A New Test for Strict Positive Realness

Brian D. O. Anderson, Mohamed Mansour, and Frantisek J. Kraus

Abstract— Suppose $w(s) = p(s)/q(s)$ is a real rational function, with $p(s), q(s)$ polynomials. This paper presents a test for the strict positive realness of $w(\cdot)$ expressed in terms of the values assumed by $p(s), q(s)$ at a finite set of points $s = j\omega_i, i = 0, \cdots, N, \omega_o = 0$ and $\omega_N = \infty$.

I. Introduction

We are concerned with testing a prescribed real rational function $w(s)$ for strict positive realness. This is defined as followed: $w(s) = p(s)/q(s)$ with $p(\cdot)$ and $q(\cdot)$ polynomial is strictly positive real if [1]

$$\text{all finite zeros of } q(s) \text{ lie in } Re[s] < 0 \tag{1}$$

$$Re\frac{p(j\omega)}{q(j\omega)} > 0 \text{ for all real } \omega \tag{2}$$

These requirements can be used to conclude that the order of p is one less than, equal to or one greater than the order of q. In applications, the first two cases (corresponding to properness of w) are generally more relevant. When w and w^{-1} are proper, the second requirement becomes

$$Re\frac{p(j\omega)}{q(j\omega)} \geq \delta > 0 \text{ for all real } \omega \text{ and some } \delta \tag{3}$$

Strict positive real functions arise in a number of applications, for example, adaptive systems [2], [3] and robust systems [4]. Other tools for working with robust systems include the concept of a *value set*, which is typically the set of values assumed by a transfer function or a polynomial at a fixed frequency when the coefficients vary over some defined set. While theoretically knowledge of a value set at all frequencies may be required for some purpose, practicalities dictate that it can only be computed at a finite number of frequencies. This fact prompts the general question: to what extent can properties such as stability of a polynomial or strict positive realness of a transfer function be inferred by examining behavior at isolated frequencies.

In [5]–[8], the stability problem is examined. This is reviewed in Section II. In Section III, we state a result for strict positive real functions. Some additional observations for discrete-time systems are contained in Section IV, and Section V contains concluding remarks.

One notational convention to be used in the paper is as follows. For $p(s)$ a polynomial, assumed to be nonzero for $s = j\omega, \omega \in [\omega_1, \omega_2]$,

a) $\arg \frac{p(j\omega_2)}{p(j\omega_1)}$ is assumed to lie in $(-\pi, \pi]$

b) $\Delta_{\omega_1}^{\omega_2} \arg p(j\omega)$ denotes the actual change in argument as ω moves from ω_1 to ω_2

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If $p(j\omega)$ can be zero in $[\omega_1, \omega_2]$, we proceed as follows:

c) $\arg \frac{p(j\omega_2)}{p(j\omega_1)}$ and $\Delta_{\omega_1}^{\omega_2} \arg p(j\omega)$ are not defined if $p(j\omega_1) = 0$ or $p(j\omega_2) = 0$

d) at each zero of $p(j\omega)$ in (ω_1, ω_2) we shall say that $\Delta_{\omega_1}^{\omega_2} \arg p(j\omega)$ undergoes a jump decrease of π as ω moves through the zero.

Note that d) is a convention; also, it is tantamount to regarding $j\omega$ axis zeros as limiting cases of *right* half plane zeros.

Last, we shall suppose

e) when $\omega_N = \infty$, $\arg \frac{p(j\omega_N)}{p(j\omega_{N-1})}$ means $\lim_{\omega\to\infty} \arg \frac{p(j\omega)}{p(j\omega_{N-1})}$

Obviously

$$\Delta_{\omega_1}^{\omega_2} \arg p(j\omega) = \arg\frac{p(j\omega_2)}{p(j\omega_1)} + 2n\pi \tag{4}$$

for some integer n (depending on ω_1 and ω_2). In the sequel, we shall work with quantities like $\arg \frac{p(j\omega_2)}{p(j\omega_1)}$, which are evaluated using the values assumed by $p(j\omega)$ at discrete frequencies. One of our concerns will be to ensure that these values are in fact identical with $\Delta_{\omega_1}^{\omega_2} \arg p(j\omega)$, i.e., $n = 0$ in (4).

II. Stability Testing

The following result is a modified version of the result of [5]:

Theorem 2.1: Let $p(s)$ be a dth degree real polynomial. Then a necessary and sufficient condition that $p(s)$ is Hurwitz, i.e., all roots lie in $Re[s] < 0$, is that there exist frequencies $0 = \omega_0 < \omega_1 < \cdots < \omega_N = \infty$ such that

$$0 < \arg\frac{p(j\omega_k)}{p(j\omega_{k-1})} < \pi \tag{5}$$

$$\sum_{k=1}^{N} \arg\frac{p(j\omega_k)}{p(j\omega_{k-1})} = \frac{d\pi}{2} \tag{6}$$

Note that in [5], the condition (5) is not present.[1] It is then possible to obtain a counterexample. Consider the polynomial $p(s) = (s-1)^4$, and select $\omega_0 = 0, \omega_1 = 1$ and $\omega_2 = \infty$. Then $\arg p(j\omega_1) = \pi$ and in accord with the convention set out in Section I,

$$\arg\frac{p(j\omega_1)}{p(j\omega_0)} + \arg\frac{p(j\omega_2)}{p(j\omega_1)} = 2\pi = 4(\pi/2) \tag{7}$$

The problem is that

$$\Delta_{\omega_1}^{\omega_2} \arg p(j\omega) = -\pi$$

but this negative change of argument is interpreted as a positive change π due to the $\mathrm{mod} 2\pi$ effect always present when arguments are computed just using data at discrete points. Correct proofs of Theorem 2.1 may be found in [7] and [8].

Notice that the theorem says nothing about how to choose the ω_i, see [5]. Nevertheless, from (5) and (6) one can conclude the necessity of the following condition:

$$N > \frac{d}{2} \tag{8}$$

[This condition is violated in the counterexample]

[1] Reference [5] quotes an earlier version, [6] in which the upper bound of (5) is assumed, but [6] asserts without full substantiation that $j\omega$-axis zeros of $p(\cdot)$ are easily excluded.

The following Corollary will be used in the next section.

Corollary 2.1: Let $p(s)$ be a dth degree real polynomial. Suppose (5) and (6) hold. Then

$$\arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} = \Delta^{\omega_k}_{\omega_{k-1}} \arg p \tag{9}$$

Proof: Let n_k be such that

$$\Delta^{\omega_k}_{\omega_{k-1}} \arg p = \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} - 2\pi n_k \tag{10}$$

The condition of the corollary ensures that p is stable and so $\Delta^{\omega_k}_{\omega_{k-1}} \arg p > 0$. In view of (5), $n_k \le 0$. Adding the identities (10) yields

$$\Delta^{\omega_N}_{\omega_0} \arg p = \sum_{k=1}^{N} \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} - 2\pi \sum n_k$$

The left-hand side is $d\pi/2$ since p is stable. Using (6), it follows that $\sum n_k = 0$. Since $n_k \le 0, n_k = 0$ for all k, as required.

Remarks: The theorem can be extended in various directions, [7]. Thus one can consider other stability regions, among which the unit circle will be of significant interest (see Section IV). Also, one could treat polynomials with zeros to the left and to the right of the imaginary axis, and relate expressions such as that on the left of (6) to the zero distribution, without necessarily being able to obtain it exactly.

III. Strict Positive Realness Test

In this section, we shall establish the following result.

Theorem 3.1: Let $p(s), q(s)$ be real polynomials of degree d, with positive leading coefficients. Then $p(s)/q(s)$ is strictly positive real if and only if there exist frequencies $\omega_o = 0 < \omega_1 < \cdot < \omega_N = \infty$ such that

$$0 < \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} \le \frac{\pi}{4} \tag{11}$$

$$0 < \arg \frac{q(j\omega_k)}{q(j\omega_{k-1})} \le \pi/4 \tag{12}$$

$$\sum_{k=1}^{N} \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} = \frac{d\pi}{2} \tag{13}$$

$$\sum_{k=1}^{N} \arg \frac{q(j\omega_k)}{q(j\omega_{k-1})} = \frac{d\pi}{2} \tag{14}$$

$$-\pi/2 < \arg \frac{p(j\omega_{k-1})}{q(j\omega_k)} < \pi/2 \tag{15}$$

$$-\pi/2 < \arg \frac{p(j\omega_k)}{q(j\omega_{k-1})} < \pi/2 \tag{16}$$

Proof: Assume $p(s)/q(s)$ is SPR. This implies $p(s)$ and $q(s)$ are stable. Then it is possible to find $\omega_0, \omega_1, \cdots$, with (11) and (12) holding, just because $\arg p(j\omega)$ and $\arg q(j\omega)$ are continuous and monotone increasing in ω, and are always well defined; also, (15) and (16) follow by the continuity of $p(j\omega)/q(j\omega), p(j\omega)$ and $q(j\omega)$, and the fact that (due to the SPR property) $\arg \frac{p(j\omega)}{q(j\omega)} \in (-\pi/2, \pi/2)$ for all ω. Last, (13) and (14) follow from the stability of q (part of the SPR definition) and of p (an easy and standard consequence of the SPR definition).

To prove the contrary, observe that from (12) and (14), we can conclude that $q(\cdot)$ is stable [as is $p(\cdot)$, from (11) and (13)]. Also, with the equality following from the Corollary of Section II and the inequalities from stability and the theorem hypotheses

$$0 \le \Delta^{\omega}_{\omega_{k-1}} \arg p \le \Delta^{\omega_k}_{\omega_{k-1}} \arg p = \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} \le \pi/4, \quad \omega \in [\omega_{k-1}, \omega_k]$$

$$0 \le \Delta^{\omega_k}_{\omega} \arg q \le \Delta^{\omega_k}_{\omega_{k-1}} \arg q = \arg \frac{q(j\omega_k)}{q(j\omega_{k-1})} \le \pi/4, \quad \omega \in [\omega_{k-1}, \omega_k]$$

Now observe the three inequalities

$$-\pi/2 < \arg \frac{p(j\omega_{k-1})}{q(j\omega_k)} < \pi/2 \tag{17}$$

$$0 \le \Delta^{\omega}_{\omega_{k-1}} \arg p \le \pi/4 \qquad \omega \in [\omega_{k-1}, \omega_k] \tag{18}$$

$$0 \le \Delta^{\omega_k}_{\omega} \arg q \le \pi/4 \qquad \omega \in [\omega_{k-1}, \omega_k] \tag{19}$$

yield, on summation

$$-\pi/2 < \arg \frac{p(j\omega)}{q(j\omega)} < \pi \tag{20}$$

Likewise, we can sum the inequality

$$-\pi/2 < \arg \frac{p(j\omega_k)}{q(j\omega_{k-1})} < \pi/2 \tag{21}$$

with the easily established inequalities

$$-\pi/4 \le \Delta^{\omega}_{\omega_k} \arg p \le 0 \qquad \omega \in [\omega_{k-1}, \omega_k] \tag{22}$$

$$-\pi/4 \le \Delta^{\omega_{k-1}}_{\omega} \arg q \le 0 \qquad \omega \in [\omega_{k-1}, \omega_k] \tag{23}$$

We obtain on summation

$$-\pi < \arg \frac{p(j\omega)}{q(j\omega)} < \pi/2 \tag{24}$$

From (20) and (24), we obtain

$$-\pi/2 < \arg \frac{p(j\omega)}{q(j\omega)} < \pi/2 \tag{25}$$

and the SPR property is established.

As noted earlier in the introduction $p(s)/q(s)$ can be SPR with $\deg p$ and $\deg q$ differing by zero or ± 1. In case the two degrees are not equal, the statement of Theorem 3.1 needs to be amended so that the strict inequalities in (17) and (16) with $k = N$ are replaced by nonstrict inequalities. The proof is changed marginally. We shall consider the case when $\deg q = \deg p + 1$. Then (17) and (18, 19) become for $k = N$ and $\omega_{N-1} \le \omega < \infty$

$$-\pi/2 \le \arg \frac{p(j\omega_{N-1})}{q(j\omega_N)} \le \pi/2$$

$$0 \le \Delta^{\omega}_{\omega_{N-1}} \arg p \le \pi/4$$

$$0 < \Delta^{\omega_N}_{\omega} \arg q \le \pi/4$$

when

$$-\pi/2 < \arg \frac{p(j\omega)}{q(j\omega)} \le \pi \tag{26}$$

Similarly, the following modification of (16) is established by slight adjustment of the preceding inequalities:

$$-\pi \leq \arg \frac{p(j\omega)}{q(j\omega)} < \pi/2 \tag{27}$$

From (26) and (27) comes the SPR condition (2).

Notice that the integers N and d are necessarily related by

$$N \geq 2d \tag{28}$$

in distinction to the condition for testing stability, which was $N > d/2$, see (8).

IV. Discrete-Time (Unit Circle) Results

The following results can be achieved by the same arguments as led to Theorem 2.1 and 3.1. Proofs are omitted.

Theorem 4.1a: Let $p(z)$ be a dth degree real polynomial in z. Then a necessary and sufficient condition that all roots of $p(z)$ lie in $|z| < 1$ is that there exist frequencies $0 = \omega_o < \omega_1 < \cdots < \omega_N = \pi$ such that

$$0 < \arg \frac{p(\exp j\omega_k)}{p(\exp j\omega_{k-1})} < \pi \tag{29}$$

$$\sum_{k=1}^{N} \arg \frac{p(\exp j\omega_k)}{p(\exp j\omega_{k-1})} = d\pi \tag{30}$$

Note that for (29) and (30) to hold, it is necessary that $N > d$, in distinction to the continuous time problem. Since discrete time transfer functions and polynomials often use z^{-1} as the variable, we note the following variant of Theorem 4.1a:

Theorem 4.1b: Let $p(z^{-1})$ be a dth degree real polynomial in z^{-1} with a nonzero constant term. Then a necessary and sufficient condition that $p(z_0^{-1}) = 0$ implies $|z_0| < 1$ is that there exist frequencies $0 = \omega_0 < \omega_1 \cdots < \omega_N = \pi$ such that

$$0 < \left[d(\omega_k - \omega_{k-1}) + \arg \frac{p(\exp -j\omega_k)}{p(\exp -j\omega_{k-1})}\right] < \pi \tag{31}$$

and

$$\sum_{k=1}^{N} \arg \frac{p(\exp -j\omega_k)}{p(\exp -j\omega_{k-1})} = 0 \tag{32}$$

where $\arg \frac{p(\exp -j\omega_k)}{p(\exp -j\omega_{k-1})}$ is assigned a value in the set $(-\pi - d(\omega_k - \omega_{k-1}), \pi - d(\omega_k - \omega_{k-1})]$.

Corresponding to Theorem 3.1, we have

Theorem 4.2: Let $p(z), q(z)$ be real polynomials of degree d in z. Then $p(z)/q(z)$ is strictly positive real if and only if there exist frequencies $\omega_o = 0 < \omega_1 < \cdots < \omega_N = \pi$ such that

$$0 < \arg \frac{p(\exp j\omega_k)}{p(\exp j\omega_{k-1})} \leq \pi/4 \tag{33}$$

$$0 < \arg \frac{q(\exp j\omega_k)}{q(\exp j\omega_{k-1})} \leq \pi/4 \tag{34}$$

$$\sum_{k=1}^{N} \arg \frac{p(\exp j\omega_k)}{p(\exp j\omega_{k-1})} = d\pi \tag{35}$$

$$\sum_{k=1}^{N} \arg \frac{q(\exp j\omega_k)}{q(\exp j\omega_{k-1})} = d\pi \tag{36}$$

$$-\pi/2 < \arg \frac{p(\exp j\omega_{k-1})}{q(\exp j\omega_k)} < \pi/2 \tag{37}$$

$$-\pi/2 < \arg \frac{p(\exp j\omega_k)}{q(\exp j\omega_{k-1})} < \pi/2 \tag{38}$$

Of particular interest is the case where $p(z)/q(z)$ corresponds to a finite impulse response:

$$\begin{aligned} w(z) &= w_o + w_1 z^{-1} + \cdots + w_d z^{-d} \\ &= \frac{w_o z^d + w_1 z^{d-1} + \cdots + w_d}{z^d} \end{aligned}$$

In this case, $q(z) = z^d$. The conditions (33) and (35) are unchanged. The others specialize to

$$0 \leq \omega_k - \omega_{k-1} \leq \frac{\pi}{4d} \tag{39}$$

$$-\pi/2 < \arg\left[\exp(-jd\omega_k)p(\exp j\omega_{k-1})\right] < \pi/2 \tag{40}$$

$$-\pi/2 < \arg\left[\exp(-jd\omega_{k-1})p(\exp j\omega_k)\right] < \pi/2 \tag{41}$$

Note that (39) imposes a necessary (but not sufficient) condition on the closeness of the frequencies at which p has to be evaluated [The requirement (33) imposes another condition, which may or may not be met when (39) is used]. In particular, (39) implies that $N \geq 4d$. The real dth degree polynomial $p(\cdot)$ can be interpolated from knowledge of its values at 2 real points and $\frac{1}{2}(d-1)$ complex points, but when it comes to checking the positive realness of $p(z^{-1})$ one must consider at least two real points and $4d-1$ complex points, ie. about 8 times as many points.

V. Conclusion

In this paper, we have clarified the statement of a result on the checking of polynomial stability via examination of the polynomial values at a finite set of points, and we have extended the ideas to present tests for strict positive realness. There are a number of directions in which the results could be further extended. In the stability area, these could include zero distribution determination, degree of stability verification, generation of Lyapunov functions (say for a matrix constructed with interpolated values of a polynomial $p(s)$), and inclusion of interpolation data off the imaginary axis. Further, if a polynomial has a prescribed degree of stability, it is probably possible to derive insights into the spacing of the ω_i that would assist in their choice. A number of these ideas could also be taken up for the SPR problem. In addition, one could write down (fairly easily in fact) a strictly bounded real condition using data at discrete points.

References

[1] B. D. O. Anderson, "A simplified viewpoint of hyperstability," *IEEE Trans. Automat. Contr.*, vol. AC-13, pp. 292–294, 1968.

[2] I. D. Landau, *Adaptive Control - The Model Reference Approach.* New York: Dekken, 1979.

[3] B. D. O. Anderson *et al.*, *Stability Adaptive Systems: Passivity and Averaging Analysis.* Cambridge, MA: MIT, 1986.

[4] B. R. Barmish, *New Tools for Robustness of Linear Systems.* New York: McMillan, 1994.

[5] A. Rantzer, "Parameter uncertainty and feedback complexity in linear control systems," Rep. TRITA-MAT-91-0035, Royal Inst. Technol., Stockholm, Oct. 1991

[6] ——, "A finite zero exclusion principle," in *Robust Control of Linear Systems*, M. A. Kaashoek, J. H. van Schuppen, and A. C. M. Ran, Eds. Cambridge, MA: Birkhauser, 1990, pp. 239-245.
[7] R. D. Kaminsky and T. E. Djaferis, "The finite inclusions theorem," in *Proc. 32nd IEEE Conf. Decision and Control*, San Antonio, TX, 1993, pp. 508-518.
[8] ——, "A novel approach to the analysis and synthesis of controllers for parametrically uncertain systems," *IEEE Trans. Auto. Contr.*, vol. 39, pp. 874-876, 1994.

Effect of Loop Delay on Stability of Discrete-Time PLL

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***Abstract*—The stability regions of the first- and second-order discrete-time PLL in the presence of a loop delay are analyzed. The region of the second-order PLL is compared with that of its continuous-time counterpart.**

I. Introduction

Digital PLL's inherently have a discrete-time (DT) nature and often contain a loop delay. This delay may take the form of latency that is introduced to facilitate implementation at high sampling rates [7], [5], [2], or it may form an integral part of the phase detector, as in certain timing recovery schemes [8], [9]. In both cases the presence of the delay will restrict the stability region of the PLL. This restriction can be critical to system performance when loop bandwidths are high, as they often are during acquisition [2]. It is exactly in these circumstances that the behavior of the DT PLL cannot readily be approximated by that of its continuous-time (CT) counterpart [7]. Past results regarding the stability range of CT PLL's [10], [6], [1] are, therefore, not necessarily relevant to DT PLL's.

In this paper we analyze the stability region of the first- and second-order DT PLL in the presence of a loop delay of $M \geq 0$ sampling intervals. The analysis is an extension of one that was used by Kabal to assess the effect of delayed adjustment in adaptive filters [4]. These filters are similar to the first-order DT PLL, and the stability range admits a similar analytic description (Section II). The second-order PLL is of greater practical interest than the first-order PLL because of its ability to handle frequency errors with no steady-state phase error [6]. For this PLL we derive a parametric description of the stability region (Section III) and we compare this region with that of the second-order CT PLL as given by Lindsey [6] (Section IV).

For the sake of completeness we mention that the statistical effects of a loop delay on the first- and second-order DT PLL have been studied in [5] for the specific case $M = 1$. That paper, however, is not concerned with stability regions.

II. First-Order PLL

Fig. 1 depicts the phase-domain model of a first-order DT PLL with a loop delay of M sampling intervals. The input phase ϕ_k is tracked by a VCO that is modeled as an ideal integrator with output phase

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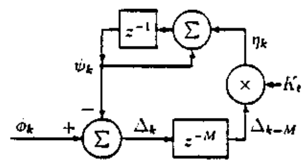

Fig. 1. Phase-domain model of first-order PLL with a loop delay of M sampling intervals.

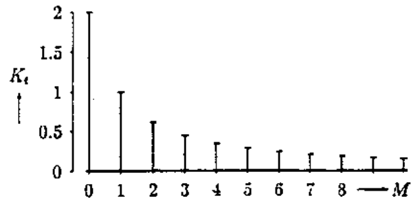

Fig. 2. Stability range of first-order DT PLL as a function of the loop delay M.

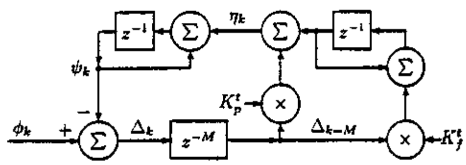

Fig. 3. Phase-domain model of high-gain second-order PLL with a loop delay of M sampling intervals.

ψ_k. The operator z^{-1} denotes a delay of one sampling interval. The phase error $\Delta_k \triangleq \phi_k - \psi_k$ serves as the VCO input η_k after being delayed across M sampling intervals and scaled by the total open-loop gain K_t. Frequency offsets and additive noise are immaterial to stability ranges and are therefore not modeled. For $M = 0$ there is still a delay of one sampling interval in the loop. This delay is due to the integrator and is needed to keep the loop physically realizable. In Appendix A it is shown that the loop is stable provided that

$$0 \leq K_t \leq 2 \sin \frac{\pi}{2(2M+1)}. \tag{1}$$

This expression is equivalent to the one of [4] for the stability range of the LMS adaptive filter with delayed adjustment. Ranges are depicted as vertical bars in Fig. 2. For $M = 1$ the stability range is already halved with respect to a loop without delay ($M = 0$). For $M \gg 1$ the upper edge of the range is well approximated by $K_t \simeq \pi/(2M+1)$.

III. Second-Order PLL

For brevity we restrict attention to the high-gain second-order PLL (Fig. 3). Stability regions of other second-order PLL's are identical because only the numerator of the closed-loop transfer function differs [7]. The loop filter now has a proportional and an integrating path with total open-loop gains K_p^t and K_f^t, respectively. Any gain of the phase detector and/or VCO is accommodated in K_p^t and K_f^t. Except for the loop filter the PLL model is identical to that of Fig. 1. Stability regions are analyzed in Appendix B and are depicted in Fig. 4 for various delays M. Even a small loop delay reduces the stability region rather dramatically as compared to a loop without delay ($M = 0$). For the sake of a comparison with the second-order CT PLL we will rephrase these regions in terms of parameters ω_n^d