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Multiplicative approximation of transfer functions with frequency weighting[☆]

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Abstract

An algorithm for transfer function order reduction is presented, which generalizes the balanced stochastic truncation algorithm to allow for input weighting. An example illustrates use of the algorithm to secure smaller dB error in selected frequency bands through the introduction of the weighting.

Keywords: Model reduction; Multiplicative error; Balanced stochastic truncation; Frequency weighting; Lyapunov equation; Riccati equation

1. Introduction

In this paper, we consider a class of frequency weighted model reduction which finds an rth order transfer function $V_r(s)$ aimed at minimizing the following error index:

$$E_w = \| V^{-1}[V - Vr] W \|_\infty . \quad (1.1)$$

Here $V(s)$ is a given stable transfer function of order greater than r, and $W(s)$ is a given stable weighting function. With $W(s)$ the identity, balanced stochastic truncation (BST) [1, 4–6] is one method that can be used to find a $V_r(s)$ which is approximately minimizing. Our scheme with non-constant $W(s)$ generalizes BST and we term it therefore weighted BST. We also term E_w of (1.1) the weighted multiplicative error.

The BST method which finds a reduced order transfer function minimizing or approximately minimizing a multiplicative error E_w with $W(s) = 1$ was initiated by Desai and Pal [1] and generalized by Green [4, 5] and Green and Anderson [6]. The unweighted error index, involving so-called multiplicative approximation, is to be contrasted with the index $\| V - Vr \|_\infty$, which involves additive approximation. The former tends to produce an error which is flat in the dB sense (i.e. as a percentage) with frequency, while the latter tends to produce an absolute error with flat magnitude. Often a multiplicative approximation is preferred to the additive approximation, and a great many magnitude specifications are given in decibels [8]; ± 1 dB correspond to a multiplicative error of about 2%.

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Specification involving phase shift can also be regarded as multiplicative error statements; an error of ± 0.5 radians of phase shift is like a 5% multiplicative error also. Multiplicative error approximation rather than additive error approximation is also important in reducing high order models of plants, prior to design of a controller.

The 'robustness theorem' of Safonov et al. [10, 11] also provides a compelling case for the importance of multiplicative error reduction of a plant model in control system design. Use of an unweighted criterion causes the multiplicative error to be approximately uniformly small, including at frequencies well above the closed-loop cut-off frequency, where larger multiplicative error in the plant model could be tolerated with no risk to stability. The introducing of frequency weighting to reflect this fact then allows smaller error to be obtained in the pass-band (which is helpful) at the expense of large error at the stop-band (which can be tolerable). Actually, in digital filter design, where the use of a multiplicative error is in general logical in, say, approximating a high order FIR filter by a low order IIR filter, deep in the stop-band multiplicative error probably also has much reduced relevance. Again therefore, there is case for introducing weighting into the multiplicative criterion.

The problem which finds $V_r(s)$ minimizing the weighted multiplicative error E_w could also (at least in principle) be solved by the two-sided weighted balanced truncation method [2] in which the input weighting is $W(s)$ and the output weighting $V^{-1}(s)$. However the calculation of this approximation is more complicated than that of the weighted balanced stochastic truncation, because the degrees with which one is working are certainly higher. Furthermore this method is not applicable when $V(s)$ is non-minimum phase because then the output weighting is unstable. Even when $V(s)$ is minimum phase, so that $V^{-1}(s)$, $V(s)$ and $W(s)$ are all stable, the two-sided weighted balanced truncation method has never been proven to yield a stable $V_r(s)$, in contrast to the $V_r(s)$ weighting when the scheme of this paper is used.

The principal contribution of this paper is to develop an algorithm for the input-weighted balanced stochastic truncation method. This algorithm can be extended to the output-weighted and the two-sided weighted cases.

An outline of this paper is as follows. In Section 2, the algorithm for non-weighted balanced stochastic truncation is reviewed and the algorithm for the input-weighted case is presented. An example is given in Section 3, followed by some concluding remarks in Section 4.

2. Algorithm for input-weighted BST

In this section, the algorithm for non-weighted balanced stochastic truncation is reviewed [1, 4–6] and then the algorithm for the input-weighted case is presented.

Consider a possibly non-minimum phase, stable nth order ($n > r$) square transfer function matrix $V(s)$, given by a minimal state-space realization

$$V(s) = D_v + H(sI - A)^{-1}B \tag{2.1}$$

with $\det[V(j\omega)] \neq 0$ for all $\omega \in [0, \infty]$. For any transfer function matrix $G(s)$, the notation $G^*(s)$ denotes the complex conjugate transpose of $G(-\bar{s})$ where $\bar{s}$ denotes the complex conjugate of s. A power spectrum matrix $\Phi(s)$ can be defined along with two further transfer function matrices $W(s)$ and $Z(s)$ related to $\Phi(s)$ by

$$\Phi(s) = V(s)V^*(s) = W^*(s)W(s) = Z(s) + Z^*(s) \tag{2.2}$$

with $W(s)$ stable and minimum phase and $Z(s)$ the stable part of $\Phi(s)$. Notice that $W(s)$ is unique to within left multiplication by an orthogonal matrix and $Z(s)$ is unique to within addition of a real skew matrix. The functions $V(s)$, $W(s)$ and $Z(s)$ together with a function $F_c(s)$ have the forms

$$\begin{bmatrix} Z(s) & V(s) \\ W(s) & F_c(s) \end{bmatrix} = \begin{bmatrix} D_z & D_v \\ D_w & 0 \end{bmatrix} + \begin{bmatrix} H \\ C \end{bmatrix}(sI - A)^{-1}[G \;\; B], \tag{2.3}$$

where $D_w = D_v^T$, $D_z + D_z^T = D_v D_v^T = D_w^T D_w$, and C and G can be calculated as follows. Define P as the solution of the Lyapunov equation

$$AP + PA^T + BB^T = 0. \tag{2.4}$$

Then

$$G = PH^T + BD_v^T. \tag{2.5}$$

Next, let Q be the stabilizing solution of the Riccati equation

$$QA + A^TQ + (H^T - QG)D_v^{-T}D_v^{-1}(H - G^TQ) = 0 \tag{2.6}$$

and then

$$C = D_v^{-1}(H - G^T Q). \tag{2.7}$$

Notice that as a result,

$$QA + A^T Q + C^T C = 0. \tag{2.8}$$

Definition. Given the nth order transfer function matrix $V(s)$, the minimal realizations $V(s)$, $W(s)$ and $Z(s)$ in (2.3) are balanced stochastic realizations if

$$P = Q = \Sigma = \mathrm{diag}(\sigma_i), \quad i = 1, \ldots, n, \ \sigma_i \geqslant \sigma_{i+1}, \tag{2.9}$$

that is, the minimal realization $F_c = C(sI - A)^{-1}B$ in (2.3) is internally balanced.

It is of course easy to change the coordinate basis so that balanced stochastic realizations are present. Suppose this is done. Approximation proceeds as follows.

Let Σ be partitioned into two blocks,

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{2.10}$$

where $\Sigma_2 = \mathrm{diag}[\sigma_{r+1}, \ldots, \sigma_n]$, and let the other matrices A, B, C, H and G be partitioned correspondingly as

$$\begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, [C_r \ C_2], [H_r \ H_2], \begin{bmatrix} G_r \\ G_2 \end{bmatrix}. \tag{2.11}$$

The BST model-reduction method simply keeps the upper-left corner sub-blocks of all the matrices above to form the reduced transfer functions

$$\begin{bmatrix} Z_r(s) & V_r(s) \\ W_r(s) & F_{cr}(s) \end{bmatrix} = \begin{bmatrix} D_z & D_v \\ D_w & 0 \end{bmatrix} + \begin{bmatrix} H_r \\ C_r \end{bmatrix} (sI - A_r)^{-1} [G_r \ B_r]. \tag{2.12}$$

While $V_r(s)$ in general will not minimize the unweighted version of (1.1), it will generally be a good approximant [7, 10]. Error bound formulas are available in [4, 12].

Weighted balanced stochastic truncation: Consider now a stable input-weighting function and associated minimal state-space realization

$$W_I(s) = D_I + C_I(sI - A_I)^{-1}B_I. \tag{2.13}$$

Let us motivate the approach of the algorithm before presenting the details. The calculations in unweighted BST inter alia compute for $V(s)$ a transfer function $F_c(s)$ which is then reduced by (ordinary additive error) balanced truncation to yield an approximant $F_{cr}(s)$. Then $V_r(s)$ is found from $F_{cr}(s)$, by reversing the process which yielded $F_c(s)$ from $V(s)$. Evidently, the multiplicative approximation problem is replaced by an additive approximation problem. This is what is done in the weighted case too – and the same weighting is used for the multiplicative problem as for the additive problem.

Proceeding as for the unweighted case, matrices P, G, Q and C are all found. The matrices Q and P are the observability and controllability gramians for the realization $\{A, B, C\}$ of $F_c(s)$. The basic idea is now to change the controllability gramian of the realization of $F_c(s)$ (equivalently of the realization of $V(s)$) to reflect the presence of the input weighting $W_I(s)$. The observability gramian Q is unaltered.

The input-weighted transfer function $V(s)W_I(s)$ has a representation with the following state-space matrices:

$$\bar{A} = \begin{bmatrix} A & BC_I \\ 0 & A_I \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} BD_I \\ B_I \end{bmatrix}, \quad \bar{C} = [C \ DC_I],$$

$$\bar{D} = DD_I, \tag{2.14}$$

and $F_c(s)W_I(s)$ has a realization with the same $\bar{A}, \bar{B}$ pair.

Now let

$$\bar{P} = \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_I \end{bmatrix} \tag{2.15}$$

be the solution of the following Lyapunov equation:

$$\bar{P}\bar{A}^T + \bar{A}\bar{P} + \bar{B}\bar{B}^T = 0. \tag{2.16}$$

Assuming there is no pole-zero cancellation in VW_I, $\bar{P}$ is positive definite. So is its principal submatrix P. Now P can be regarded as the frequency weighted controllability gramian for the original transfer function $V(s)$.

Now introduce the coordinate basis change to $\{A, B, C, H, G\}$ which makes P of (2.15) $= Q$ of (2.8) $= \Sigma = \mathrm{diag}[\sigma_1, \ldots, \sigma_n]$, $\sigma_i \geqslant \sigma_{i+1}$, $i = 1, \ldots, n-1$. This new realization $\{A, B, C, H, G\}$ is called an input-weighted balanced stochastic realization. (The coordinate basis change is easy to determine [5].)

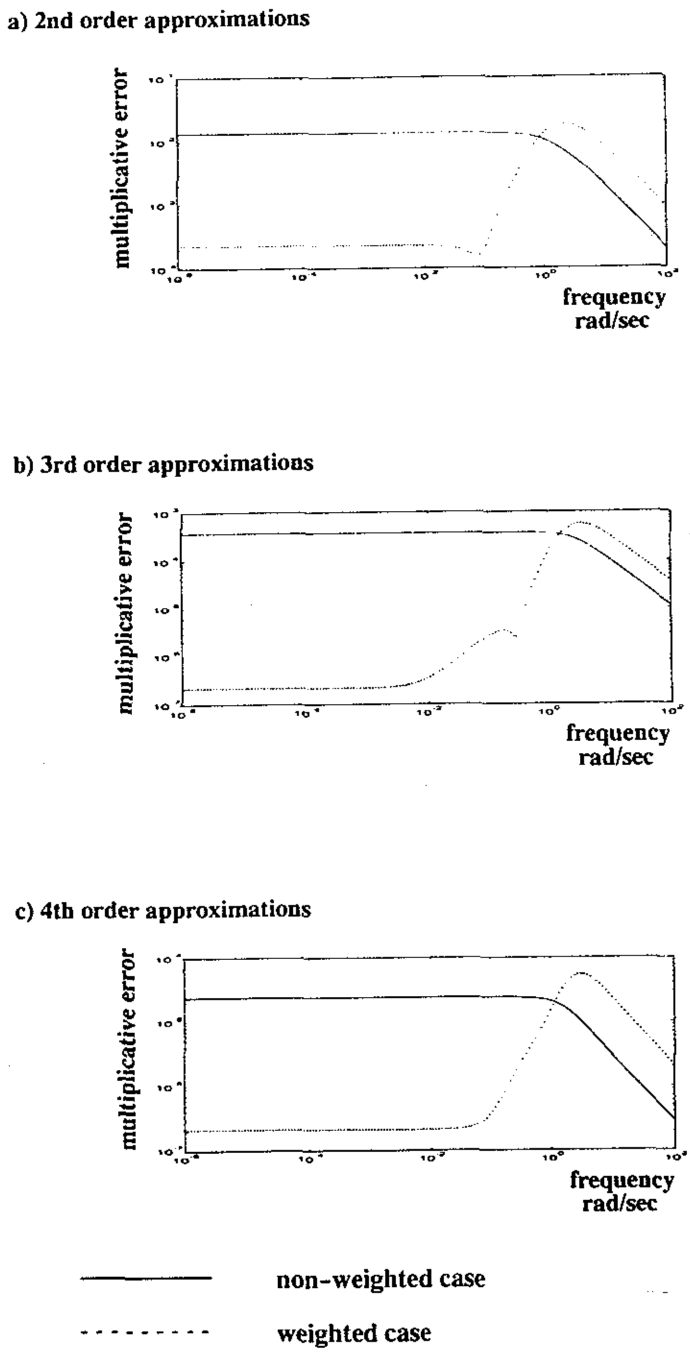

Fig. 1. Plots of approximation errors.

Partition $\{A, B, C, H, G\}$ as in (2.11) and Σ as in (2.10). Now, as previously, the reduction is achieved by eliminating the rows and columns of A, B and H corresponding to Σ_2 in Σ. The reduced order transfer functions of order r are given by (2.12), which includes of course $V_r(s)$.

3. Example

In this section we present an example to illustrate how a reduced order transfer function obtained via weighted balanced stochastic truncation differs from that obtained in the non-weighted case.

We consider a stable transfer function with two unstable zeros at $z_{01} = 3.5$ and $z_{02} = 4$ given by $V(s) = (s + 1.5)^2(s + 2.5)(s - z_{01})(s - z_{02})/(s + 1)^2 \times (s + 2)^2(s + 3)$. In the non-weighted case, the reduced order transfer functions of order 2, 3 and 4 are

$$V_2(s) = (s - z_{01})(s - z_{02})/(s^2 + 3.48s + 2.1054),$$

$$V_3(s) = (s + 1.4629)(s - z_{01})(s - z_{02})/ (s^3 + 4.9618s^2 + 7.1315s + 3.1198),$$

$$V_4(s) = (s + 1.1765)(s + 3.146)(s - z_{01})(s - z_{02})/ (s^4 + 7.8226s^3 + 20.83s^2 + 21.855s + 7.8966).$$

(It is a standard result that non-minimum phase zeros are preserved [5].) Now we introduce an input-weighting function $W_I(s) = 1/(s + 0.1)^2$. Then the reduced order transfer functions are

$$V_{W2}(s) = (s - z_{21})(s - z_{22})/(s^2 + 3.4168s + 2.1323),$$

$$V_{W3}(s) = (s + 1.2952)(s - z_{31})(s - z_{32})/ (s^3 + 4.7916s^2 + 6.5542s + 2.7630),$$

$$V_{W4}(s) = (s + 1.2294)(s + 4.0225)(s - z_{41})(s - z_{42})/ (s^4 + 8.7521s^3 + 25.324s^2 + 28.084s + 10.55),$$

where $z_{21} - z_{01} = 1.2093 \times 10^{-2}$, $z_{22} - z_{02} = -1.4849 \times 10^{-2}$, $z_{31} - z_{01} = 3.7653 \times 10^{-6}$, $z_{32} - z_{02} = -4.7569 \times 10^{-6}$, $z_{41} - z_{01} = 3.4066 \times 10^{-8}$, $z_{42} - z_{02} = -4.7531 \times 10^{-8}$. This fact shows that the number of unstable zeros of the reduced order transfer function is equal to that of the original transfer function, but the locations of unstable zeros are slightly changed.

Now let us define two approximation error functions as

$$e_i(\omega) = |V^{-1}(j\omega)[V(j\omega) - V_i(j\omega)]|, \quad i = 2, 3, 4,$$

$$e_{wi}(\omega) = |V^{-1}(j\omega)[V(j\omega) - V_{wi}(j\omega)]|, \quad i = 2, 3, 4.$$

Fig. 1 shows the plots of $\{e_i(\omega), e_{wi}(\omega)\}$ with respect to $\omega \in [10^{-6}, 10^2]$ for $i = 2, 3, 4$. This figure shows that the weighted balanced stochastic truncation method can approximate a given transfer function better in some frequency range than the non-weighted balanced stochastic truncation method, if the weighting function is properly chosen. Of course, at the frequencies where the weighting function has small amplitude, the error is higher, as expected.

4. Conclusions

An algorithm for frequency weighted balanced stochastic truncation was suggested and its performance compared with the non-weighted case via an example. The example illustrates the fact that the reduced order transfer functions obtained with the frequency weighted BST method can approximate the original transfer function much better than a reduced order transfer function with non-weighted BST at certain frequencies which are dependent on the weighting. This example also showed that the number of unstable zeros of the reduced order transfer functions was the same as that of the original transfer function but the zero locations were slightly changed.

Issues of interest which remain open are the calculation of error bound formulas and a proof of the preservation of the number of unstable zeros. One could also examine weighted multiplicative Hankel norm order reduction, generalizing the unweighted ideas of [3, 9].

References

[1] U.B. Desai and D. Pal, A transformation approach to stochastic model reduction, *IEEE Trans. Automat. Control* **29** (1984) 1097–1100.

[2] D.F. Enns, Model reduction with balanced realizations: an error bound and a frequency weighted generalization, *Proc. 23rd IEEE Conf. on Decision and Control*, Las Vegas, NV (1984) 127–132.

[3] K. Glover, Multiplicative approximation of linear multivariable systems with L^∞ error bounds, *Proc. Amer. Control Conf.*, Seattle, WA (1986) 1705–1709.

[4] M. Green, A relative-error bound for balanced stochastic truncation, *IEEE Trans. Automat. Control* **33** (1988) 961–965.

[5] M. Green, Balanced stochastic realization, *Linear Algebra Appl.* **98** (1988) 211–247.

[6] M. Green and B.D.O. Anderson, Generalized balanced stochastic truncation, *Proc. 29th Conf. on Decision and Control*, Honolulu (1990) 476–481.

[7] P. Harshavardhana, E.A. Jancheere and L.M. Silverman, Stochastic balancing and approximation – stability and minimality, *IEEE Trans. Automat. Control* **29** (1985) 744–746.

[8] Integrated Systems Inc., MATRIX_X Model Reduction Module, 1991.

[9] J. B. Matson, L. Lam, B. D. O. Anderson and B. James, Multiplicative Hankel norm approximation of multivariable systems, *Internat. J. Control* **58** (1993) 129–167.

[10] M.G. Safonov and R.Y. Chiang, Model reduction for robust control: a Schur relative-error method, *Internat. J. Adaptive Control Signal Process.* **2** (1989) 252–272.

[11] M.G. Safonov, R.Y. Chiang and H. Flashner, H^∞ robust control synthesis for a large space structure, *Proc. Amer. Control Conf.*, Atlanta, GA (1988) 2038–2045.

[12] W. Wang and M.G. Safonov, A tighter relative-error bound for balanced stochastic truncation, *Systems Control Lett.* **14** (1990) 307–317.