in Theorem 2. Let the parameters of the HMM's $\lambda_i = (A^{(i)}, B^{(i)}, \Pi^{(i)})$ and $\lambda_j = (A^{(j)}, B^{(j)}, \Pi^{(j)})$ be

$$A^{(i)} = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}, \quad B^{(i)} = \begin{bmatrix} 0.3 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.8 \end{bmatrix},$$

$$A^{(j)} = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}, \quad B^{(j)} = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.1 \\ 0.3 & 0.2 & 0.8 \end{bmatrix}.$$

$$\Pi^{(i)} = [0.379, 0.238, 0.195, 0.186].$$

$$\Pi^{(j)} = [1/4, 1/4, 1/4, 1/4].$$

Table I shows the logarithm of the values of the moments $M_{ji}(k,T)$ of the HMM's given above for different values of k and T. These values were calculated using the results given in Theorem 1 and 2.

As can be seen from Table I, the approximation error for this example is very small even (less than 0.02%) for $T = 5$ and negligible for $T = 50$.

[1] As in [2], an easy variation of the theorem allows a treatment of continuous symbol HMM's.

[2] "Left-to-right" models, common in speech recognition, yield imprimitive F. But note that more complicated versions of (20) exist and can be used (see [4]).

V. Conclusion

In this correspondence, the algorithm given in [2] to calculate certain moments that arise in the context of the classification of discrete output symbol HMM's is reformulated in a matrix algebra framework. A simple way of calculating these moments is presented and the asymptotic analysis of the algorithm is carried out. This work can also be extended to continuous output symbol HMM's as well.

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Complex, Linear-Phase Filters for Efficient Image Coding

Ben Belzer, Jean-Marc Lina, and John Villasenor

Abstract—**With the exception of the Haar basis, real-valued orthogonal wavelet filter banks with compact support lack symmetry and therefore do not possess linear phase. This has led to the use of biorthogonal filters for coding of images and other multidimensional data. There are, however, complex solutions permitting the construction of compactly supported, orthogonal linear phase QMF filter banks. By explicitly seeking solutions in which the imaginary part of the filter coefficients is small enough to be approximated to zero, real symmetric filters can be obtained that achieve excellent compression performance.**

I. Introduction

Wavelets and other multiresolution techniques have received significant attention as a means to perform efficient coding of images and other multidimensional data [1], [2]. Wavelet transforms utilize a perfect reconstruction filter bank whose filters satisfy certain regularity constraints to furnish a subband decomposition of an input signal. There are two broad classes of filters that permit such a decomposition—orthogonal and biorthogonal. Real orthogonal filters are energy preserving, but lack linear phase, and require use of discontinuous periodic boundary extensions that introduce high frequency artifacts in the reconstructed image. Biorthogonal filters are linear phase and permit use of continuous symmetric boundary extensions, but do not conserve energy in the transform domain. While the number of possible biorthogonal or orthogonal filters is of course infinite, restricting consideration to filters of reasonable length limits the number of distinct filter banks to several thousand. Most of these filter banks lead to very poor image compression, but the few that do enable good compression are now well known [3]. Given this rather limited set of known high-quality filters, new filter classes that would allow exploration of a new and useful set of filters would be of great interest in image coding.

When complex solutions are permitted, a new set of filters that show high promise for use in image coding emerges. From a mathematical standpoint, the advantage of complex filters is that the properties of linear phase, compact support, and orthogonality can all be retained. Linear phase is desirable for reasons of both computational complexity and image quality. The symmetry of linear phase filters leads to a lower complexity hardware implementation because the multiplies involving origin-symmetric coefficient pairs can be combined. This savings is especially important in image and video coding applications, where the required processing rate is quite high. Additionally, it is known that the phase distortions due to information loss that occurs in compression using nonlinear-phase filters can lead to visually objectional artifacts [4].

While use of complex filters may seem to compound the image coding problem because mapping a real image into a complex domain will double the data storage requirements, efficient compression can be obtained by seeking complex filters in which the energy in the imaginary part of the filter coefficients is as small as possible. These "almost-real" filters can be approximated by simply discarding the imaginary portion of the filter coefficients. In effect, the complex filters are used as a means to obtain almost perfect reconstruction linear phase real filters. Using this approach, we have identified and present several filters that perform as well for image coding as the best-known real biorthogonal filters.

Complex filters have been discussed previously in a paper by Lawton [5], who noted that complex solutions were possible and explored the utility of a few specific complex filters using a simple coding approach in which high-frequency wavelet coefficients are discarded. What is new in the present work is that we have performed a systematic exploration of the lower order complex orthogonal filters, and have demonstrated that some of these filters allow extremely efficient image compression. The approach we present of selecting a complex filter with very little energy in the imaginary part of the coefficients is new, as are the numerical results concerning the loss in image quality this incurs.

II. Design of Complex Wavelet Filters

As discussed by Daubechies [6] and others [7], orthogonal wavelet filters can be designed from a maximally flat FIR filter with spectral factors (not including the zeros at $z = -1$) of $p(z)$ and $p^*(1/z^*)$. The lowpass filter $H(z)$ of an orthogonal wavelet filter bank is then obtained by retaining one of the spectral factors and half of the zeros at $z = -1$, i.e.

$$H(z) = \left(\frac{1+z^{-1}}{2}\right)^{N+1} p(z). \tag{1}$$

The integer N is arbitrary and controls the size of the filter as well as the order of the root at $z = -1$. It has been shown [8] that the spectral factor $p(z)$ can be expressed

$$p(z) = \sum_{n=0}^{N} \rho_n (z^{-1}+1)^{N-n}(z^{-1}-1)^n. \tag{2}$$

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