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Brief Paper

Design of Reduced-order Multirate Output Linear Functional Observer-based Compensators*

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Key Words—Reduced-order; multirate output sampling; linear functional observer-based compensator.

Abstract—Often in control system design, it is only necessary to estimate a single (but prespecified) linear function of the system's state for the purpose of implementing a feedback control law. This can be achieved by a linear functional observer which may have order one less than the observability index of the plant, but generically cannot have smaller order. In this paper, we explore the concept of multirate output sampling and show that a linear functional observer-based compensator employing multirate sampling of the plant output can be designed with dimension much smaller than that of the linear functional observer-based compensator employing single-rate sampling of the plant output. Necessary and sufficient conditions for the existence of the single-rate and multirate output linear functional observer-based compensators are found. Design procedures for constructing these observer-based compensators are also outlined. Furthermore, both types of compensators are strictly causal and open-loop stable for sufficiently small frame period, T_0.

1. *Introduction*

It is well-known that simple (low-order) linear compensators are normally to be preferred to complex (high-order) linear compensators for finite-dimensional linear time-invariant (FDLTI) plants. Reasons for this include the higher reliability associated with lower complexity in the hardware, the lesser complexity in the software, and higher computational efficiency associated with the reduced computational burden. Simple compensators are likely to be easier to understand at a conceptual level so that they are more likely to be accepted by design engineers and managers. Accordingly, there is a desire to have methods available to design a low-order compensator for a higher-order plant.

The search for reduced-order observer-based and nonobserver-based compensator designs has been an ongoing process (see Luenberger, 1966; Tse and Athans, 1970; Fortmann and Williamson, 1972; Murdoch, 1973; Hagiwara *et al.*, 1990). Controllers which do not employ observers but employ multirate sampling of the plant output were studied in Hagiwara *et al.* (1990). In Hagiwara *et al.* (1990), the authors showed that arbitrary pole assignment is possible by generalized multirate output controllers of reduced order. Their approach essentially follows the discrete-time version of designing dynamic compensators developed in Pearson and Ding (1969). An augmented system is formed by connecting L delay elements, as opposed to L differentiators, to each input of the discrete-time system. A multi-frame control law is then devised by employing the concept of multirate output sampling and it is shown to be equivalent to realizing the dynamic control law for the augmented system in the absence of measurement of the plant's state. One of the main results is that for a single-input system, the order of the controller, L, the i-th output-rate multiplicity,§ N_i^O and the observability index, n_i^o are related by $LN_i^O \geq n_i^o$. This implies that in order to achieve arbitrary pole assignment for an n-th-order SISO system, the smallest order of the controller is n with single-rate sampling while with multirate output sampling, the smallest order becomes $\left\lceil \frac{n}{N^O} \right\rceil$, with N^O, of course, the ratio of the output sampling rate to the input sampling rate. Here, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

In this paper, we exploit the observer theory developed by Luenberger and show that in the case of estimating a single (but pre-specified) linear functional of a system's state, a multirate output linear functional observer-based compensator (employing multirate sampling of the plant output) of dimension much smaller than that of a single-rate output linear functional observer-based compensator (employing single-rate sampling of the plant output) can be designed by exploiting the multirate output sampling mechanism developed in Hagiwara and Araki (1988). In Hagiwara and Araki (1988), it is shown that for a m-input ρ-output system, if $N_i^O \geq n_i^o$ $(i = 1, 2, \ldots, \rho)$, then a realization of the state feedback law is possible with no dynamics. In our approach, N_i^O satisfies $1 \leq N_i^O < n_i^o$. Furthermore, the multirate output sampling scheme employed here has uniform output-rate multiplicity. As a consequence, we are able to obtain a controller of order L satisfying $LN_i^O \geq n_i^o$, just as in Hagiwara *et al.* (1990). The controller of this paper can be regarded as a combined estimator and state feedback law, which is not really the same as the controller in Hagiwara *et al.* (1990). Here, we can choose the dynamics of the estimator separately from those of the closed-loop system with true state feedback.

The structure of the paper is as follows: Section 2 considers continuous-time single-input multiple-output (SIMO) FDLTI systems and derives the structures and the design procedures for the single-rate and multirate output linear functional observer-based compensators. The structures and design

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§ The output-rate multiplicity, N_i^O is defined as the rational ratio of the frame period, T_0 and the output sampling time, T_i which indicate the intervals in which the control signals are applied and the outputs are detected, respectively.

procedures for the two types of observer-based compensators for this case are also presented here. An example of a SISO system appears in Section 3 to illustrate the ideas and methods described. Section 4 contains concluding remarks.

2. *SIMO case*

2.1. *Single-rate output linear functional observer-based compensator.* In this section, we begin by presenting the general structure of and the design procedure for the single-rate output linear functional observer-based compensator for a SIMO system. It also serves as a basis of deriving the multirate output linear functional observer-based compensator and comparing the reduction in the dimension of the compensator in the subsections.

2.1.1. *Structure of compensator.* Without loss of generality, we assume that the SIMO discretized plant

$$x_d(k+1) = A_s x_d(k) + b_s u_d(k), \quad x_d(0) = x_0 \qquad (1)$$

$$y_d(k) = Cx_d(k) \qquad (2)$$

inherits the controllability and observability properties of its continuous-time counterpart. Here

$$A_s = \exp(AT_0), \quad b_s = \int_0^{T_0} \exp(At)b\, dt. \qquad (3)$$

where the triple (A, b, C) represents the continuous-time plant. We shall also assume that C has full row rank. Since not all the states of equations (1) and (2) are directly measurable, we propose a compensator (comprising a linear functional observer and a rule for computing the control) of the form

$$z_d(k+1) = Fz_d(k) + Gy_d(k) + dw_d(k) \qquad (4)$$

$$w_d(k+1) = py_d(k) + qz_d(k) + rw_d(k), \qquad (5)$$

where $w_d(k)$ is a scalar which approximates a desired linear feedback $k'x_d(k)$ which we choose to express for later notational convenience as $a'A_s^{-1}x_d(k)$. (Note that we consider only the case where the external input $r_d(k) = 0$ here. It is straightforward but notationally more intricate to allow $r_d(k) \neq 0$.) The connection of the plant and the single-rate output linear functional observer-based compensator is shown in Fig. 1.

The first equation (4) follows directly from the continuous-time equivalent. The second equation (5) is used in lieu of the discrete-time equivalent of the observer-based compensator proposed in Luenberger (1966) and (1971), which would be given by

$$w_d(k) = py_d(k) + qz_d(k). \qquad (6)$$

The problem with the above equation is that it becomes non-causal if one were to extend it to the multirate output case, since $y_d(k)$ must be replaced a vector $\bar{y}_d(k)$, some entries of which become available after time kT_0 though before $(k+1)T_0$. This will become clear in the later subsection where we deal with the multirate output linear functional observer-based compensator. The following result is a trivial variant on that of a continuous-time system in Luenberger (1971).

Lemma 1. The state $z_d(k)$ of equations (4) and (5) is an estimate of $Tx_d(k)$ with $x_d(k)$ defined in equations (1) and (2) if and only if the following conditions hold:

Condition a: F is stable, i.e. $|\lambda_i(F)| < 1$ for all eigenvalues λ_i of F.

Condition b: $TA_s - FT = GC$.

Condition c: $d = Tb_s$.

Next, we have:

Lemma 2. Suppose that the plant given by equations (1) and (2) is controllable. Then, $w_d(k)$ of equation (5) estimates $a'A_s^{-1}x_d(k)$ if there exists a linear transformation T such that $z_d(k)$ estimates $Tx_d(k)$ with error $e_d(k)$ and

$$a' = pC + qT$$
$$r = a'A_s^{-1}b_s.$$

Proof. Let $e_d(k) = z_d(k) - Tx_d(k)$ and observe that

$$\begin{aligned} w_d(k+1) - a'A_s^{-1}x_d(k+1) &= py_d(k) + qz_d(k) + rw_d(k) \\ &\quad - a'x_d(k) - a'A_s^{-1}bw_d(k) \\ &= pCx_d(k) + qTx_d(k) - a'x_d(k) \\ &\quad + qe_d(k) \\ &= qe_d(k). \end{aligned}$$ □

2.1.2. *Order of compensator.* As a basis for comparison of different compensators through their orders, we make:

Definition 1. The order of the compensator given by equations (4) and (5) is the dimension of the vector $[z_d'(k)\ w_d'(k)]'$.

Note that the continuous-time result in Chen (1970) translates to discrete time to imply that any single linear functional of the state, $a'A_s^{-1}x_d(k)$ can be estimated with a reduced-order discrete-time linear functional observer whose dimension of $z_d(k)$ is $\left(\left\lceil \frac{n}{p} \right\rceil - 1\right)$ where n and p are the dimensions of the plant state and the row rank of the output matrix, C, respectively. Such an estimate has a state update equation like (3) and an output equation like (5) rather than equation (4). With an output equation like (4), the dimension must be increased by $\dim(w_d(k)) = 1$.

2.1.3. *Design procedure for compensator.* The problem at hand boils down to finding F, G, d, p, q and r such that $w_d(k)$ estimates $a'A_s^{-1}x_d(k)$. To facilitate the construction of the single-rate output linear functional observer-based compensator, we provide here a design procedure for its construction.

(1) Select T_0 according to the recommendations given in Åström and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990).
(2) Discretize the continuous-time plant with the selected T_0. Let the discretized plant be represented by (A_s, b_s, C).
(3) Choose a stable F (for simplicity, choose F to be diagonal with distinct eigenvalues) and $q = [1\ 1 \cdots 1] \in \mathbb{R}^{1\times(\lceil n/p\rceil-1)}$. Solve for $p \in \mathbb{R}^{1\times p}$, $G \in \mathbb{R}^{(\lceil n/p\rceil-1)\times p}$

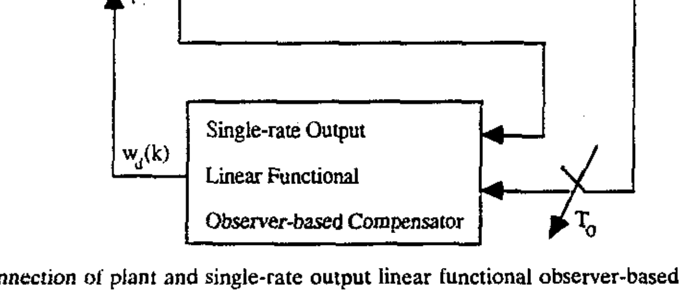

Fig. 1. Connection of plant and single-rate output linear functional observer-based compensator.

and $T \in \mathbb{R}^{(\lceil n/\rho \rceil - 1) \times n}$ from $TA_s - FT = GC$ and $a' = pC + qT$, using the algorithm of Murdoch (1973). Note that there always exists a unique triple p, G and T solving these equations if A_s and F do not have common eigenvalues.

(4) Compute $d = Tb_s$ and $r = a'A_s^{-1}b_s$.

2.2. *Multirate output linear functional observer-based compensator.* For a multiple-output system with ρ outputs, multirate output sampling with uniform output-rate multiplicity, i.e. $N_1^O = N_2^O = \cdots = N_\rho^O = N^O$ effectively produces N^O successive and independent values of each of the outputs over each time interval $[(i-1)T_0, iT_0]$, $i = 1, 2, 3, \ldots$. Intuitively, this is like maintaining the original T_0 but increasing the output dimension and the row rank of the output matrix, thereby reducing the observability index of the discretized plant. It follows that further reduction in the order of the compensator should be possible with multirate output sampling. This inspires us to look into the feasibility of applying multirate output sampling to achieve reduced-order compensators. In the sequel, we shall present the structure of the multirate output linear functional observer-based compensator and a design procedure for its construction.

2.2.1. *Multirate output sampling.* Before we deal with the structure and design procedure for a multirate output linear functional observer-based compensator, let us briefly review the concept of multirate output sampling. As foreshadowed earlier, for multiple-output systems, multirate output sampling with uniform output-rate multiplicity means that each of the plant outputs is sampled N^O times over the time interval $[kT_0, (k+1)T_0]$, $k = 0, 1, 2, \ldots$ where the integer N^O is termed the output-rate multiplicity.

As a result of multirate output sampling, the discretized plant becomes

$$x_d(k+1) = A_s x_d(k) + b_s u_d(k) \tag{7}$$

$$\bar{y}_d(k) = \bar{C}x_d(k) + \bar{d}u_d(k), \tag{8}$$

where A_s and b_s are given by equation (3) and

$$\bar{y}_d(k) = \begin{bmatrix} y_d(k) \\ y_d\left(k + \frac{1}{N^O}\right) \\ \vdots \\ y_d\left(k + \frac{N^O - 1}{N^O}\right) \end{bmatrix} \tag{9}$$

$$\bar{C} = \begin{bmatrix} C \\ C\exp(AT_0/N^O) \\ \vdots \\ C\exp(A(N^O-1)T_0/N^O) \end{bmatrix}$$
$$\bar{d} = \begin{bmatrix} 0 \\ C\int_0^{T_0/N^O} \exp(At)b\,dt \\ \vdots \\ C\int_0^{(N^O-1)T_0/N^O} \exp(At)b\,dt \end{bmatrix}. \tag{10}$$

2.2.2. *Structure of compensator.* In the same spirit as for the single-rate compensator, we propose the following structure for the multirate output compensator:

$$z_d(k+1) = Fz_d(k) + G\bar{y}_d(k) + dw_d(k) \tag{11}$$

$$w_d(k+1) = p\bar{y}_d(k) + qz_d(k) + rw_d(k). \tag{12}$$

A few important observations can be made about equations (11) and (12). First, notice that they are obtained by replacing $y_d(k)$ in equations (4) and (5) by $\bar{y}_d(k)$. Second, we shall take no account of the possible need to store entries of $\bar{y}_d(k)$ (which become available at different times) in defining the state dimension of the compensator which in accordance with Definition 1 is $\dim(z_d(k)) + \dim(w_d(k))$. Third, as mentioned in the previous section, there is a need to introduce a delay in the $w_d(k)$ equation so that the multirate output compensator is causal. This can be easily seen by substituting equation (9) into $w_d(k) = p\bar{y}_d(k) + qz_d(k)$ (which does not contain the delay) and examining the time index on both sides of the equality.

The condition for $z_d(k)$ to be an estimate of $Tx_d(k)$ is noted in the following lemma which is a trivial variant of Lemma 1.

Lemma 3. The state $z_d(k)$ of equations (11) and (12) is an estimate of $Tx_d(k)$ with $x_d(k)$ defined in equations (7) and (8) if and only if the following conditions hold:

Condition a: F is stable, i.e. $|\lambda_i(F)| < 1$.

Condition b: $TA_s - FT = G\bar{C}$.

Condition c: $d = Tb_s - G\bar{d}$.

The construction of $w_d(k)$ proceeds virtually the same way as in Lemma 2.

Lemma 4. Suppose that the plant given by equations (7) and (8) is observable. Then, $w_d(k)$ of equations (11) and (12) estimates $a'A_s^{-1}x_d(k)$ if there exists a linear transformation T such that $z_d(k)$ estimates $Tx_d(k)$ and

$$a' = p\bar{C} + qT \tag{13}$$

$$r = a'A_s^{-1}b_s - p\bar{d}. \tag{14}$$

2.2.3. *Order of compensator.* Before stating a theorem concerning the order of the compensator, we note a minor point concerning the order counting of the compensator. Since $w_d(k)$ *increases the dimension* of both a single-rate output linear functional observer-based compensator as well as a multirate output linear functional observer-based compensator by at most one, it suffices to consider the dimension of $z_d(k)$ and show that there is a reduction in this quantity using a multirate output linear functional observer-based compensator.

Theorem 1. For almost all T_0, the dimension of the state, $z_d(k)$ of the multirate output linear functional observer-based compensator (11) and (12) is $\left\lceil \frac{n}{N^O\rho} \right\rceil - 1$ with $N^O < n_i^o$ where n, ρ, N^O and n_i^o are the dimensions of the plant state, the row rank of the full row rank output matrix, C, the output-rate multiplicity and the observability indices, respectively.

Proof. It follows from the proof of Lemma 1 of Hagiwara and Araki (1988) that for $N^O < n_i^o$ and almost all T_0

$$\operatorname{rank}\bar{C} = \operatorname{rank}\begin{bmatrix} C \\ C\exp(AT_0/N^O) \\ \vdots \\ C\exp((N^O-1)AT_0/N^O) \end{bmatrix} = N^O\rho. \tag{15}$$

The argument of Subsection 2.1.2 applies with $\bar{C}$ replacing C to yield the condition. □

2.2.4. *Design procedure for compensator.* The design procedure uses the ideas of the preceding lemmas, together with the scheme of Murdoch (1973).

(1) Select an appropriate T_0 as in Subsection 2.1.3, choose an output-rate multiplicity N^O and discretize the continuous-time plant. Let the discretized plant be represented by $(A_s, b_s, \bar{C}, \bar{d})$.

(2) Using Theorem 1, the dimension of $z_d(k)$ is $\left(\left\lceil \frac{n}{N^O\rho} \right\rceil - 1\right)$.

(3) Choose a stable F (again for simplicity, choose F to be diagonal with distinct eigenvalues) and $q = [1\ 1\ \cdots\ 1] \in \mathbb{R}^{1 \times (\lceil n/N^O\rho \rceil - 1)}$. Solve for $p \in \mathbb{R}^{1 \times N^O\rho}$, $G \in \mathbb{R}^{(\lceil n/N^O\rho \rceil - 1) \times N^O\rho}$ and $T \in \mathbb{R}^{(\lceil n/N^O\rho \rceil - 1) \times n}$ from $TA_s - FT = G\bar{C}$ and $a' = p\bar{C} + qT$, using the algorithm of Murdoch (1973).

(4) Compute $d = Tb_s - b\bar{d}$ and $r = a'A_s^{-1}b_s - p\bar{d}$.

We remark that in this paper, we only deal with the noiseless case. In the case where there is output noise, based on the results of Er and Anderson (1992), one should expect some deterioration in performance with the magnitude of the control signal depending on the particular case.

3. Example

To illustrate the ideas presented, we give an example involving a model of an electrical circuit consisting of resistors, capacitors and inductors used in Kaufman (1973) which is given by

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t),$$

where

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$c = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0].$$

Assume that the states are not available for measurement.

3.1. *Single-rate output linear functional observer-based compensator.* Using $T_0 = 0.2$ with single-rate output sampling, the discretized plant becomes

$$x_d(k+1) = A_s x_d(k) + b_s u_d(k)$$
$$y_d(k) = cx_d(k),$$

where

$$A_s = \begin{bmatrix} 0.6837 & 0.1339 & 0.0135 & 0.0020 & 0.0154 & -0.0153 \\ 0.1339 & 0.6666 & 0.1359 & 0.0308 & 0.1646 & -0.1624 \\ 0.0135 & 0.1359 & 0.6983 & 0.1503 & 0.0155 & 0.0010 \\ 0.0020 & 0.0308 & 0.1503 & 0.8168 & 0.0022 & 0.1790 \\ -0.0154 & -0.1646 & -0.0155 & -0.0022 & 0.9824 & 0.0175 \\ 0.0153 & 0.1624 & -0.0010 & -0.1790 & 0.0175 & 0.9639 \end{bmatrix}$$

$$b_s = \begin{bmatrix} 0.1669 \\ 0.0330 \\ 0.0021 \\ 0.0002 \\ 0.1977 \\ 0.0023 \end{bmatrix}.$$

The open-loop poles of the discretized plant are 0.5083, 0.6440, $0.8509 \pm 0.2343i$ and $0.9788 \pm 0.0958i$. Suppose that the closed-loop poles were assigned to 0.2472, 0.6183, $0.8150 \pm 0.2342i$ and $0.9332 \pm 0.1142i$ via a linear feedback gain

$$a'A_s^{-1} = [-1.5255 \;\; 0.3548 \;\; -1.5366 \;\; 0.2746 \;\; -1.0297 \;\; 0.005]$$

or

$$a' = [-1 \;\; 0 \;\; -1 \;\; 0 \;\; -1 \;\; 0].$$

Since the states are not available for measurement, we shall attempt to find a minimal-order observer to estimate the control law, $a'A_s^{-1}x_d(k)$. As will be shown in the sequel, this is accomplished by a sixth-order observer (with $z_d(k)$ of dimension five and with strict causality) using single-rate output sampling. (Note that if strict causality were not required, we would use an observer of dimension five.) However, with multirate output sampling, a second-order strictly causal observer is sufficient.

The structure of the single-rate output linear functional observer-based compensator is given by

$$z_d(k+1) = Fz_d(k) + gy_d(k) + dw_d(k),$$

where

$$TA_s - FT = gc, \quad d = Tb_s.$$

Choose

$$F = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad q = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1].$$

Since F and A_s have no common eigenvalues, there is a unique triple p, q and T satisfying $TA_s - FT = gc$ and $pc + qT = a'$. Using the algorithm of Murdoch (1973), we get

$$p = -108.32, \quad g = 10^3 \begin{bmatrix} 3.9371 \\ -6.3123 \\ 3.0116 \\ -0.4018 \\ 0.0023 \end{bmatrix}$$

$$T = 10^4 \begin{bmatrix} -0.0040 & 0.0361 & -0.1470 & 0.5518 & -0.0033 & -0.1074 \\ 0.0182 & -0.1084 & 0.3430 & -1.0379 & 0.0137 & 0.2197 \\ -0.0275 & 0.1101 & -0.2666 & 0.6066 & -0.0184 & -0.1363 \\ 0.0165 & -0.0427 & 0.0761 & -0.1128 & 0.0093 & 0.0235 \\ -0.0033 & 0.0048 & -0.0056 & 0.0032 & -0.0014 & 0.0004 \end{bmatrix}.$$

Furthermore

$$d = Tb_s = \begin{bmatrix} -5.7416 \\ 31.5805 \\ -53.2774 \\ 33.8502 \\ -6.7547 \end{bmatrix}, \quad r = a'A_s^{-1}b_s = -0.4497.$$

Hence, the desired single-rate output linear functional observer-based compensator is

$$\begin{bmatrix} z_{d,1}(k+1) \\ z_{d,2}(k+1) \\ z_{d,3}(k+1) \\ z_{d,4}(k+1) \\ z_{d,5}(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} z_{d,1}(k) \\ z_{d,2}(k) \\ z_{d,3}(k) \\ z_{d,4}(k) \\ z_{d,5}(k) \end{bmatrix} + 10^3 \begin{bmatrix} 3.9371 \\ -6.3123 \\ 3.0116 \\ -0.4018 \\ 0.0023 \end{bmatrix} y_d(k) + \begin{bmatrix} -5.7416 \\ 31.5805 \\ -53.2774 \\ 33.8502 \\ -6.7547 \end{bmatrix} w_d(k),$$

$$w_d(k+1) = -108.32 y_d(k) + [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1] \times \begin{bmatrix} z_{d,1}(k) \\ z_{d,2}(k) \\ z_{d,3}(k) \\ z_{d,4}(k) \\ z_{d,5}(k) \end{bmatrix} - 0.4497 w_d(k).$$

Note that the compensator is open-loop stable and strictly causal.

To further demonstrate the performance of the single-rate observer, the closed-loop response of the observer-based compensator is simulated. The initial values of $x_d(k)$ and $[z_d'(k)\, w_d'(k)]'$ are set to $A^{-1}b$ and zero, respectively. The

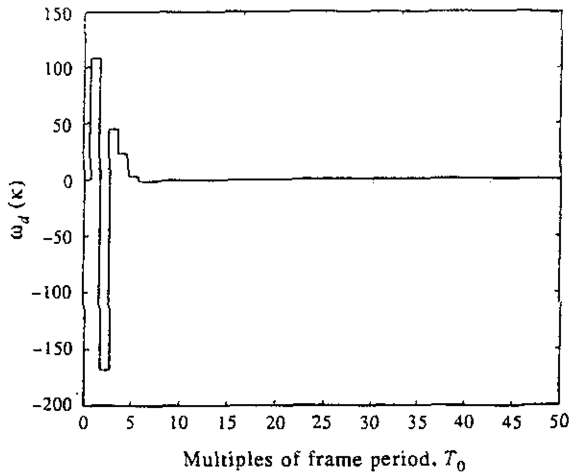

Fig. 2. Trajectory of $w_d(k)$ vs multiples of frame period, T_0.

trajectories of $w_d(k)$ and $y_d(k)$ with respect to multiples of the frame period, T_0 are shown in Figs 2 and 3, respectively.

3.2. *Multirate output linear functional observer-based compensator.* Using a multirate output linear functional observer-based compensator with the same T_0 and output-rate mutliplicity $N^O = 3$, we obtain the following discretized plant:

$$x_d(k+1) = A_s x_d(k) + b_s u_d(k)$$
$$\bar{y}_d(k) = \bar{C} x_d(k) + \bar{d} u_d(k),$$

where

$$\bar{y}_d(k) = \begin{bmatrix} y_d(k) \\ y_d(k+1/3) \\ y_d(k+2/3) \end{bmatrix} \quad \bar{d} = 10^{-4}\begin{bmatrix} 0 \\ 0.0308 \\ 0.4618 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0.0001 & 0.0041 & 0.0604 & 0.9354 & 0.0001 & 0.0644 \\ 0.0006 & 0.0149 & 0.1099 & 0.8745 & 0.0007 & 0.1241 \end{bmatrix}.$$

From Theorem 1, the dimension of $z_d(k)$ is one. Hence, a second-order observer is sufficient to estimate the same control law.

Choose $f = 0.1$ and $q = 1$. Solving $tA_s - ft = g\bar{C}$ and $p\bar{C} + qt = a'$ for the triple p, g and t, we obtain

$$p = 10^3[9.5084 \quad -2.8670 \quad -0.0573]$$
$$g = 10^4[-0.0082 \quad 0.5417 \quad -1.1263]$$

$t =$

$$10^3[-0.0007 \quad 0.0125 \quad 0.1785 \quad -6.7765 \quad -0.0007 \quad 0.1917].$$

Also

$$d = tb_s - g\bar{d} = -0.0104, \quad r = a'A_s^{-1}b_s - p\bar{d} = -0.4382.$$

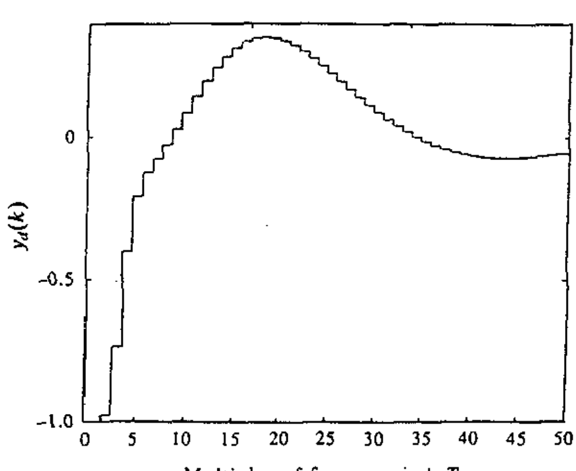

Fig. 3. Trajectory of $y_d(k)$ vs multiples of frame period, T_0.

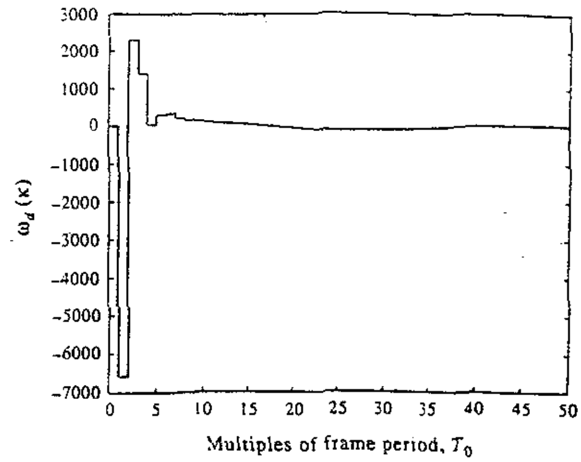

Fig. 4. Trajectory of $w_d(k)$ vs multiples of frame period, T_0.

Hence, the desired multirate linear functional observer-based compensator is given by

$$z_d(k+1) = 0.1z_d(k) + 10^4[-0.0082 \quad 0.5417 \quad -1.1263] \times \begin{bmatrix} y_d(k) \\ y_d(k+1/3) \\ y_d(k+2/3) \end{bmatrix} - 0.0104w_d(k)$$

$$w_d(k+1) = 10^3[9.5084 \quad -2.8670 \quad -0.0573] \times \begin{bmatrix} y_d(k) \\ y_d(k+1/3) \\ y_d(k+2/3) \end{bmatrix} + z_d(k) - 0.4382w_d(k),$$

which is open-loop stable and strictly causal.

In order to compare the performance with the single-rate observer, simulation studies are also carried out for the multirate output linear functional observer-based compensator using the same initial conditions as before. The trajectories of $w_d(k)$ and $\bar{y}_d(k)$ are plotted in Figs 4 and 5, respectively. The plots show that the values of $w_d(k)$ and $\bar{y}_d(k)$ are much bigger than those of the single-rate observer. This is a result of the large gains in the multirate observer.

4. *Conclusions*

In this paper, we have given a new insight into using multirate output sampling in designing reduced-order observers for estimating a single linear functional of the system's state for the purpose of implementing a feedback control law. Specifically, we have shown via theory and an example that reduction in the order of the linear functional observer-based compensator is possible using multirate output sampling with uniform output-rate multiplicity for single-input systems. It turns out that the order of the

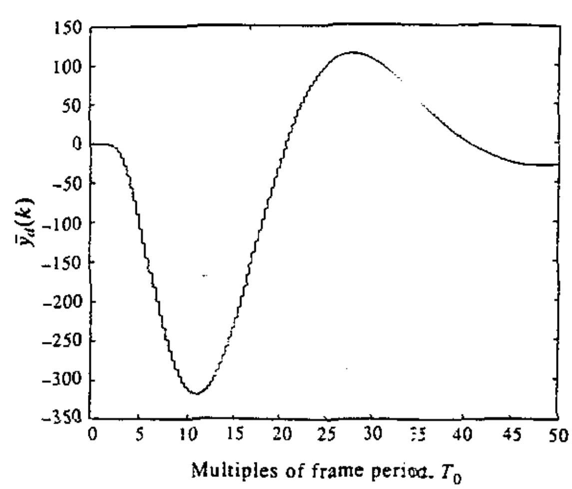

Fig. 5. Trajectory of $\bar{y}_d(k)$ vs multiples of frame period, T_0.

compensator only depends on the observability index of the discretized plant induced via sampling of the continuous-time plant. The multirate output compensator is strictly causal and open-loop stable for sufficiently small sampling time, T_0. The same type of ideas could of course be used to achieve dimension reduction in the multiple-input multiple-output (MIMO) case. The algorithm of Murdoch (1974) would be relevant here.

Two points on practical applications need to be highlighted. First, the value of $w_d(k)$ for the multirate observer is much bigger than that for the single-rate case. This is due to the large gains in the multirate observer which will have the effect of amplifying any problems associated with noise. Second, it is more attractive to use multirate sampling than additional sensors from the point of view of collection of information. The feasibility of doing so will depend on issues like the compensator gains found and the deleterious effects of noise. This has to be examined in a particular case.

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