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Error bound for transfer function order reduction using frequency weighted balanced truncation†

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Abstract

An error bound for transfer function order reduction is derived, when frequency weighted balanced truncation is the order reduction method. The bound is valid for both one-sided (input or output) and two-sided weighted balancing approximations with stable weights, which can otherwise be arbitrary. The error bound formula is valid for both discrete-time and continuous-time problems. Examples are studied to demonstrate effectiveness of the error bound.

Keywords: Controller order reduction; Balanced truncation; Frequency domain error bound; Frequency weighting

1. Introduction

Controller design methods for physical systems with high-order models normally result in a high-order controller and, for many reasons, it is desirable to reduce the controller order, i.e. to find a controller of a lower-order, performing satisfactorily in certain sense. Examples of these performance criteria include (but are not limited to) preserving of the closed-loop stability robustness and closed-loop transfer function [1]. All the controller reduction problems aimed at achieving these goals can be stated as problems of frequency weighted transfer function order reduction with the frequency weighting implying that it is important for the reduced order controller to approximate the original full order controller better at some frequencies, than at others.

One of the methods for order reduction in the frequency weighted case is the frequency weighted balanced reduction technique.

It is highly desirable to be able to predict an error in approximating an original full order controller by a reduced one, or at least to know an error bound. If such a prior error bound is known, it allows an intelligent estimation of the effect of a given degree reduction. For example, given a maximum acceptable error value, one may be able to determine a minimum order the controller can be reduced to, or, given

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a desired order of a reduced controller, one can predict an error this reduction brings (or at least bound it). This makes trade-off between the order of a reduced controller and the related approximation error easier to deal with.

Furthermore, knowledge of a prior error formula might allow one to compare alternative approaches to controller reduction, employing different frequency weightings aimed at achieving different objectives. For example, it allows one to optimise a trade-off between a stability margin achieved and the closed-loop transfer function being preserved, both possible objectives of the order reduction process [1].

Lower and, more importantly, upper frequency domain error bounds for the balanced truncation approximation in the nonweighted case are well known and have been described in [2, 3, 4]. However, no error bound formula has been available for the balanced truncation frequency-weighted problem.

The contribution of this paper is that an upper error bound for frequency weighted balanced controller reduction is obtained. The bound is valid for both one-sided (input or output) and two-sided weighted balancing approximation for stable weights which can otherwise be arbitrary. The error bound formula is valid for both discrete-time and continuous-time problems.

An outline of this paper is as follows. In Section 2 the algorithm for weighted balanced reduction is reviewed. The main result, the error bound formula itself, is presented in Section 3. An example (showing tightness of the bound) is given in Section 4, followed by some concluding remarks in Section 5.

2. Background

In this section the algorithm for nonweighted balanced reduction and the error bound are reviewed. Also, the algorithm for balanced weighted reduction is recalled.

Let us consider a stable transfer function K, given by a minimal state-space realization:

$$K(s) = C(sI - A)^{-1}B + D. \tag{2.1}$$

Also, consider a stable input weight $V(s)$ and a stable output weight $W(s)$, realized in their minimal state-space form as

$$V(s) = C_V(sI - A_V)^{-1}B_V + D_V, \tag{2.2}$$

$$W(s) = C_W(sI - A_W)^{-1}B_W + D_W. \tag{2.3}$$

Then the weighted reduction problem is to find a stable lower-order transfer function K_r (of order r), such that the norm $\| W(s)[K(s) - K_r(s)]V(s)\|_\infty$ is minimal, or at least is approximately minimal.

A key application of interest is when $K(s)$ is a controller and $V(s)$ and $W(s)$ are obtained by one of the methods described in [1]. (In a number of the methods of [1], $V(s)$ or $W(s)$ is the identity.) Such a controller may be open loop unstable; the scheme presented here is restricted to reducing the order of the stable part of $K(s)$, to yield the stable part of $K_r(s)$; the unstable part of $K(s)$ is copied with $K_r(s)$.

Let us recall first the nonweighted case.

Definition. Given an nth order, linear time invariant, asymptotically stable system with transfer function matrix $K(s)$, a minimal realization of $K(s) = C(sI - A)^{-1}B + D$ is internally balanced if $\{A, B, C\}$ satisfy the following Lyapunov equations

$$A\Lambda^{\mathrm{T}} + \Lambda A + BB^{\mathrm{T}} = 0, \tag{2.4a}$$

$$\Lambda A + A^{\mathrm{T}}\Lambda + C^{\mathrm{T}}C = 0 \tag{2.4b}$$

and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \text{where } \lambda_i \geqslant \lambda_{i+1} > 0, \quad i = 1, 2, \ldots, n-1. \tag{2.5}$$

In Eq. (2.4a), Λ is the controllability gramian, and in (2.4b), Λ is the observability gramian. Thus, a system is balanced when its controllability and observability gramians are equal and have a diagonal form.

Partition the system $\{A, B, C\}$ and Λ as

$$A = \begin{pmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_r \\ B_2 \end{pmatrix}, \quad C = (C_r \quad C_2), \quad \Lambda = \begin{pmatrix} \Lambda_r & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \tag{2.6}$$

where $A_r, \Lambda_r \in \mathbb{R}_{r\times r}$, $B_r \in \mathbb{R}_{r\times p}$, $C_r \in \mathbb{R}_{m\times r}$ and $r < n$. Then the reduced-order system $\{A_r, B_r, C_r\}$ is a good approximation of the system $\{A, B, C\}$ if $\lambda_r \gg \lambda_{r+1}$. In fact, the following two properties are true:

Lemma 2.1 (Pernebo and Silverman [6]). *For a balanced asymptotically stable system $\{A, B, C\}$ satisfying* (2.1), *and with Λ in the form of* (2.5) *satisfying* (2.4) *and partitioned as in* (2.6), *if $\lambda_r > \lambda_{r+1}$, then both subsystems $\{A_r, B_r, C_r\}$ and $\{A_{22}, B_2, C_2\}$ are asymptotically stable.*

Lemma 2.2 (Enns, Glover [2,4]). *With the same hypothesis as Lemma* 2.1, *there holds a frequency error bound*

$$\| C(j\omega I - A)^{-1}B - C_r(j\omega I - A_r)^{-1}B_r \|_\infty \leqslant 2(\lambda_{r+1} + \cdots + \lambda_n) = 2\,\mathrm{tr}(\Lambda_2). \tag{2.7}$$

Now, consider asymptotically stable frequency-weighting functions and associated minimal state-variable realizations $W(s) = C_w(sI - A_w)^{-1}B_w + D_w$ and $V(s) = C_v(sI - A_v)^{-1}B_v + D_v$. The basic idea is to change the gramians to reflect the introduction of the frequency weighting, to diagonalize these "weighted" gramians and then to truncate.

The frequency-weighted transfer function $W(s)K(s)V(s)$ has a representation with the following state-space matrices:

$$\bar{A} = \begin{pmatrix} A_W & B_W C & B_W D C_V \\ 0 & A & B C_V \\ 0 & 0 & A_V \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B_W D D_V \\ B D_V \\ B_V \end{pmatrix}, \quad \bar{C} = (C_W \quad D_W C \quad D_W D C_V), \quad \bar{D} = D_W D D_V.$$

Let

$$\bar{P} = \begin{pmatrix} P_W & P_{12} & P_{13} \\ P_{12}^{\mathrm{T}} & P & P_{23} \\ P_{13}^{\mathrm{T}} & P_{23}^{\mathrm{T}} & P_V \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} Q_W & Q_{12} & Q_{13} \\ Q_{12}^{\mathrm{T}} & Q & Q_{23} \\ Q_{13}^{\mathrm{T}} & Q_{23}^{\mathrm{T}} & Q_V \end{pmatrix}$$

be the solutions of the following Lyapunov equations:

$$\overline{PA}^{\mathrm{T}} + \overline{AP} + \overline{BB}^{\mathrm{T}} = 0, \tag{2.8a}$$

$$\overline{QA} + \bar{A}^{\mathrm{T}}\bar{Q} + \bar{C}^{\mathrm{T}}\bar{C} = 0. \tag{2.8b}$$

Now, P and Q can be regarded as the frequency-weighted controllability and observability gramians for the original transfer function $K(s)$. For later reference, we note that P_V is determined for A_V, B_V alone and Q_W is determined for A_W, C_W alone.

Consider a coordinate basis change to $\{A, B, C\}$ which makes $P_{\mathrm{new}} = Q_{\mathrm{new}} = \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_i \geqslant \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$. This new realization $\{A, B, C\}$ is called a frequency-weighted balanced realization. (The coordinate basis change is easy to determine.)

Partition $\{A, B, C\}$ as in (2.6) and Σ as

$$\Sigma = \begin{pmatrix} \Sigma_r & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{2.9}$$

where $\Sigma_r \in \mathbb{R}_{r\times r}$ and $r < n$.

Now, as previously, the reduction is achieved by eliminating the rows and columns of A, B and C corresponding to Σ_2 in Σ. The reduced order transfer function of order r is given by

$$K_r = C_r(sI - A_r)^{-1}B_r + D. \tag{2.10}$$

A detailed, computer oriented description of this weighted balanced truncation algorithm is given in [5].

3. Main result

The major aim of this section is to derive an upper error bound for balanced frequency-weighted controller reduction. This result is stated in the following theorem.

Theorem 3.1. *Let $K(s)$, $V(s)$ and $W(s)$ be a stable transfer function of order n and stable weighting functions, respectively. The minimal state-space realizations are given by* (2.1), (2.2) *and* (2.3), *respectively. Also, let $K_r(s)$ (given by* (2.10)) *be a reduced order transfer function of order r, obtained by the frequency-weighted (with the weights $V(s)$ and $W(s)$) balanced reduction technique. Assume that K_r is stable, which is guaranteed if $V(s)$ or $W(s)$ is constant, see* [2, 3]. *Then the following error bound holds [compare with* (2.7)]:

$$\| W(s)[K(s) - K_r(s)]\, V(s) \|_\infty \leqslant 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k^2 + (\alpha_k + \beta_k)\sigma_k^{3/2} + \alpha_k \beta_k \sigma_k},$$

where

$$\alpha_k = \| \Xi_{k-1} \|_\infty \, \| C_V \Phi_V P_V^{1/2} \|_\infty \quad \text{and} \quad \beta_k = \| Q_W^{1/2} \Phi_W B_W \|_\infty \, \| \Gamma_{k-1} \|_\infty$$

and

$$\Xi_{k-1}(s) = A_{21}^{k-1} \phi_{k-1}(s) B_{k-1} + b_k, \qquad \Gamma_{k-1}(s) = C_{k-1} \phi_{k-1}(s) A_{12}^{k-1} + c_k,$$

$$\phi_{k-1}(s) = (sI - A_{k-1})^{-1}, \qquad \Phi_W(s) = (sI - A_W)^{-1}, \qquad \Phi_V(s) = (sI - A_V)^{-1},$$

$$A_k = \begin{pmatrix} A_{k-1} & A_{12}^{k-1} \\ A_{21}^{k-1} & a_{kk} \end{pmatrix}, \qquad B_k = \begin{pmatrix} B_{k-1} \\ b_k \end{pmatrix}, \qquad C_k = (C_{k-1} \;\; c_k)$$

and b_k and c_k are the kth row of B_k and the kth column of C_k, respectively, and $A_n = A$, $B_n = B$, $C_n = C$.

For proof, see the appendix.

Remark 3.2. Since the L_∞ bound on the error is expressed in terms of other L_∞ bounds, it might be thought that the advantage of the bound is minor. Several points should however be noted.

- The order of the transfer functions Ξ_{k-1} and Γ_{k-1} (viz. $k-1$ for $k = r+1, \ldots, n$) will often be much less than that of $W(s)[K(s) - K_r(s)]V(s)$, viz. $(n + r) + \deg W + \deg V$. Accordingly, the L_∞ bounds will be much easier to compute.
- The transfer functions $C_V \Phi_V P_V^{1/2}$ and $Q_W^{1/2} \Phi_W B_W$ are independent of $K(s)$, depending just on the weights $V(s)$ and $W(s)$, and so their norms only need to be computed once.
- In the light of the above points, the bound formula lends itself to easy examination of a number of different trial values for r, leading to a subsequent selection.

Remark 3.3. The parameters α_k and β_k are finite. Indeed, $\| \Xi_{k-1} \|_\infty$ and $\| \Gamma_{k-1} \|_\infty$ are finite since the reduced controller is stable. Also, $\| \Phi_V \|_\infty$ is bounded since the weight V is stable and the unique solution P_V of the 3–3 block of Lyapunov equation (2.8a) is bounded since (A_V, B_V) is a controllable pair. Therefore, $\| C_V \Phi_V P_V^{1/2} \|_\infty$ is bounded. Furthermore, $\| C_V \Phi_V P_V^{1/2} \|_\infty$ depends on the input weight V only since P_V depends on A_V and B_V only. Similarly, $\| Q_W^{1/2} \Phi_W B_W \|_\infty$ is bounded and depends on the output weight W only.

It is easy to check also that the quantities $C_V \Phi_V P_V^{1/2}$ and $Q_W^{1/2} \Phi_W B_W$ do not depend on the coordinate basis choice for $V(s)$ and $W(s)$.

We can actually express a bound on $\| C_V \Phi_V P_V^{1/2} \|_\infty$ by using the upper bound of the solution P_V of the Lyapunov equation. As has been shown in [6] and [8], $P_V \leqslant \| G \|_2 S S^{\mathrm{T}}$, where G is a positive definite symmetric matrix dependent on A_V only, and $S = [B_V \, A_V B_V \ldots A_V^{N_V - 1} B_V]$ is the controllability matrix of

$\{A_V, B_V\}$. Thus, the norm $\| C_V \Phi_V P_V^{1/2} \|_\infty$ can be bounded in terms of A_V, B_V and C_V as follows:

$$\| C_V \Phi_V P_V^{1/2} \|_\infty \leqslant \| G \|_2^{1/2} \| C_V \Phi_V S \|_\infty.$$

A similar result can be derived for a bound of the norm $\| Q_W^{1/2} \Phi_W B_W \|_\infty$ in terms of A_W, B_W and C_W.

We can also argue that if a different frequency weighted balanced realization is used, the same norm $\| \Xi_{k-1} \|_\infty$ results. Indeed, the frequency-weighted balanced realization is unique to within a sign change of a state variable when the singular values of the balanced gramian are distinct. Different balanced realizations are related by a transformation $T_k = \mathrm{diag}[t_i, i = 1, 2, \ldots, k]$ and $t_i = \pm 1$. It is not hard to conclude from this that $\| \Xi_{k-1} \|_\infty$ is invariant under a transformation T_k.

Actually, examples suggest that $\| \Xi_{n-1} \|_\infty$ and $\| \Gamma_{n-1} \|_\infty$ are bounded by quantities proportional to $\sqrt{\sigma_n}$ as $\sigma_n \to 0$. However, a proof of this has yet to be established. If true, the bound formula of the theorem would depend on σ_k linearly, for $k = r + 1, \ldots, n$.

Many frequency-weighted approximation problems have just a one-sided weighting. Then the main result becomes

Corollary 3.4. *Let $K(s)$, $V(s)$ and $W(s)$ be a stable controller of order n and stable weighting functions. Then a reduced order transfer function $K_r(s)$ of order r, obtained by single-sided frequency weighting (with either input weight $V(s)$ or output weight $W(s)$) is stable (see [3]) and the following error bounds are true:*

$$\| [K(s) - K_r(s)] V(s) \|_\infty \leqslant 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k^2 + \alpha_k \sigma_k^{3/2}},$$

$$\| W(s) [K(s) - K_r(s)] \|_\infty \leqslant 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k^2 + \beta_k \sigma_k^{3/2}}.$$

Remark 3.5. In the nonweighted case, when $W(s) = V(s) = I$, $\| K(s) - K_r(s) \|_\infty \leqslant 2 \sum_{k=r+1}^{n} \sigma_k$.

4. Examples

We now present three examples to illustrate how the bound on the weighted controller reduction error, obtained in accordance with the above theorem, compares to the actual weighted controller reduction error.

4.1. Example 1

This example has been studied in [2]. The stable controller to be reduced is given by its transfer function

$$K(s) = (s^2 + 2.8s + 1.6)/(s^3 + 2.9s^2 + 3.1s + 1.5).$$

The stable input weighting is in the form

$$V(s) = (s^3 + 2.9s^2 + 3.1s + 1.5)/(s^3 + 3.8s^2 + 4.4s + 1.6)$$

and there is no output weighting.

The weighted Hankel singular values are (0.53999, 0.12355, 0.0042758) and the reduced controllers of order 2 and 1 are

$$K_2(s) = 1.0135(s + 1.1373)/(s^2 + 1.3384s + 1.0715) \quad \text{and} \quad K_1(s) = 1.1694/(s + 0.83068),$$

respectively.

The actual weighted controller reduction errors are

$$E_2 = \| [K(s) - K_2(s)] V(s) \|_\infty = 0.0085342 \quad \text{and} \quad E_1 = \| [K(s) - K_1(s)] V(s) \|_\infty = 0.31977$$

for the reduced controllers of order 2 and 1 respectively.

When we estimate these values by calculating upper bounds of the errors using Theorem 3.1, we have:

$$\| C_V \Phi_V(j\omega) P_V^{1/2} \|_\infty = 0.31911, \qquad \| \Xi_2(j\omega) \|_\infty = 0.18476, \qquad \| \Xi_1(j\omega) \|_\infty = 0.75851,$$

$$\alpha_2 = 0.24205, \qquad \alpha_3 = 0.058959$$

and the bounds are

$$\bar{E}_2 = 2(\sigma_3^2 + \alpha_3 \sigma_3^{3/2})^{1/2} = 0.011793, \qquad \bar{E}_1 = \bar{E}_2 + 2(\sigma_2^2 + \alpha_2 \sigma_2^{3/2})^{1/2} = 0.33290.$$

Comparing the actual error values with their calculated upper bounds, we can see that the bound differs 4% from its actual value for the reduced controller of order 1, and the difference is 38% for the reduced controller of order 2. Thus, we can say that the approximated values are close to the real ones.

4.2. Example 2

As a second example, we consider a plant, given by its transfer function

$$G(s) = (s + 0.8)(s + 2)/(s + 1.5)(s^2 + 1.4s + 1)$$

or, in a state-space form

$$A_g = \begin{pmatrix} -2.9 & -3.1 & -1.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B_g = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad C_g = (1 \quad 2.8 \quad 1.6), \qquad D_g = 0.$$

An LQG compensator was designed to control this plant. There were used state weighting $Q_C = I_3 + 100 C_g^{\mathrm{T}} C_g$, control weighting $R_C = 1$, state noise covariance $Q_f = I_3 + 100 B_g B_g^{\mathrm{T}}$ and measurement noise covariance $R_f = 1$.

The design procedure resulted in the controller

$$K(s) = 10.3544(s + 1.86183)(s + 0.745649)/(s + 19.8229)(s + 2.00134)(s + 0.800627).$$

Weighting functions, chosen to preserve (as far as possible) the closed-loop transfer function (see e.g. [1, 5]) are in the form:

- input weighting

$$V(s) = \frac{(s + 0.80062709)(s + 1.5)(s + 2.00134)(s + 19.8229)(s + 1.4s + 1)}{(s + 0.800687)(s + 1.30002)(s + 2.00147)(s + 19.279)(s^2 + 2.14368s + 1.75884)},$$

- output weighting

$$W(s) = \frac{(s + 19.8229)(s + 2.00134)(s + 2)(s + 0.800627)(s + 0.8)}{(s + 0.800687)(s + 1.30002)(s + 2.00147)(s + 19.279)(s^2 + 2.14368s + 1.75884)}.$$

(Because $K(s)$ is scalar, we could work with a single-sided weighting $V(s)\,W(s)$. However, our goal here is to indicate the effect of two-sided weighting.)

The weighted Hankel singular values σ are (0.052428, 0.011097, 0.00048095) and the weighted balanced reduced controllers of order 1 and 2 are

$$K_1(s) = 10.372/(s + 21.312) \quad \text{and} \quad K_2(s) = (10.384s + 11.916)/(s^2 + 21.299s + 26.205),$$

respectively.

The poles of the transfer functions with the controllers of reduced order are

$$(-19.4486,\ -1.0836 \pm 0.7547\mathrm{j},\ -1.2917 \pm 0.2295\mathrm{j}) \quad \text{and} \quad (-20.8123,\ -1.29,\ -1.0546 \pm 0.8346\mathrm{j})$$

for the loops with the 2nd order and 1st order controllers respectively. Thus, the closed loops are stable.

The actual weighted balanced controller reduction errors are

$$E_1 = \| W(s)[K(s) - K_1(s)]V(s)\|_\infty = 0.016581 \quad \text{and} \quad E_2 = \| W(s)[K(s) - K_2(s)]V(s)\|_\infty = 0.0010472$$

for the reduced controllers of order 1 and 2, respectively.

When we estimate these values by calculating upper bounds of the errors using Theorem 3.1, we have

$$\| C_V \Phi_V(j\omega) P_V^{1/2} \|_\infty = 0.22893, \qquad \| Q_W^{1/2} \Phi_W(j\omega) B_W \|_\infty = 0.0023564,$$

$$\| \Xi_1(j\omega) \|_\infty = 0.20246, \qquad \| \Xi_2(j\omega) \|_\infty = 0.054046, \qquad \| \Gamma_1(j\omega) \|_\infty = 2.8548, \qquad \| \Gamma_2(j\omega) \|_\infty = 0.8166,$$

$$\alpha_2 = 0.046348, \qquad \alpha_3 = 0.012373, \qquad \beta_2 = 0.006727, \qquad \beta_3 = 0.0019243$$

and the bounds are

$$\bar{E}_2 = 2(\sigma_3^2 + (\alpha_3 + \beta_3)\sigma_3^{3/2} + \alpha_3\beta_3\sigma_3)^{1/2} = 0.0012547,$$

$$\bar{E}_1 = \bar{E}_2 + 2(\sigma_2^2 + (\alpha_2 + \beta_2)\sigma_2^{3/2} + \alpha_2\beta_2\sigma_2)^{1/2} = 0.029353.$$

Comparing the actual error values with their calculated upper bounds, we can see that the bound differs 77% from its actual value for the reduced controller of order 1, and the difference is 80% for the reduced controller of order 2. Thus, we can conclude that the bound in the double-side weighted case is not as tight as for single-side weighting (Example 1). But, anyway, the approximation is very good.

4.3. Example 3

The third example deals with a controller having its poles close to the $j\omega$-axis.

Let us consider the stable controller to be reduced, given by its transfer function

$$K(s) = (s^2 + 2.8s + 1.6)/(s^5 + 2.911s^4 + 3.1319s^3 + 1.5341s^2 + 0.01653s + 0.000015).$$

The poles are $\{-1.5, -0.7 \pm j0.71414, -0.01, -0.001\}$. Thus two poles approach the $j\omega$-axis very closely.

The stable input weighting is given in the form $V(s) = K^{-1}(s)/[(s+1)^2(s+2)]$.

The weighted Hankel singular values σ are $\{797.19, 1.6265, 0.07408, 0.0004583\}$ and the weighted balanced reduced controllers of order 1, 2, 3 and 4 are

$$K_1(s) = -0.151/(s + 2.21 \cdot 10^{-9}), \quad K_2(s) = (-0.154s + 0.876)/(s^2 + 0.00742s + 9.98 \cdot 10^{-6}),$$

$$K_3(s) = (0.00985s^2 - 0.102s + 1.47)/(s^3 + 1.21s^2 + 0.0171s + 1.21 \cdot 10^{-5}),$$

$$K_4(s) = (0.00145s^3 - 0.0138s^2 + 1.06s + 1.04)/(s^4 + 1.33s^3 + 0.992s^2 + 0.0108s + 9.78 \cdot 10^{-6}),$$

respectively.

The actual weighted balanced controller reduction errors $E_i = \| [K(j\omega) - K_i(j\omega)]V(j\omega)\|_\infty$, $i = 1,2,3,4$ are

$$E_1 = 321.03, \qquad E_2 = 0.13124, \qquad E_3 = 0.06691, \qquad E_4 = 0.0009187.$$

When we estimate these values by calculating upper bounds of the errors using results of Theorem 3.1, we have: $\| C_V \Phi_V(j\omega) P_V^{1/2} \|_\infty = 0.41013$, $\| \Xi_1(j\omega) \|_\infty = 2.5504 \cdot 10^9$, $\| \Xi_2(j\omega) \|_\infty = 9.4683 \cdot 10^4$, $\| \Xi_3(j\omega) \|_\infty = 4.8518 \cdot 10^4$, $\| \Xi_4(j\omega) \|_\infty = 8.0189 \cdot 10^3$, $\alpha_2 = 1.046 \cdot 10^9$, $\alpha_3 = 3.8832 \cdot 10^4$, $\alpha_4 = 1.9899 \cdot 10^4$, $\alpha_5 = 3.2888 \cdot 10^3$ and the bounds are

$$\bar{E}_4 = 2(\sigma_5^2 + \alpha_5\sigma_5^{3/2})^{1/2} = 0.35927, \qquad \bar{E}_3 = \bar{E}_4 + 2(\sigma_4^2 + \alpha_4\sigma_4^{3/2})^{1/2} = 22.203,$$

$$\bar{E}_2 = \bar{E}_3 + 2(\sigma_3^2 + \alpha_3\sigma_3^{3/2})^{1/2} = 78.166, \qquad \bar{E}_1 = \bar{E}_2 + 2(\sigma_2^2 + \alpha_2\sigma_2^{3/2})^{1/2} = 93.239.$$

As we can see, actual errors may be large when the controller has poles close to the imaginary axis. Also in this case the upper bound, derived in Theorem 3.1 and based on the inequality

$$\|\Xi_i(j\omega)C_V\Phi_V(j\omega)P_{23i+1}^{T}\|_\infty \leqslant \|\Xi_i(j\omega)\|_\infty \|C_V\Phi_V(j\omega)P_V^{1/2}\|_\infty\sqrt{\sigma_{i+1}},$$

becomes very conservative.

Let us try to make the bound tighter by using the inequality

$$\|\Xi_i(j\omega)C_V\Phi_V(j\omega)P_{23i+1}^{T}\|_\infty \leqslant \|\Xi_i(j\omega)\|_\infty \|C_V\Phi_V(j\omega)P_{23i+1}^{T}\|_\infty.$$

That gives us the following reduction error bounds:

$$\bar{E}_4^a = 0.015098, \quad \bar{E}_3^a = 3.3371, \quad \bar{E}_2^a = 18.906, \quad \bar{E}_1^a = 51.388.$$

As we can see, this approach gives us better bounds, though still very conservative.

The bounds are worse when $|\Xi_i(j\omega)|$ and $|C_V\Phi_V(j\omega)P_{23i+1}^{T}|$ assume their maximum values at mutually distant frequencies.

Finally, let us try to improve the results by bounding the reduction errors by the norms

$$\|\Xi_i(j\omega)C_V\Phi_V(j\omega)P_{23i+1}^{T}\|_\infty.$$

That gives us extremely tight reduction error bounds:

$$\bar{E}_4^b = 0.00091892, \quad \bar{E}_3^b = 0.074513, \quad \bar{E}_2^b = 0.25299, \quad \bar{E}_1^b = 321.25.$$

In this case the bounds differ from the actual errors by 0.07%, 93%, 11% and 0.02% for reduced controllers of order 1, 2, 3 and 4 respectively.

5. Conclusions

The controller reduction approach discussed in this paper involves a frequency weighted error between the full and reduced order transfer functions. The weighted balanced reduction technique was applied to reduce the order of the transfer function. An error bound formula for frequency weighted balanced truncation was obtained.

Three examples were used to demonstrate the effectiveness of the error bound formula. The first example involves an input weighting, the second one double-sided weighting. The calculated error bounds were close to the actual error values for both examples.

The third example studies the case of a transfer function having its poles close to the imaginary axis. It has been shown that the bound derived in the main theorem of the paper is conservative, but a way to improve the bound has been shown as well. It is widely held that in the unweighted case, the standard bounds are also conservative when poles are close to the $j\omega$-axis.

Issues of interest which remain open are the possibility of improving the error bound formula and of determining cases when the bound is tight/weak. In [2], the question is considered for the unweighted reduction problem of when the bound is tight and weak. It is probably weak when the weighted system's transfer function has alternating poles and zeros almost along the $j\omega$-axis and tight when there are alternating poles and zeros along the negative real axis. This conclusion may carry over to the weighted case. The examples allow no real definitive conclusion on this point, although they tend to support the carry-over hypothesis.

Appendix A. Proof of Theorem 3.1

With the partitions of $\{A, B, C\}$ and Σ as in (2.6) and (2.9), introduce the following notation:

$$\phi_r(s) = (sI - A_r)^{-1}, \quad \Delta_r(s) = sI - A_{22} - A_{21}\phi_r(s)A_{12}, \quad \Xi_r(s) = A_{21}\phi_r(s)B_r + B_2,$$

$\Gamma_r(s) = C_r\phi_r(s)A_{12} + C_2$.

As it has been shown in [3], $K - K_r = \Gamma_r\Delta_r^{-1}\Xi_r$. Thus,

$$\| W(K - K_r)V\|_\infty^2 = \max_\omega \lambda_{\max}[W\Gamma_r\Delta_r^{-1}\Xi_r VV^{\mathrm{H}}\Xi_r^{\mathrm{H}}\Delta_r^{-\mathrm{H}}\Gamma_r^{\mathrm{H}}W^{\mathrm{H}}] = \max_\omega \lambda_{\max}[\Delta_r^{-1}\Pi_r\Delta_r^{-\mathrm{H}}\Theta_r],$$

where $\Pi_r(s) = \Xi_r(s)V(s)V^{\mathrm{H}}(s)\Xi_r^{\mathrm{H}}(s)$ and $\Theta_r(s) = \Gamma_r^{\mathrm{H}}(s)W^{\mathrm{H}}(s)W(s)\Gamma_r(s)$. Let us make the further definition: $\Phi(s) = (sI - A)^{-1}$.

Then, as is shown below, the following formulae are true:

$$\Delta_r^{-1}\Pi_r = \Sigma_2 + [0\ \ I]P_{23}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}\Xi_r^{\mathrm{H}} + \Delta_r^{-1}(\Sigma_2 + \Xi_r C_V\Phi_V P_{23}^{\mathrm{T}}[0\ \ I]^{\mathrm{T}})\Delta_r^{\mathrm{H}},$$

$$\Delta_r^{-\mathrm{H}}\Theta_r = \Sigma_2 + [0\ \ I]Q_{12}^{\mathrm{T}}\Phi_W B_W\Gamma_r + \Delta_r^{-\mathrm{H}}(\Sigma_2 + \Gamma_r B_W^{\mathrm{T}}\Phi_W^{\mathrm{H}}Q_{12}[0\ \ I]^{\mathrm{T}})\Delta_r.$$

Indeed, from the 3–3, 2–3 and 2–2 blocks of the Lyapunov equation (2.8a), one can obtain:

$$P_VA_V^{\mathrm{T}} + A_VP_V + B_VB_V^{\mathrm{T}} = 0,\ P_{23}A_V^{\mathrm{T}} + AP_{23} + BC_VP_V + BD_VB_V^{\mathrm{T}} = 0,$$

$$\Sigma A^{\mathrm{T}} + A\Sigma + P_{23}C_V^{\mathrm{T}}B^{\mathrm{T}} + BC_VP_{23}^{\mathrm{T}} + BD_VD_V^{\mathrm{T}}B^{\mathrm{T}} = 0.$$

Then, using the equations above, one can obtain:

$$\begin{aligned}\Pi_r &= \Xi_r VV^{\mathrm{H}}\Xi_r^{\mathrm{H}} = [A_{21}\phi_r\ \ I]BVV^{\mathrm{H}}B^{\mathrm{T}}[A_{21}\phi_r\ \ I]^{\mathrm{H}}\\ &= [A_{21}\phi_r\ \ I]B\{C_V\Phi_VB_VB_V^{\mathrm{T}}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}} + D_VB_V^{\mathrm{T}}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}} + C_V\Phi_VB_VD_V^{\mathrm{T}} + D_VD_V^{\mathrm{T}}\}B^{\mathrm{T}}[A_{21}\phi_r\ \ I]^{\mathrm{H}}\\ &= [A_{21}\phi_r\ \ I]\{-[P_{23}A_V^{\mathrm{T}} + AP_{23} + BC_VP_V + BC_V\Phi_V(P_VA_V^{\mathrm{T}} + A_VP_V)]\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}B^{\mathrm{T}}\\ &\qquad - BC_V\Phi_V[A_VP_{23}^{\mathrm{T}} + P_{23}^{\mathrm{T}}A^{\mathrm{T}} + P_VC_V^{\mathrm{T}}B^{\mathrm{T}}]\\ &\qquad - \Sigma A^{\mathrm{T}} - A\Sigma - P_{23}C_V^{\mathrm{T}}B^{\mathrm{T}} - BC_VP_{23}^{\mathrm{T}}\}[A_{21}\phi_r\ \ I]^{\mathrm{H}}.\end{aligned}$$

Then, Π_r can be rewritten as

$$\begin{aligned}\Pi_r &= [A_{21}\phi_r\ \ I]\{-[P_{23}A_V^{\mathrm{T}} + AP_{23}]\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}B^{\mathrm{T}} - BC_V\Phi_V[A_VP_{23}^{\mathrm{T}} + P_{23}^{\mathrm{T}}A^{\mathrm{T}}]\\ &\qquad - \Sigma A^{\mathrm{T}} - A\Sigma - P_{23}C_V^{\mathrm{T}}B^{\mathrm{T}} - BC_VP_{23}^{\mathrm{T}}\}[A_{21}\phi_r\ \ I]^{\mathrm{H}}\\ &= [A_{21}\phi_r\ \ I]\{-[P_{23}A_V^{\mathrm{T}} + AP_{23}]\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}B^{\mathrm{T}} + BC_V\Phi_VP_VC_V^{\mathrm{T}}B^{\mathrm{T}}\\ &\qquad - BC_V\Phi_V[A_VP_{23}^{\mathrm{T}} + P_{23}^{\mathrm{T}}A^{\mathrm{T}}] - BC_V\Phi_VP_VC_V^{\mathrm{T}}B^{\mathrm{T}} - \Sigma A^{\mathrm{T}} - A\Sigma\\ &\qquad - P_{23}C_V^{\mathrm{T}}B^{\mathrm{T}} - BC_VP_{23}^{\mathrm{T}}\}[A_{21}\phi_r\ \ I]^{\mathrm{H}},\end{aligned}$$

and using the same equations above,

$$\begin{aligned}\Pi_r &= [A_{21}\phi_r\ \ I]\{\Phi^{-1}P_{23}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}B^{\mathrm{T}} + BC_V\Phi_VP_{23}^{\mathrm{T}}\Phi^{-\mathrm{H}} - \Sigma A^{\mathrm{T}} - A\Sigma\}[A_{21}\phi_r\ \ I]^{\mathrm{H}}\\ &= \Delta_r\Sigma_2 + \Sigma_2\Delta_r^{\mathrm{H}} + \Xi_rC_V\Phi_VP_{23}^{\mathrm{T}}[0\ \ \Delta_r]^{\mathrm{H}} + [0\ \ \Delta_r]P_{23}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}\Xi_r^{\mathrm{H}}.\end{aligned}$$

Thus,

$$\Delta_r^{-1}\Pi_r = \Sigma_2 + [0\ \ I]P_{23}\Phi_V^{\mathrm{H}}C_V^{\mathrm{T}}\Xi_r^{\mathrm{H}} + \Delta_r^{-1}(\Sigma_2 + \Xi_rC_V\Phi_VP_{23}^{\mathrm{T}}[0\ \ I]^{\mathrm{T}})\Delta_r^{\mathrm{H}}.$$

Similarly, using the Lyapunov equation (2.8b), one can obtain

$$\Delta_r^{-\mathrm{H}}\Theta_r = \Sigma_2 + [0\ \ I]Q_{12}^{\mathrm{T}}\Phi_WB_W\Gamma_r + \Delta_r^{-\mathrm{H}}(\Sigma_2 + \Gamma_rB_W^{\mathrm{T}}\Phi_W^{\mathrm{H}}Q_{12}[0\ \ I]^{\mathrm{T}})\Delta_r.$$

Now consider the case when the state dimension is reduced by one, i.e. $r = n - 1$. Then

$$\Delta_{n-1}^{-1}\Pi_{n-1} = (\sigma_n + \Xi_{n-1}C_V\Phi_VP_{23}^{\mathrm{T}}[0\ldots0\ \ 1]^{\mathrm{T}})(1 + \Delta_{n-1}^{-1}\Delta_{n-1}^{\mathrm{H}}),$$

$$\Delta_{n-1}^{-\mathrm{H}}\Theta_{n-1} = (\sigma_n + [0\ldots0\ \ 1]Q_{12}^{\mathrm{T}}\Phi_WB_W\Gamma_{n-1})(1 + \Delta_{n-1}^{-\mathrm{H}}\Delta_{n-1}),$$

and

$$\| W(K - K_{n-1})V\|_\infty^2 = \|(\sigma_n + \Xi_{n-1}C_V\Phi_V P_{23}^{\mathrm T}[0\ldots0\ \ 1]^{\mathrm T})(\sigma_n + [0\ldots0\ \ 1]Q_{12}^{\mathrm T}\Phi_W B_W\Gamma_{n-1})$$
$$\times(1 + \Delta_{n-1}^{-1}\Delta_{n-1}^{\mathrm H})(1 + \Delta_{n-1}^{-\mathrm H}\Delta_{n-1})\|_\infty$$
$$\leqslant 4(\sigma_n + \|\Xi_{n-1}\|_\infty \|C_V\Phi_V P_{23}^{\mathrm T}[0\ldots0\ \ 1]^{\mathrm T}\|_\infty)$$
$$\times(\sigma_n + \|[0\ldots0\ \ 1]Q_{12}^{\mathrm T}\Phi_W B_W\|_\infty \|\Gamma_{n-1}\|_\infty).$$

At this point we need the following inequalities:

$$\|C_V\Phi_V P_{23}^{\mathrm T}[0\ldots0\ \ 1]^{\mathrm T}\|_\infty \leqslant \sqrt{\sigma_n}\,\|C_V\Phi_V P_V^{1/2}\|_\infty, \tag{A1}$$

$$\|[0\ldots0\ \ 1]Q_{12}^{\mathrm T}\Phi_W B_W\|_\infty \leqslant \sqrt{\sigma_n}\,\|Q_W^{1/2}\Phi_W B_W\|_\infty. \tag{A2}$$

To show that they are true, let us first consider the following positive definite submatrix of the gramian $\bar{Q}$:

$$\left(\begin{array}{c|ccc} Q_W & q_{12,1}\ldots q_{12,n} & & \\ \hline q_{12,1}^{\mathrm T} & \sigma_1 & \ddots & 0 \\ \vdots & & & \sigma_n \\ q_{12,n}^{\mathrm T} & & 0 & \end{array}\right) > 0.$$

Then, it follows that $\sigma_n Q_W - q_{12,n}q_{12,n}^{\mathrm T} > 0$. Hence, we obtain

$$\|[0\ldots0\ \ 1]Q_{12}^{\mathrm T}\Phi_W B_W\|_\infty = \|q_{12,n}^{\mathrm T}\Phi_W B_W\|_\infty \leqslant \sqrt{\sigma_n}\,\|Q_W^{1/2}\Phi_W B_W\|_\infty.$$

Similar derivations lead to the first inequality (A1).

According to these inequalities, $\|W(K - K_{n-1})V\|_\infty^2 \leqslant 4(\sigma_n + \alpha_n\sqrt{\sigma_n})(\sigma_n + \beta_n\sqrt{\sigma_n})$, where $\alpha_n = \|\Xi_{n-1}\|_\infty \|C_V\Phi_V P_V^{1/2}\|_\infty$ and $\beta_n = \|Q_W^{1/2}\Phi_W B_W\|_\infty \|\Gamma_{n-1}\|_\infty$. Thus, the reduction error can be bounded as

$$\|W(K - K_{n-1})V\|_\infty \leqslant 2\sqrt{\sigma_n^2 + (\alpha_n + \beta_n)\sigma_n^{3/2} + \alpha_n\beta_n\sigma_n}.$$

The result can be extended to the general case of $r \leqslant n - 1$. In fact,

$$\|W(K - K_r)V\|_\infty = \|W(K - K_{n-1} + K_{n-1} - K_{n-2} + \cdots + K_{r+1} - K_r)V\|_\infty$$
$$= \left\|\sum_{k=r+1}^{n}[W(K_k - K_{k-1})V]\right\|_\infty \leqslant \sum_{k=r+1}^{n}\|W(K_k - K_{k-1})V\|_\infty.$$

As we have shown, $\|W(K_k - K_{k-1})V\|_\infty \leqslant 2\sqrt{\sigma_k^2 + (\alpha_k + \beta_k)\sigma_k^{3/2} + \alpha_k\beta_k\sigma_k}$. Then the main result of the theorem follows immediately. □

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