Correspondence

A Multiport State-Space Darlington Synthesis

Two recent papers[1],[2] have described a network synthesis of rational positive real functions or matrices via what may be loosely termed "state-space techniques." Brockett[1] actually restricted consideration to the case of positive-real functions $z(s)$ for which $z(\infty)$ was zero; the synthesis was of the Darlington type, i.e., the state-space equations of a 2-port lossless network were given, where the lossless network presented the impedance $z(s)$ at one port when its other port was terminated in a unit resistor.

In this correspondence we give extensions, without detailed proofs, to cover the synthesis of positive-real matrices $Z(s)$ subject to the inessential restriction $Z(\infty) < \infty$. Extensions of the Darlington synthesis to the multiport situation have been given by Bayard[3] for the reciprocal case and Newcomb[4] for the nonreciprocal case, using classical approaches.

The state-space interpretation of these results rests ultimately on the following lemma.[5],[6]

Lemma 1

Let $Z(s)$ be a rational positive-real matrix with $Z(\infty) < \infty$. Suppose $\{F, G, H, J\}$ is a minimal realization for $Z(s)$ in the sense that

$$Z(s) = J + H'(sI - F)^{-1}G \tag{1}$$

and F has minimal dimension. The superscript prime denotes matrix transposition. Then there exist real matrices $P' = P > 0$, L, and W_0 such that

$$PF + F'P = -LL' \tag{2a}$$

$$PG = H - LW_0 \tag{2b}$$

$$W_0'W_0 = J + J'. \tag{2c}$$

The significance of this lemma in frequency domain terms is as follows. If $W(s)$ is a spectral factor of $Z(s) + Z'(-s)$, i.e.,

$$Z(s) + Z'(-s) = W'(-s)W(s), \tag{3}$$

it can be taken to be of the form

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{4}$$

for some matrices W_0 and L; and there is positive definite symmetric matrix P such that (2a) and (2b) hold.

In Anderson,[5],[6] methods for computing the matrices P, L, and W_0 are discussed; one method is essentially algebraic, and does not require explicit determination of a spectral factor of $Z(s) + Z'(-s)$.

In preparation for a presentation of the synthesis procedure, we introduce the simple Lemma 2.

Lemma 2

With the relevant matrices defined as in Lemma 1, let T be a matrix such that

$$T'T = P. \tag{5}$$

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Define

$$F_1 = TFT^{-1},\quad G_1 = TG,\quad H_1' = H'T^{-1},\quad L_1' = L'T^{-1}. \tag{6}$$

Then

$$F_1 + F_1' = -L_1L_1' \tag{7a}$$

$$G_1 = H_1 - L_1W_0 \tag{7b}$$

$$W_0'W_0 = J + J'. \tag{7c}$$

To fix ideas, suppose the prescribed $Z(s)$ is an $n \times n$ matrix, and that the matrix L_1 has r columns. Let us identify an n vector u with the currents at the first n ports of an $(n + r)$-port network to be prescribed later, an r vector u_1 with the currents at the remaining r ports, an n vector y with the voltages at the first n ports, and an r vector y_1 with the voltages at the remaining r ports (see Fig. 1). Then, (see outline proof below):

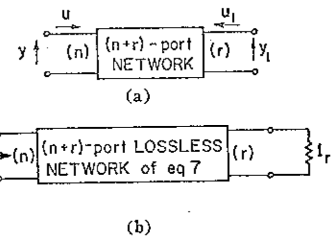

Fig. 1. (a) Lossless network of Lemma 4. (b) Idea of Darlington–Bayard–Newcomb synthesis (1_r denotes r unit resistors).

Lemma 3

With F_1, G_1, H_1, L_1 defined as in Lemma 2, and J and W_0 defined as in Lemma 1, the following state-space equations define an $(n + r)$-port lossless network.

$$\dot{x} = \tfrac{1}{2}(F_1 - F_1')x + \left(G_1 + \frac{L_1W_0}{2}\right)u - \frac{L_1}{\sqrt{2}}u_1 \tag{8a}$$

$$y = \left(G_1 + \frac{L_1W_0}{2}\right)'x + \left(\frac{J - J'}{2}\right)u + \frac{W_0'}{\sqrt{2}}u_1 \tag{8b}$$

$$y_1 = -\frac{L_1'}{\sqrt{2}}x - \frac{W_0}{\sqrt{2}}u. \tag{8c}$$

The significance of this lossless network is contained in the following theorem, yielding in state-space terms a multiport derivation of the Darlington synthesis.

Theorem

Let $Z(s)$ be a positive-real matrix with $Z(\infty) < \infty$. Starting with a minimal realization F, G, H, J of $Z(s)$, suppose an associated lossless network is defined by (8). If the last ports of this network are terminated in unit resistors, corresponding to setting

$$u_1 = -y_1, \tag{9}$$

then at the remaining n ports the impedance $Z(s)$ is observed.

Proof: Setting in (8a) and (8b) u_1 equal to $-y_1$, and then substituting for y_1 from (8c) leads to

$$\dot{x} = \tfrac{1}{2}(F_1 - F_1' - L_1L_1')x + G_1u \tag{10a}$$

$$y = (G_1 + L_1W_0)'x + \tfrac{1}{2}(J - J' + W_0'W_0)u. \tag{10b}$$

On using (7), these become

$$\dot{x} = F_1x + G_1u \tag{11a}$$

$$y = H_1'x + Ju \tag{11b}$$

and evidently the transfer function matrix relating $U(s)$ to $Y(s)$ is $J + H_1'(sI - F_1)^{-1}G_1$, or $Z(s)$.

In summary, the synthesis procedure is as follows. From $Z(s)$ a minimal realization is formed, and corresponding to this realization matrices P, L, and W_0 are found by any of the available methods, which include an algebraic procedure and a spectral factorization procedure. The matrices of the minimal realization, together with P, L, and W_0, can be used to define new matrices F_1, G_1, H_1, and L_1, and in terms of these the equations of a lossless network can be written down. This lossless network has the property that when some of its ports are terminated in unit resistors, $Z(s)$ is seen at the remaining ports.

Remarks

1) The degree of the lossless network is the same as that of $Z(s)$; consequently, a minimal reactive synthesis of the lossless network gives a minimal reactive synthesis of $Z(s)$.

2) A synthesis of $Z(s)$ using a minimal number of resistors corresponds to taking L (Lemma 1) of minimal dimension. Procedures are available[2],[5],[6] for ensuring that this is the case.

3) The procedure spelled out above is merely designed to reduce any lossy synthesis problem to a considerably simpler lossless synthesis problem. The form of (8) actually makes an immediate solution of the lossless synthesis problem available.

Sketch of Proof of Lemma 3

With u, y, u_1, having the meanings depicted in Fig. 1, the lossless network defined by (8) may be synthesized by terminating all but the first $(n + r)$ ports of the nondvnamic network N_1, defined below, in unit inductors.

The network N_1 is lossless, memoryless, and defined via the constant impedance matrix, evidently skew,

$$Z_{n_1} = \begin{array}{c} \begin{matrix} \quad (n) & \quad\quad (r) & \quad\quad \end{matrix} \\ \left[\begin{matrix} \dfrac{J - J'}{2} & \dfrac{W_0}{\sqrt{2}} & -\left(G_1 + \dfrac{L_1W_0}{2}\right)' \\ -\dfrac{W_0'}{\sqrt{2}} & 0 & \dfrac{L_1'}{\sqrt{2}} \\ G_1 + \dfrac{L_1W_0}{2} & -\dfrac{L_1}{\sqrt{2}} & \tfrac{1}{2}(F_1' - F_1) \end{matrix}\right] \begin{matrix} (n) \\ (r). \\ \\ \end{matrix} \end{array} \tag{12}$$

4) It is suggested by (2a) that $x'Px$ is a Liapunov function for the network. The maneuvers of Lemma 2 are evidently designed to ensure that $x'x$ is a Liapunov function, with $-(L_1'x)^2$ its derivative. Since the decrease in the Liapunov function is related to the presence of resistors which dissipate energy, it is not surprising to see from (8) and (9) the coupling of the resistors to the lossless $(n + r)$-port in terms of the matrix L_1.

5) The following restatement of Lemmas 1 and 2 together with a sufficiency statement, from Anderson,[5] has potential application in those areas of system theory outside of network synthesis where positive-real matrices appear.

Lemma 4

Let $Z(s)$ be a matrix of rational functions of s, with $Z(\infty) < \infty$. Then $Z(s)$ is positive real if, and only if, it possesses a minimal realization $\{F_1, G_1, H_1, J\}$ such that

$$F_1 + F_1' = -L_1L_1' \tag{7a}$$

$$G_1 = H_1 - L_1W_0 \tag{7b}$$

$$W_0'W_0 = J + J' \tag{7c}$$

for some matrices L_1 and W_0.

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