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Easily Testable Sufficient Conditions for the Robust Stability of Systems with Multilinear Parameter Dependence*

B. D. O. ANDERSON,† F. KRAUS,‡ M. MANSOUR‡ and S. DASGUPTA§

Simple sufficient conditions are presented for checking the stability of a family of polynomials with coefficients depending multilinearly on a number of parameters.

Key Words—Stability criteria; polynomials; robust control.

Abstract—A number of robust stability problems take the following form. A polynomial has real coefficients which are multilinear in real parameters that are confined to a rectangular axis-parallel box in parameter space. An efficient method is required for checking the stability of this set of polynomials. We present two sufficient conditions in this paper, which can be applied in many cases. They involve checking certain properties at the corners and edges of the parameter space box.

1. INTRODUCTION

In this paper, we are concerned with the robust stability of polynomials with coefficients, which are multilinear in certain parameters. There are a number of motivations for this problem, as set out in the References. More precisely, we consider real polynomials

$$f(s, \underline{\gamma}) = s^n + a_1(\underline{\gamma})s^{n-1} + \cdots + a_n(\underline{\gamma}), \quad (1)$$

where the $a_i(\underline{\gamma})$ are multilinear in m scalar parameters $\gamma_1, \gamma_2, \ldots, \gamma_m$. We shall suppose, without loss of generality, that

$$0 \leq \gamma_j \leq 1 \quad j = 1, 2, \ldots, m. \quad (2)$$

Call this m-dimensional box Γ. We actually allow the a_i to be multiaffine, not multilinear, in the sense that if the values of all but one of the γ_j are fixed, then the a_i are affine in the remaining γ_j.

Examples of uncertain systems whose characteristic polynomials are multiaffine in the uncertain parameters include: (i) systems having state variable description $\{A, b, c, d\}$ with certain elements of A known and others known to be in independent intervals; (ii) systems whose uncertain parameters are certain physical parameters representing elements, such as moments of inertia, which do not allow cross-coupling of energy storage devices (Dasgupta and Anderson, 1987); (iii) systems operating under multiple gain feedback with the feedback gains uncertain and lying in independent intervals (de Gaston and Safonov, 1988); and (iv) most interconnections of such systems.

An important tool for addressing such problems is the concept of the value set, e.g. (Barmish, 1988). For each ω, this is the set $\{f(j\omega, \underline{\gamma}) : \underline{\gamma} \in \Gamma\}$. If (1) is stable for some, $\underline{\gamma} \in \Gamma$ and if 0 is never in the value set for any $\omega \in R$, robust stability follows (Barmish, 1988). If 0 is not in the value set for $\omega = 0$, and is in the value set for some nonzero ω_0, then a continuity argument shows that 0 must be on the boundary of the value set for some $\omega_1 < \omega_0$, implying $f(j\omega_1, \bar{\gamma}) = 0$ for some $\bar{\gamma} \in \Gamma$. Accordingly, stability is often considered by arranging that 0 is never on the boundary of the value set. Clearly, it is of interest to know when the value set or at least its boundary can be simply characterized.

When a_i are affine, rather than multiaffine, in γ_j, the value set has a nice description following immediately from results of (Bartlett *et al.*, 1987): it is a (convex) polygon, all of whose

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† Department of Systems Engineering and Cooperative Research Centre for Robust and Adaptive Systems, Research School of Information Sciences and Engineering, Australian National University, Canberra, ACT 0200, Australia.

‡ Institute for Automatic Control, ETH, CH-8006, Gloriastrasse 35, Zürich, Switzerland.

§ Department of Electrical & Computer Engineering, University of Iowa, Iowa City, IA 52242, U.S.A.

edges are images of edges of Γ (an edge of Γ being obtained by fixing all but one γ_i at 0 or 1, and letting the remaining γ_j vary in $[0, 1]$). This is an easy consequence of the convex polytopic nature of Γ, and the affine dependence of f on γ_j. We note that if the value set has this property, then to verify stability of all members of $f(s, \Gamma)$ it is sufficient to check the stability of edge polynomials only.

An obvious question in relation to the multiaffine case now presents itself: when will the value set be a (convex) polygon, with edges which are images of edges of Γ? [See the title of (Hollot and Xu, 1989)!] If this property holds for the multiaffine case for all ω then it will be, as in the affine case, very easy to check the stability of $f(s, \Gamma)$ just by verifying the stability of edge polynomials.

In searching for an answer to this question, one result is available which helps simplify the question, that is the Mapping Theorem of Zadeh and Desoer (Zadeh and Desoer, 1963; Saeki, 1986; de Gaston and Safonov, 1986).

Mapping Theorem. Let $f(j\omega, \gamma)$ be a multiaffine function of γ_j, $j = 1, \ldots, m$ with $\gamma \in \Gamma$, defined by $0 \leq \gamma_j \leq 1$. Let Γ_0 denote the corners of Γ. Let $\mathrm{conv}\, A$ denote the convex hull of a set $A \subset R^2$. Then

$$\mathrm{conv}\, f(j\omega, \Gamma) = \mathrm{conv}\, f(j\omega, \Gamma_0).$$

There is an immediate consequence.

Corollary. With hypothesis as in the Mapping Theorem, suppose that the edges of $\mathrm{conv}\, f(j\omega, \Gamma_0)$ are images of edges of Γ and that $f(j\omega, \Gamma)$ is simply connected. Then $f(j\omega, \Gamma)$ is a convex polygon.

Proof. By the Mapping Theorem, the boundary of $\mathrm{conv}\, f(j\omega, \Gamma)$ is defined by the edges of a convex polygon, viz. $\mathrm{conv}\, f(j\omega, \Gamma_0)$, and these edges themselves lie in $f(j\omega, \Gamma)$. Since $f(j\omega, \Gamma)$ is simply connected, the result follows. ∎

We remark that, as discussed further in Section 5, the requirement that $f(j\omega, \Gamma)$ be simply connected is essential, though this point was not explicitly discussed in Hollot and Xu (1989).

The preceding argument has now raised the issue: can we find simple sufficient conditions on a multiaffine f to ensure that its value set is a convex polygon, and thus has a boundary constructible as the images of various edges of Γ?

In the next section, we shall analyze the case of $m = 2$. A number of the results have appeared already in Kraus *et al.* (1989). In Section 3, we state some preliminary facts concerning Jacobians associated with the map $R^m \to R^2 : \Gamma \to f(j\omega, \Gamma)$. In Section 4, we present a new result providing a sufficient condition for the convex polytopic nature of f at a particular value of ω. The condition is in terms of the signs of Jacobians evaluated at the (finite number of) corners Γ_0 of Γ. Thus, for any fixed ω, the test can be executed with a finite number of calculations. Various remarks concerning the result of Section 4, as well as examples, are presented in Section 5.

Section 6 presents a different form of a new result: we examine a conjecture of Hollot and Xu (1989) and show that a modified form of the conjecture is true. This means that an easily checked property of the images of the corners of Γ determines whether or not the outer boundary† of the value set is a convex polygon boundary that can be mapped from the edges of Γ. Suppose, that for all ω, this outer boundary is a convex polygon boundary. Then, again, stability checking is easily performed by checking the stability of the edge polynomials. To see this, suppose one does have stability of the edge polynomials. Then, as the set of polynomials we are concerned with is real, the value set at $\omega = 0$, is a *connected* straight line segment each of whose elements have at least one pre-image in the edges of Γ. Stability of the edge polynomials then ensures that at $\omega = 0$ the value set does not include the origin. Then, one can lose stability if and only if for some $\omega > 0$ the outer boundary of the value set contains the origin. This is because it is impossible (by continuity) for the origin to be outside the value set at $\omega = 0$ (a straight line of finite extent) and then to lie, for arbitrarily small $\omega > 0$, in the interior hole of a value set topologically equivalent to a doughnut. Thus, in such a setting, stability of all edge polynomials ensures that of the entire set $f(s, \Gamma)$.

Let us emphasise that our results concern the value set at a single frequency ω. In order to check stability, in principle, it is necessary to examine all frequencies. It is, however, important to note that if the sufficient conditions of Section 4 hold at one frequency, it is easy (by finding the zero crossings at real values of ω of certain polynomials in ω that are Jacobian

† Suppose Γ is not a simple connected set. The outer boundary is that part of the boundary which can be connected to the part at ∞ by a curve entirely in the complement of Γ.

determinants) to identify an interval containing this frequency in which they hold. In this sense, the construction of the value set for each frequency can be greatly speeded up. If there are no real values of ω for which the Jacobian determinants are zero (a fact which can be established via a finite number of rational calculations) and the sufficient conditions are fulfilled for all frequencies (if it is fulfilled at one), then the stability problem is resolved in the same manner as when the edge theorem is used.

It is important that the results of this paper are placed in the context of related results that are available. Anderson *et al.* (1992a, 1992b) are conference versions of this paper, with the proofs omitted. Poljak (1992) and Tsing and Tits (1992) examine the same question as this paper. They were first presented at the same meeting as Anderson *et al.* (1992a). Among these, Poljak (1992) presents the same Jacobian-based condition for checking the convexity of the value set as is presented here. However, Poljak (1992) does not examine the conjecture of Hollot and Xu (1989). Further, our results provide strong indications on constructing the value set boundary, indications that are not apparent in the results of Poljak (1992). Unlike the results presented here, and indeed in Poljak (1992), the results of Tsing and Tits (1992) are not explicitly algebraic. Rather, Tsing and Tits (1992) gives geometric conditions on the location of images in the value set space of certain corners and edges of Γ.

Another result of note, is the result of Djaferis and Hollot (1989) who postulated certain 'shaping conditions' on polynomial sets involving more general parametrizations than just multiaffine ones. Also of interest, is the work of Djaferis (1988), where he considers a simplified version of these shaping conditions when applied to the two-parameter multiaffine case. In Section 5, we demonstrate the connection between these shaping conditions and the Jacobian-based conditions presented here.

2. THE CASE OF TWO PARAMETERS

Let a polynomial $f(s, \underline{\gamma})$ depend in a multiaffine way on two parameters, γ_1 and γ_2. We can then write

$$f(s, \underline{\gamma}) = f_0(s) + \gamma_1 f_1(s) + \gamma_2 f_2(s) + \gamma_1\gamma_2 f_3(s). \quad (3)$$

It is assumed that $\underline{\gamma} \in \Gamma$, i.e.

$$0 \le \gamma_j \le 1. \quad (4)$$

Let $s = j\omega$, where ω is real, be fixed. Denote the

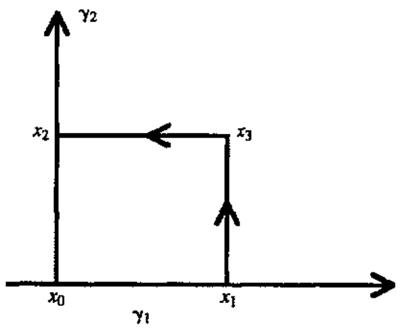

Fig. 1. The region Γ.

four corners Γ_0 of Γ by x_j, $j = 0, 1, 2, 3$, where the subscript j has a binary representation (γ_2, γ_1), (Fig. 1). Thus, x_2 corresponds to $\gamma_1 = 0$, $\gamma_2 = 1$. Let $\bar{x}_j \in R^2$ denote $f(j\omega, x_j)$, the coordinates of $\bar{x}_j$ being the real part and the imaginary part of $f(j\omega, x_j)$. At an arbitrary $\underline{\gamma} \in \Gamma$, we can evaluate the Jacobian determinant J_{12} of the mapping $\underline{\gamma} \mapsto f(j\omega, \underline{\gamma})$ as follows. (The reason for the subscripting on J will become apparent later, in considering the case of more than two parameters.) Let

$$f_i(j\omega) = g_i(j\omega) + jh_i(j\omega) \quad i = 0, 1, 2, 3 \quad (5)$$

be a decomposition of f_i into its real and imaginary parts.

Then

$$J_{12} = \det\begin{bmatrix} \dfrac{\partial \operatorname{Re} f}{\partial \gamma_1} & \dfrac{\partial \operatorname{Re} f}{\partial \gamma_2} \\ \dfrac{\partial \operatorname{Im} f}{\partial \gamma_1} & \dfrac{\partial \operatorname{Im} f}{\partial \gamma_2} \end{bmatrix}$$
$$= \det\begin{bmatrix} g_1(j\omega) + \gamma_2 g_3(j\omega) & g_2(j\omega) + \gamma_1 g_3(j\omega) \\ h_1(j\omega) + \gamma_2 h_3(j\omega) & h_2(j\omega) + \gamma_1 h_3(j\omega) \end{bmatrix}$$
$$= (g_1h_2 - g_2h_1) + \gamma_1(g_1h_3 - g_3h_1) + \gamma(g_3h_2 - g_2h_3). \quad (6)$$

Before proceeding further, we make the following observation from which flow the main algebraic results of this paper.

Fact 2.1. The ratio

$$\frac{\dfrac{\partial \operatorname{Im} f}{\partial \gamma_i}}{\dfrac{\partial \operatorname{Re} f}{\partial \gamma_i}}, \quad (7)$$

evaluated at a given γ, is the slope in the value set space of the image of any line in Γ that is parallel to the γ_i axis and passes through γ.

Thus, for example, if x_0 and x_1 are the respective points with coordinates $(\gamma_1 = 0,\ \gamma_2 = 0)$ and $(\gamma_1 = 1,\ \gamma_2 = 0)$, then (7) evaluated at x_0 is the slope of the segment joining $f(j\omega, x_0)$ and $f(j\omega, x_1)$.

We can now state the following result.

Theorem 2.1. With notation as defined above, the following conditions are equivalent:

(i) $f(j\omega, \Gamma)$ is a four-cornered convex polygon;

(ii) $\operatorname{conv} f(j\omega, \Gamma_0)$ has edges forming a quadrilateral, and these edges are images of edges of Γ;

(iii) $J_{12}(x_j)$ is a nonzero and has the same sign for $j = 0, 1, 2, 3$; and

(iv) $J_{12}(\gamma) \neq 0$ for any $\gamma \in \Gamma$, i.e. J_{12} has constant sign in Γ.

Proof. In proving (iii)→(iv) by (6), we have that J_{12} is linear, and thus convex, in γ_1 and γ_2. Since J_{12} has the same sign at the extreme values of γ_1 and γ_2, i.e. the corners of Γ, its value at any other point of Γ, which is a convex combination of the values at the corners, must also have the same sign.

For (iv)→(iii) suppose (iii) fails. Then, it is immediate that (iv) fails.

For (i)↔(iii) consider the images in the value set of the four lines x_0x_1, x_1x_3, x_3x_2 and x_2x_0. The image can take one of several forms: a convex quadrilateral with the same or opposite orientation to $x_0x_1x_3x_2$ (cases (a) and (f) of Fig. 2); a triangle or a straight line with the ordering of the vertices being immaterial for our purposes (cases (b) and (c)); a nonconvex quadrilateral (case (d)); or a figure with an intersection (case (e)). This is the full set of possibilities that needs to be considered. Ackermann (1991) includes a catalog of all possibilities, which is in effect 'indexed' by the nature of the intersection (if any) between the locus of points where the

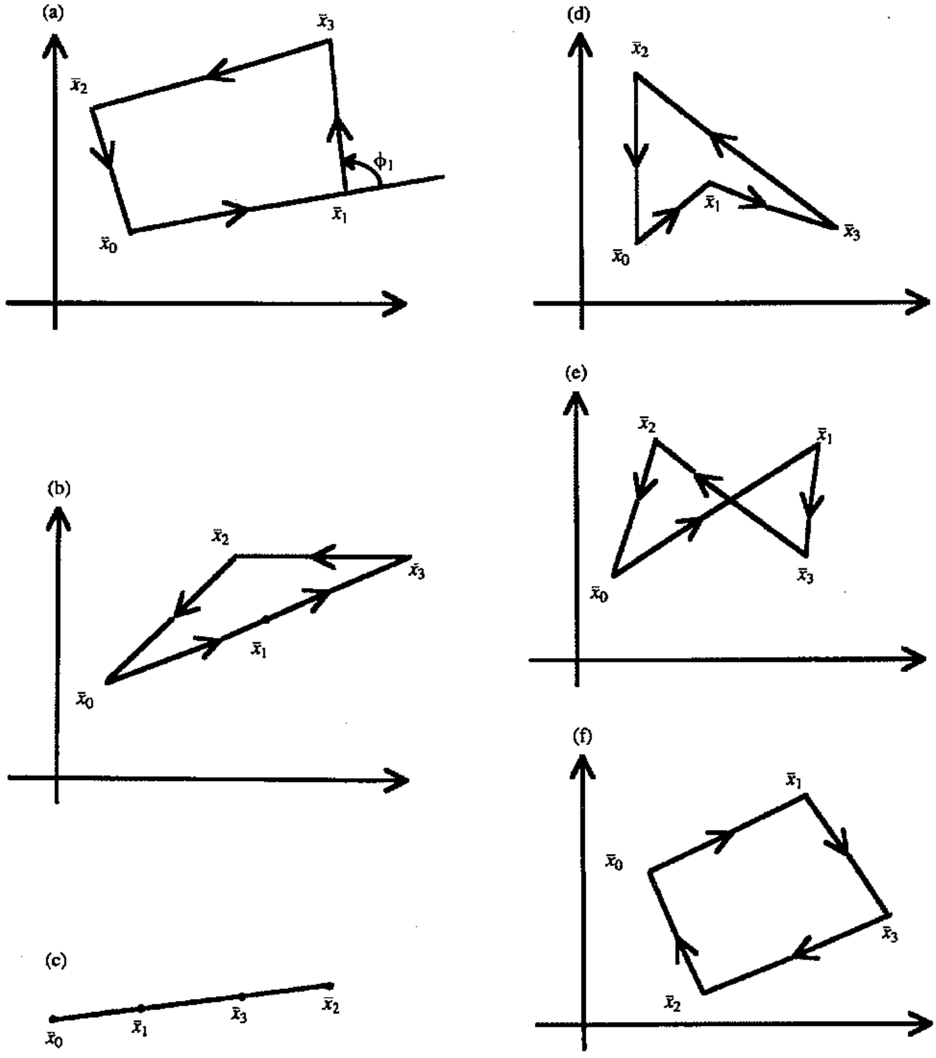

Fig. 2. Possible images of Γ considered in the proof of Theorem 2.1.

Jacobian is zero with the set Γ.

Elementary calculus and Fact 2.1, allow characterization of the Jacobian values at each corner on the basis of the angular change at the corner (the angular change being that of the images of the oriented lines x_0x_1, x_1x_3, x_3x_2, x_2x_0 where one line ends and another begins):

Case	Jacobian values
(a)	All positive;
(b)	$J_{12}(x_1)=0$, $J_{12}(x_3)>0$, $J_{12}(x_2)>0$, $J_{12}(x_0)>0$;
(c)	$J_{12}(x_1)=0$, $J_{12}(x_3)=0$, $J_{12}(x_2)>0$, $J_{12}(x_0)>0$;
(d)	$J_{12}(x_1)<0$, $J_{12}(x_3)>0$, $J_{12}(x_2)>0$ and $J_{12}(x_0)>0$;
(e)	$J_{12}(x_1)<0$, $J_{12}(x_3)<0$, $J_{12}(x_2)>0$ and $J_{12}(x_0)>0$;
(f)	All negative.

Under case (e), there exist points on x_2x_3 and x_0x_1 which have the same image. Points on x_2x_3 and x_0x_1 are of the form $(1, \gamma_{1b})$ and $(\gamma_{1a}, 0)$. So, for some γ_{1a} and γ_{1b}

$$f_0(j\omega)+\gamma_{1a}f_1(j\omega)=f_0(j\omega)+\gamma_{1b}f_1(j\omega)+f_2(j\omega)+\gamma_{1b}f_3(j\omega)$$

holds, and then it is easily verified that

$$J_{12}(\gamma_1=\gamma_{1b}, \gamma_2=0)=0.$$

Hence, under case (e), the Jacobian is zero at a point in Γ.

Evidently, condition (i) of the Theorem holds if and only if condition (iii) holds. The above argument also shows the equivalence with condition (ii). This completes the proof. ■

Note that the case of affine rather than multiaffine dependence is captured in a very simple way by the theorem. The determinant J_{12} is constant. Whenever it is nonzero, $f(j\omega, \Gamma)$ is a convex quadrilateral; whenever it is zero, $f(j\omega, \Gamma)$ is a straight line (of finite extent).

Notice that if all Jacobian values have the same sign at a particular frequency, they will have the same sign in an interval including this frequency. The Jacobian evaluated at the point x_0 changes sign, see (6) at those real values of ω (if any) for which $g_1(j\omega)h_2(j\omega)-g_2(j\omega)h_1(j\omega)$ is zero. By identifying such values, not just for x_0 but for x_1, x_2, x_3, the intervals over which all Jacobians have the same sign can be identified.

3. PRELIMINARIES FOR THE m-DIMENSIONAL PARAMETER SPACE PROBLEM

We begin by making explicit the enumeration scheme used for the vertices of $\Gamma=[0,1]^m$. Each vertex is defined by a γ, whose elements are either 1 or 0. Consider a vertex corresponding to such a binary vector $\underline{\gamma}$. We will designate this vertex as x_i where

$$i=\sum_{j=1}^{m}\gamma_j 2^{m-j}. \tag{8}$$

Thus, with $m=3$ the vertex corresponding to $\gamma=[1,1,0]$ will admit the designation x_3.

We suppose, in this section, that

$$f(j\omega, \underline{\gamma})=g(j\omega, \underline{\gamma})+jh(j\omega, \underline{\gamma}) \tag{9}$$

is a multiaffine mapping (for each fixed ω) of $\underline{\gamma}\in\Gamma$ into $R^2: \underline{\gamma}\to[g(j\omega, \underline{\gamma}), h(j\omega, \underline{\gamma})]$. We shall use the notation

$$J_{\alpha\beta}=\det\begin{bmatrix}\dfrac{\partial g}{\partial\gamma_\alpha} & \dfrac{\partial g}{\partial\gamma_\beta}\\ \dfrac{\partial h}{\partial\gamma_\alpha} & \dfrac{\partial h}{\partial\gamma_\beta}\end{bmatrix}. \tag{10}$$

Of course, $J_{\alpha\beta}$ depends on $\underline{\gamma}$. Notice, that $J_{\alpha\beta}=-J_{\beta\alpha}$. We observe first (and the proof is trivial):

Proposition 3.1. Any change of variables in parameter space in which the γ_i are reordered and/or γ_i is replaced by $\gamma_i'=1-\gamma_i$ preserves the pre-image set $\Gamma\subset R^n$, and preserves the multiaffine character of f.

Call such changes of variables 'allowed'. Next, we have:

Proposition 3.2. Let $\underline{\gamma}\in\Gamma$ be fixed. Suppose, that $J_{\alpha\beta}\neq 0$ for all $\alpha\neq\beta$. Then, there exists an allowed change of variables such that the Jacobian determinants computed with the new variables satisfy $J_{\lambda\mu}>0$ for all $\lambda<\mu$.

Proof. Let J be the $2\times m$ matrix

$$J=\begin{bmatrix}\dfrac{\partial g}{\partial\gamma_1} & \dfrac{\partial g}{\partial\gamma_2} & \cdots & \dfrac{\partial g}{\partial\gamma_m}\\ \dfrac{\partial h}{\partial\gamma_1} & \dfrac{\partial h}{\partial\gamma_2} & \cdots & \dfrac{\partial h}{\partial\gamma_m}\end{bmatrix}. \tag{11}$$

At most, one entry of the first row can be zero, since $J_{\alpha\beta}\neq 0$ for all $\alpha\neq\beta$. If $\dfrac{\partial g}{\partial\gamma_i}<0$ for any i, introduce the new variable $\gamma_i'=1-\gamma_i$. Then, the corresponding J matrix in the new coordinates has all the first row positive, except possibly for one entry which could be zero.

Next, reorder the variables so that after reordering,

$$\frac{\dfrac{\partial h}{\partial\gamma_1}}{\dfrac{\partial g}{\partial\gamma_1}}<\frac{\dfrac{\partial h}{\partial\gamma_2}}{\dfrac{\partial g}{\partial\gamma_2}}<\cdots<\frac{\dfrac{\partial h}{\partial\gamma_m}}{\dfrac{\partial g}{\partial\gamma_m}}. \tag{12}$$

(If, before reordering, $\dfrac{\partial g}{\partial\gamma_i}$ is zero and

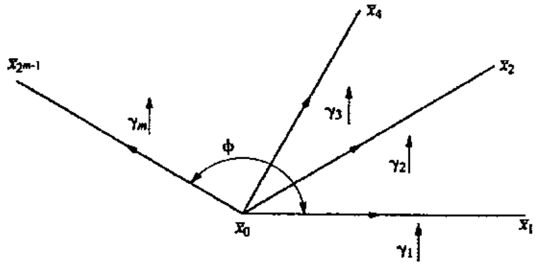

Fig. 3. Ordering property of edge images emanating from $\bar{x}_0$.

$\frac{\partial h}{\partial \gamma_i}$ is positive, γ_i becomes γ_m; otherwise it becomes γ_i. The fact that $J_{\alpha\beta} \neq 0$, for all $\alpha \neq \beta$, ensures that the inequalities above must be strict.) It is now trivial to observe that with the new (allowed) coordinates $J_{\lambda\mu} > 0$ for all $\lambda < \mu$. ■

With an assumption of positivity of Jacobian determinants at one corner of Γ, we can say something about the image in the value set of the edges emanating from this corner. Initially, let us consider the corner x_0. The m incident edges connect x_0 to $x_1, x_2, x_4, \ldots, x_{2^{m-1}}$, and one moves from x_0 to $x_{2^{i-1}}$ by increasing γ_i from 0 to 1, retaining the other γ_j at zero.

Proposition 3.3. Under the assumption that $J_{\lambda\mu}(x_0) > 0$ for all $\lambda < \mu$, the images of the edges $x_0x_{2^{i-1}}$ are a set of (non overlapping) straight lines $\bar{x}_0\bar{x}_{2^{i-1}}$, angularly ordered as shown in Fig. 3, with the angle ϕ between the directed lines $\bar{x}_0\bar{x}_1$ and $\bar{x}_0\bar{x}_{2^{m-1}}$ satisfying $0 < \phi < \pi$. More precisely, with ϕ_i the angle made with respect to the real axis by the segment $\bar{x}_0\bar{x}_{2^{i-1}}$, one has $\forall j < i$, $0 < \phi_i - \phi_j < \pi$.

Proof. Consider the face in Γ including x_0 and define by variation of γ_1, γ_2 (Fig. 1). The images of x_0x_1 and x_0x_2 are necessarily straight lines.

The angular ordering of $\bar{x}_0\bar{x}_1$ and $\bar{x}_0\bar{x}_2$ in Fig. 3 is then easily established from the positivity of the Jacobian determinant J_{12} at x_0 and Fact 2.1. Fact 2.1 also establishes the fact that the angle between these directed lines cannot exceed π.

A similar argument using the fact that $J_{1i} > 0$, shows that the angle between $\bar{x}_0\bar{x}_1$ and $\bar{x}_0\bar{x}_{2^{i-1}}$ for any i cannot exceed π; because $J_{23} > 0$, $J_{34} > 0, \ldots$, the angular ordering of $\bar{x}_1\bar{x}_2$, $\bar{x}_0\bar{x}_4$, $\bar{x}_0\bar{x}_8$, etc. follows. ■

A variation on Proposition 3.3 can be used to study the image of the edges from any corner. Suppose, for example, that at x_1, one has $J_{\alpha\beta} > 0$ for all $\alpha < \beta$. Then, the images in the value set of the edges outgoing from x_1 have the angular ordering and angular spread depicted in Fig. 4.

4. MAIN RESULT (m-DIMENSIONAL PARAMETER SPACE)

In this section, we present a sufficient condition for the value set to be a convex polygon, with edges that are images of (parameter space) edges of Γ. The result is suggested by the equivalence of conditions (i) and (iii) in Theorem 2.1, which applies to the case of two parameters.

Theorem 4.1. Let $f(j\omega, \underline{\gamma}) = g(j\omega, \underline{\gamma}) + jh(j\omega, \underline{\gamma})$ depend in a multiaffine manner on parameters $\gamma_1, \gamma_2, \ldots, \gamma_m$, with $\underline{\gamma} = [\gamma_1\ \gamma_2\ \cdots\ \gamma_m]' \in \Gamma = [0, 1]^m$. Suppose, that (with ω fixed) for each pair α, β with $\alpha \neq \beta$, $J_{\alpha\beta}(x_j)$ has the same sign for all corners x_j of Γ (with the sign possibly dependent on the pair α, β). Then, the value set (for the fixed ω) is a convex polygon with edges which are images of edges of Γ. In case $J_{\alpha\beta} > 0$ for all $\alpha < \beta$, the corners of the value set are given in (cyclic) order by $\bar{x}_0, \bar{x}_1, \bar{x}_3, \bar{x}_7, \ldots, \bar{x}_{2^m-1}, \bar{x}_{2^m-2}, \bar{x}_{2^m-4}, \bar{x}_{2^m-8}, \ldots, \bar{x}_{2^m-2^{m-1}} = \bar{x}_{2^{m-1}}$, and the successive edges in a counterclockwise direction are obtained by γ_1 increasing, γ_2 increasing, $\ldots, \gamma_m$ increasing, γ_1 decreasing, γ_2 decreasing, $\ldots, \gamma_m$ decreasing.

Figure 5 depicts the result for the case $m = 3$. Notice that for the angular ordering to hold, it is critical that $J_{\alpha\beta} > 0$ for all $\alpha < \beta$. However, even if this condition fails but the sign consistency condition holds, then Proposition 3.2 may still be

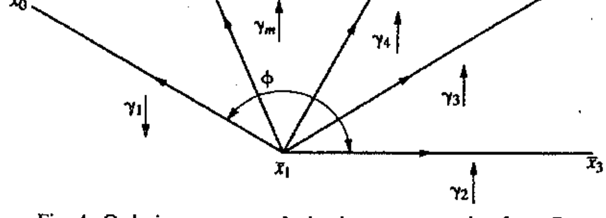

Fig. 4. Ordering property of edge images emanating from $\bar{x}_1$.

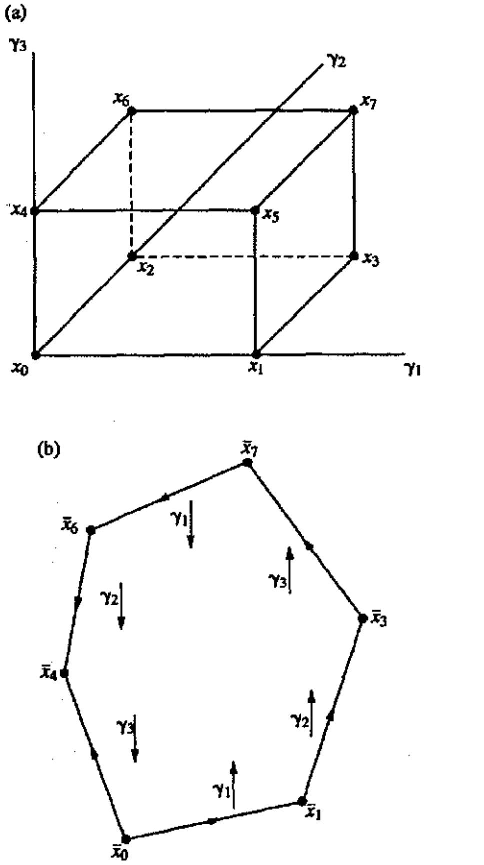

Fig. 5. (a) Three-dimensional box Γ. (b) Image of Γ given Jacobian sign condition.

exploited together with allowable changes in variables to construct the boundary of the value set.

For the proof of the theorem, observe first that (given the theorem hypotheses for each pair α, β that $J_{\alpha\beta}(x_j)$ has the same sign for all x_j) we can actually assume without loss of generality that $J_{\alpha\beta} > 0$ for all $\alpha < \beta$. If this is not initially the case, an allowed transformation of the type described in Section 3 can be introduced, and if $J_{\alpha\beta}(x_j)$ for each fixed α, β, and all x_j have the same sign before transformation, this remains so after transformation. We use the transformations to ensure that $J_{\alpha\beta}(x_0) > 0$ for all $\alpha < \beta$, and automatically then obtain $J_{\alpha\beta}(x_j) > 0$ for all x_j and $\alpha < \beta$.

The theorem proof will proceed by induction. The basic step of the induction was presented in Section 2.

Suppose, therefore, the result has been proven with $m-1$ parameters. We shall establish first the following.

Proposition 4.1. Assume the hypotheses of Theorem 4.1, together with $J_{\alpha\beta}(x_j) > 0$ for all $\alpha < \beta$ and all j. Consider the image $\bar{x}_0\bar{x}_1$ of the edge x_0x_1 of Γ. Define H as the open half plane lying to the left of an infinite prolongation of the directed line $\bar{x}_0\bar{x}_1$. Then, $\bar{x}_j \in H$ for $j = 2, 3, \ldots, 2^m - 1$. Figure 5 provides an illustration for the case $m = 3$.

Proof. Choose $i \in \{2, 3, \ldots, m\}$, and consider the $(m-1)$-dimensional subset Γ_i of Γ defined by $\gamma_i = 0$. For this $(m-1)$-dimensional subset, there holds at each corner $J_{\alpha\beta} > 0$ for $0 < \alpha < \beta \le m$, $\alpha \ne i$, $\beta \ne i$, these inequalities simply being a subset of those guaranteed by hypothesis. Then, by Theorem 4.1, using the induction hypothesis, we see that the value set $f(j\omega, \Gamma_i)$ is a convex polygon, with successive edges in a counterclockwise direction obtained by starting at $\bar{x}_0$, then increasing in turn $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_m$ and then decreasing $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_m$. The images of all corners of the $(m-1)$-dimensional subset Γ_i of Γ all lie on one side of $\bar{x}_0\bar{x}_1$, precisely because $\bar{x}_0\bar{x}_1$ is an edge of, and thus part of a supporting hyperplane of, the convex set defining the value set. The fact that $J_{12} > 0$ ensures, in fact, that all lie on the left side of $\bar{x}_0\bar{x}_1$, i.e. in H.

Since i can take any value between 2 and m, this argument shows that any $\bar{x}_j$, $j \ne 0, 1$ for which the binary expansion of j has a zero in any of positions $2, 3, \ldots, m$ is necessarily in H. the only $\bar{x}_j$ not covered are those with a 1 in the binary expansion of j in positions $2, 3, \ldots, m$, i.e. $\bar{x}_{2^m-1}$ and $\bar{x}_{2^m-2}$. We shall now prove that these also lie in H.

Consider the $(m-1)$-dimensional box Γ_1 in parameter space obtained by fixing $\gamma_1 = 0$ and letting the other γ_j vary. The induction hypothesis indicates that its value set is a convex polygon of which two successive edges are $\bar{x}_{2^{m-1}}x_0$ and $\bar{x}_0\bar{x}_2$ (see Fig. 6). The value set includes $\bar{x}_{2^m-2}$, and accordingly the straight line $\bar{x}_0\bar{x}_{2^m-2}$ (which is not the image of an edge of Γ_1) must lie between the lines $\bar{x}_0\bar{x}_2$ and $\bar{x}_0\bar{x}_{2^{m-1}}$. Refer now to Fig. 3, which is a consequence of the Jacobian condition, and shows that $\bar{x}_0\bar{x}_{2^m-2}$ also must lie between $\bar{x}_0\bar{x}_{2^{m-1}}$ and $\bar{x}_0\bar{x}_1$, and accordingly, $\bar{x}_{2^m-2} \in H$. We can consider, similarly, the $(m-1)$-dimensional box in parameter space obtained by fixing $\gamma_1 = 1$ and letting the other γ_j vary. The induction hypothesis shows that $\bar{x}_1\bar{x}_{2^m-1}$ lies between $\bar{x}_1\bar{x}_3$ and $\bar{x}_1\bar{x}_{2^{m-1}+1}$, and then refer to Fig. 4, which is a consequence of the Jacobian condition at x_1, and shows that $\bar{x}_1\bar{x}_{2^m-1}$ lies between $\bar{x}_1\bar{x}_0$ and $\bar{x}_1\bar{x}_3$. It is immediate that $\bar{x}_{2^m-1}$ lies in H. This completes the proof of the proposition. ■

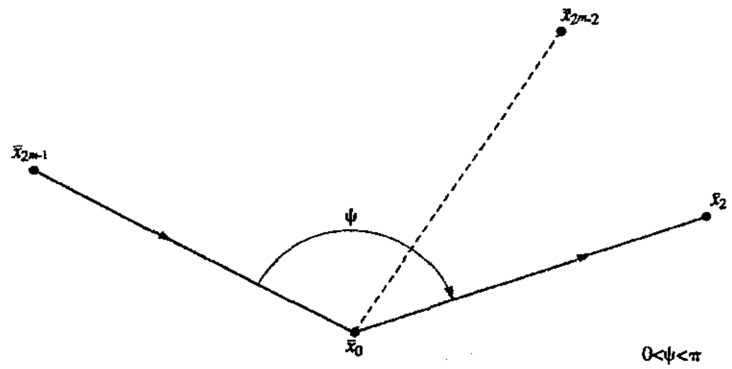

Fig. 6. Illustration of position of $\bar{x}_{2^m-2}$.

A proof of Theorem 4.1 can now be completed if (i) we can show that the value set lies on one side (the left side with the orientation we will state) of not just $\bar{x}_0\bar{x}_1$, but also $\bar{x}_1\bar{x}_3$, $\bar{x}_2\bar{x}_7, \ldots$, etc. and if (ii) we can show that $f(j\omega, \Gamma)$ is simply connected. With (i), it will follow that the boundary of $\operatorname{conv} f(j\omega, \Gamma)$ is itself part of $f(j\omega, \Gamma)$, comprising images of various edges of Γ, and then the corollary to the Mapping Theorem (Section 1) applies.

Proposition 4.1 is the tool for handling (i), together with use of a transformation which converts the problem of showing one-sidedness for an aribtrary straight line interval in the list $\bar{x}_1\bar{x}_3$, $\bar{x}_3\bar{x}_7$ etc. to a problem involving $\bar{x}_0\bar{x}_1$, as solved in Proposition 4.1.

Let $i \in \{1, 2, \ldots, m-1\}$ be arbitrary, and consider the following allowable parameter transformation:

$$\begin{aligned} \gamma_1' &= \gamma_{i+1} \\ \gamma_2' &= \gamma_{i+2} \\ &\vdots \\ \gamma_{m-i}' &= \gamma_m \\ \gamma_{m-i+1}' &= 1 - \gamma_1 \\ &\vdots \\ \gamma_m' &= 1 - \gamma_i. \end{aligned}$$

This transformation has the following properties:

(i) if $J_{\alpha\beta}$ denotes a Jacobian determinant computed using the γ_j and $K_{\lambda\mu}$ a Jacobian determinant computed using the γ_j', then

$$J_{\alpha\beta} > 0 \ \forall \alpha < \beta,$$

implies and is implied by

$$K_{\lambda\mu} > 0 \ \forall \lambda < \mu; \quad \text{and}$$

(ii) the value set computed with the $\{\gamma_k\}$ and $\{\gamma_j'\}$ is the same, while the edges labelled in cyclic order

$$\gamma_{i+1}\uparrow, \gamma_{i+2}\uparrow, \ldots, \gamma_m\uparrow, \gamma_1\downarrow, \ldots, \gamma_m\downarrow, \gamma_1\uparrow, \ldots, \gamma_i\uparrow,$$

become

$$\gamma_1'\uparrow, \gamma_2'\uparrow, \ldots, \gamma_m'\uparrow, \gamma_1'\downarrow, \ldots, \gamma_m'\downarrow.$$

For the $\{\gamma_j'\}$ system, Proposition 4.1 implies that the value set lies to the left of the images in the value set of an edge corresponding to γ_1' increasing. This is equivalent for the $\{\gamma_j\}$ system to establishing that the value set lies to the left of the image of the edge corresponding to γ_{i+1} increasing. In this way, Proposition 4.1 extends to a result for $\bar{x}_1\bar{x}_2$, $\bar{x}_3\bar{x}_7 \cdots \bar{x}_{2^{m-1}-1}\bar{x}_{2^m-1}$. Use of the transformation $\gamma_j' = 1 - \gamma_j$, $j = 1, \ldots, m$ then extends the result to the remaining edges claimed in Theorem 4.1 to contribute the boundary of the value set.

At this point we have established:

Proposition 4.2. Assume the hypotheses of Theorem 4.1 together with $J_{\alpha\beta}(x_j) > 0$ for all $\alpha < \beta$ and all j. Then the polygon $\operatorname{conv} f(j\omega, \Gamma)$ comprises the edge images $\bar{x}_0\bar{x}_1$, $\bar{x}_1\bar{x}_2$, $\bar{x}_3\bar{x}_7$, $\ldots, \bar{x}_{2^{m-1}-1}\bar{x}_{2^m-1}, \bar{x}_{2^m-1}\bar{x}_{2^m-2}, \ldots, \bar{x}_{2^{m-1}}\bar{x}_0$.

The proof of Theorem 4.1 will be completed by showing:

Proposition 4.3. Under the hypotheses of Theorem 4.1 together with $J_{\alpha\beta}(x_j) > 0$ for all $\alpha < \beta$ and all j, $f(j\omega, \Gamma)$ is simply connected.

Proof. The result will be proved by induction. The basic step was established in Section 2. Assume the result is valid for $m-1$. Consider Fig. 7. The inner convex polygon (with some solid edges and some dashed edges and corners $\cdots \bar{x}_{2^m-2}\bar{x}_0\bar{x}_1\bar{x}_3 \ \cdots \ \bar{x}_{2^{m-2}-1}\bar{x}_{2^{m-1}-1}\bar{x}_{2^{m-1}-2}\bar{x}_{2^{m-1}-4} \cdots$) is $f(j\omega, \Gamma_m)$, i.e. the image obtained when $\gamma_m = 0$. By the induction hypothesis, every point inside this polygon boundary is an image of a point in Γ_m and thus Γ. The outer convex polygon boundary is, by Proposition 4.2, the boundary of $f(j\omega, \Gamma)$. Consider, also, the $(m-1)$

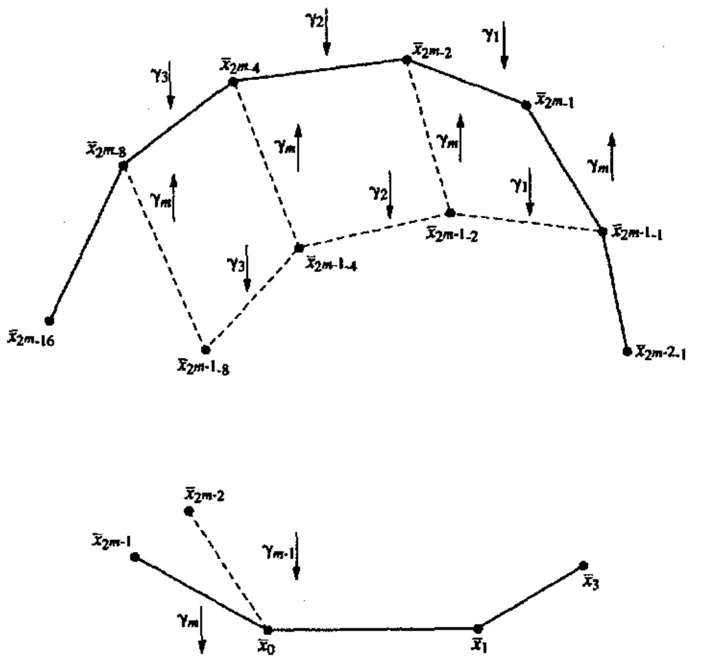

Fig. 7. Decomposition of the image of Γ used in Proposition 4.3.

quadrilateral regions defined by, e.g.

$$\bar{x}_{2^{m-1}-2}\bar{x}_{2^{m-1}-1}\bar{x}_{2^m-1}\bar{x}_{2^m-2},$$

$$\bar{x}_{2^{m-1}-4}\bar{x}_{2^{m-1}-2}\bar{x}_{2^m-2}\bar{x}_{2^m-4}, \ldots.$$

Their union together with $f(j\omega, \Gamma_m)$ makes up the whole outer polygon. These quadrilateral regions are the images of various faces of Γ; in order the face where $\gamma_2 = \gamma_3 = \cdots = \gamma_{m-1} = 1$, the face where $\gamma_1 = 0$, $\gamma_3 = \gamma_4 = \cdots = \gamma_{m-1} = 1$, the face where $\gamma_1 = \gamma_2 = 0$, $\gamma_4 = \cdots = \gamma_{m-1} = 1$, etc.

As such, every point in these quadrilaterals is the image of (at least) one point of Γ. Consequently, the whole outer polygon is identical to $f(j\omega, \Gamma)$, and the proposition and, with it, the theorem is established. ∎

Remark. Figure 7 displays a further interesting property. Define F_{ij} to be the face with $\gamma_1 = \gamma_2 = \cdots = \gamma_{i-1} = 0$, $\gamma_{i+1} = \gamma_{i+2} = \cdots = \gamma_{j-1}$, $\gamma_{j+1} = \cdots = \gamma_m = 0$. The image under f is denoted by $\bar{F}_{ij}$. Then, Fig. 7 displays

$$f(j\omega, \Gamma) = f(j\omega, \Gamma_m) + \bar{F}_{1m} \cup \bar{F}_{2m} \cdots \cup \bar{F}_{m-1,m}.$$

Now, the decomposition first applied to $f(j\omega, \Gamma)$ can be applied to $f(j\omega, \Gamma_m)$. The result is $f(j\omega, \Gamma_m) = f(j\omega$, subset of Γ where $\gamma_m = 0$ and $\gamma_{m-1} = 0) \cup \bar{F}_{1,m-1} \cup \bar{F}_{2,m-1} \cdots \cup \bar{F}_{m-2,m-1}$. In turn, $f(j\omega$, subset of Γ where $\gamma_m = 0$ and $\gamma_{m-1} = 0)$ can be further decomposed. The end result is:

$$f(j\omega, \Gamma) = \bigcup_{1 \le i < j \le m} \bar{F}_{ij}.$$

So the value set itself, as opposed to its boundary, is a union of images of $\frac{1}{2}m(m-1)$ faces; the value sets of the individual faces either intersect in a line, a point, or not at all.

As for the case $m = 2$, the determination of intervals over which the Jacobians evaluated at the corners have constant sign is simply a matter of determining the purely real zeros of certain polynomials.

5. REMARKS AND EXAMPLES

Can the Jacobian determinant condition in Theorem 4.1 be relaxed to allow $J_{\alpha\beta} \ge 0$ for all $\alpha < \beta$ rather than the strict inequality demanded by the theorem? The answer, in general, is no. Consider, the three-dimensional Γ of Fig. 5(a) and

$$f(j\omega, \Gamma) = 7\gamma_1 + 5(\gamma_2 + \gamma_3) - 6(\gamma_1\gamma_3 + \gamma_2\gamma_3) + 10\gamma_1\gamma_2\gamma_3 + j\gamma_1\gamma_2\gamma_3.$$

Then, $J_{\alpha\beta} \ge 0$ for all $\alpha < \beta$ at each corner of Γ. However, the value set (not drawn to scale) is as in Fig. 8 ($\bar{x}_0$ coincides with the origin, $\bar{x}_3 = 12$, $\bar{x}_2 = 15 + j1$ and all other $\bar{x}_i$ lie between $\bar{x}_0$ and $\bar{x}_3$.)

Nonetheless, the strict sign consistency requirement can be relaxed to the extent described in Theorem 5.1, the proof of which follows from the fact that the limit point of any sequence of convex sets is itself convex.

Theorem 5.1. Suppose, the conditions of Theorem 4.1 hold at all but isolated real values

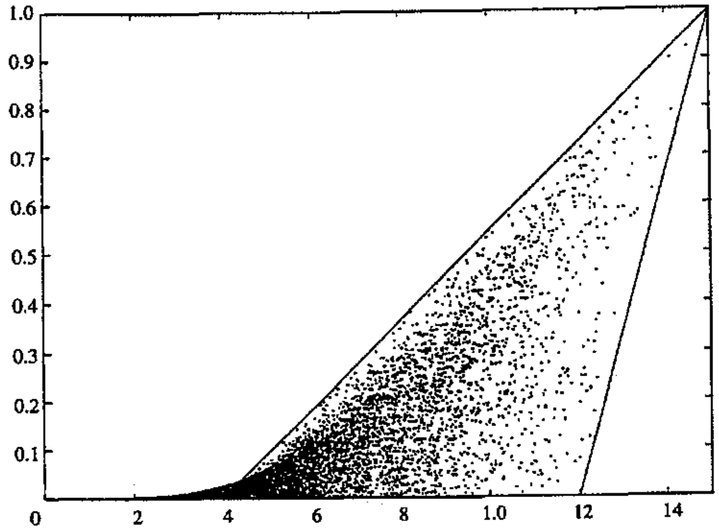

Fig. 8. Value set with determinant condition $J_{\alpha\beta} \geq 0$ for all $\alpha < \beta$.

of ω. Then, the conclusions of the theorem apply at all ω, save that three or more of the corner points may be collinear.

Is the Jacobian determinant condition 'likely' to be satisfied? Recall (or easily observe) that in the case of affine dependence of a polynomial $f(s, \gamma)$ on a parameter vector γ, the Jacobian determinants are all (for a fixed s) constant. Accordingly, in the multiaffine case with $\Gamma = [0, 1]^m$ (i.e. with scaling of the parameter introduced to normalize the region under consideration), it follows that if the nonlinearly dependent parts of f are multiplied by suitably small constants, the signs of nonzero determinants, due to continuity, will be preserved. Thus, Theorem 4.1 captures any multiaffine dependence with a small enough amount of nonlinearity, and with nonpathological dependence of the affine part on the parameters (so that for the linear problem all determinants are nonzero). The argument just given identifies a very important and wide collection of cases where the result is applicable, to an extent, at variance with one's initial intuition. (Note, the title of Poljak (1992) also.) Note, though, that there also exist arbitrarily 'large' nonlinearities where the Jacobian conditions are satisfied, as pointed out in Poljak (1992). Poljak (1992) also contains several important classes of polynomials to which the theory is applicable, e.g. $A(s) + B(s)(1 + \lambda_2 s) \cdots (1 + \lambda_m s)$, $\lambda_i \in [\underline{\lambda}_i, \bar{\lambda}^i]$, $\underline{\lambda}_i > \bar{\lambda}_{i+1}$, and $0 \notin [\underline{\lambda}_i, \bar{\lambda}^i]$.

Is the Jacobian determinant condition necessary for the value set to be a convex polygon, with edges which are images of edges of Γ? An example shows this is not the case. Consider an $f(j\omega, \gamma)$ with $\gamma \in R^3$ such that for some ω,

$$\mathrm{Re} f = -1 + 2\gamma_1 + 3\gamma_3 - 6\gamma_1\gamma_3,$$

$$\mathrm{Im} f = -1 + 2\gamma_2 - \gamma_3 + 2\gamma_2\gamma_3.$$

It is easily checked, that the mapping of the face $\gamma_2 = 1$ shown in Fig. 5(a) is as depicted in Fig. 9(a).

It is also easily verified that the image of the face $\gamma_3 = 0$ is contained in the image of the face $\gamma_3 = 1$. Observe that $\bar{x}_0, \bar{x}_1, \bar{x}_3$ and $\bar{x}_2$ fall within the image of the face $\gamma_3 = 1$. Hence, the image of the face $\gamma_3 = 1$ contains the convex hull of the $\bar{x}_i$, and thus, by the Mapping Theorem, the convex hull of the image of Γ. On the other hand, the face $\gamma_3 = 1$ is contained in Γ, and so its image is contained in the image of Γ. Hence, the image of Γ is identical with the image of Fig. 9(a).

Obviously, the value set is a convex polygon with edges which are images of edges of Γ; it is trivial to observe that the Jacobian determinant condition is not satisfied either by direct

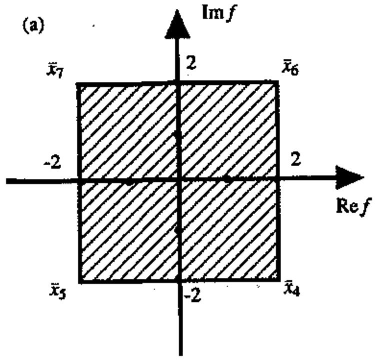

Fig. 9(a).

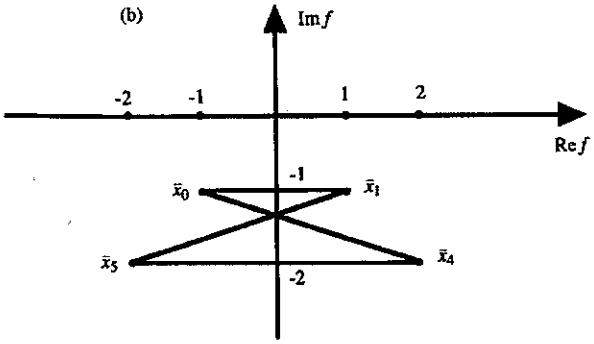

Fig. 9. (a) Image of face $\gamma_3 = 1$. (b) Images of edges of face $\gamma_2 = 0$.

calculation, or by observing the images of the edges of the face $\gamma_2 = 0$ (Fig. 9).

Is it possible to have a value set with an interior hole and with outer boundary defining a convex polygon with edges that are images of edges of Γ?

The answer is yes. Again, we consider a three-dimensional Γ:

$$\mathrm{Re} f = -(\gamma_1 + \gamma_2 + \gamma_3) + 3(\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3) - 6\gamma_1\gamma_2\gamma_3;$$

and

$$\mathrm{Im} f = 1 - (\gamma_1 + \gamma_2 + \gamma_3) + (\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3).$$

The value set is depicted in Fig. 10. The ruled part of the figure is obtained as the image of the faces, and the dotted part of the figure by selecting points in the interior of Γ. The interior boundary of the value set is the image of the diagonal in Γ joining $\gamma_1 = \gamma_2 = \gamma_3 = 0$ to $\gamma_1 = \gamma_2 = \gamma_3 = 1$. Because Re f and Im f are symmetric in γ_1, γ_2 and γ_3, two pre-image points, whose coordinates differ simply via permutation, have the same image. Hence, the six faces give rise to only two distinct images. The corners (with the usual enumeration) are mapped as follows:

$$\bar{x}_0 = \bar{x}_7 = 0 + j1;$$

$$\bar{x}_1 = \bar{x}_2 = \bar{x}_4 = -1 + j0;$$

and

$$\bar{x}_3 = \bar{x}_5 = \bar{x}_6 = 1 + j0.$$

Clearly, the outer boundary of the value set is the image of edges of Γ. Every face of Γ has a nonconvex image, and the image of Γ has a boundary consisting of straight lines together with a curve. The straight lines are images of the edges of the faces and the curve is the image of points in the faces where a Jacobian determinant is zero.

The fact that the interior and exterior boundaries of the value set meet at one point is nothing special. To avoid this situation, we could simply define a new polynomial

$$\bar{f} = \gamma_4 + f,$$

with $\gamma_4 \in [-\epsilon, \epsilon]$. The value set of $\bar{f}$ will then have an interior and exterior boundary that do not meet.

An example of this type was constructed in Poljak (1992). The characteristic equation is

$$(s - \gamma_i)(s - \gamma_2)(s - \gamma_3) + 6s^2 + 4s + 6,$$

where $|\gamma_i| \leq \sqrt{3}$, and for $\omega = 1$ a value set like Fig. 10 is obtained.

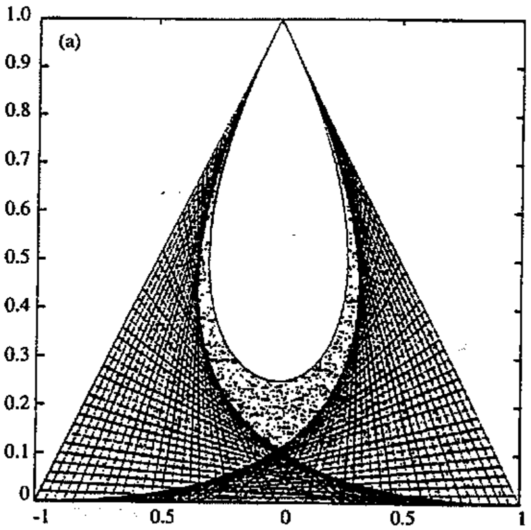

Fig. 10(a).

5.1. *Use of ideas in examining the Hurwitz stability criterion*

Consider the polynomial set

$$f(s) = s^3 + \gamma_1 s^2 + (\gamma_2 + \gamma_3)s + \gamma_1\gamma_3,$$

with $\gamma_i > 0$. The associated Hurwitz matrix is

$$H = \begin{bmatrix} \gamma_1 & 1 & \\ \gamma_1\gamma_3 & \gamma_2 + \gamma_3 & \gamma_1 \\ 0 & 0 & \gamma_1\gamma_3 \end{bmatrix}$$

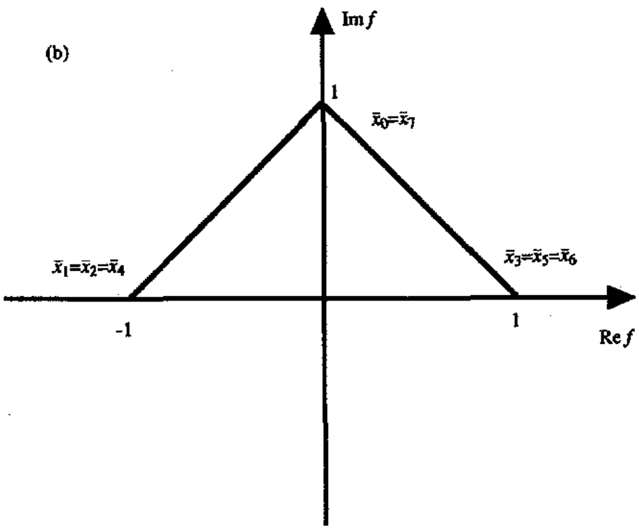

Fig. 10. (a) Value set with outer boundary defining a convex hole and with interior hole. (b) Images of edges and corners of Γ.

and the leading principal minors are positive if and only if the γ_i are all positive. It is interesting then to consider the value set when $\underline{\gamma} \in [a, b]^3$ for fixed $a < b \in (0, \infty)$. Since

$$\operatorname{Re} f(j\omega) = -\gamma_1\omega^2 + \gamma_1\gamma_3,$$

$$\operatorname{Im} f(j\omega) = -\omega^3 + (\gamma_2 + \gamma_3)\omega,$$

we obtain easily

$$J_{12}(\underline{\gamma}) = (-\omega^2 + \gamma_3)\omega,$$

$$J_{13}(\underline{\gamma}) = (-\omega^2 + \gamma_3)\omega,$$

$$J_{23}(\gamma) = -\gamma_1\omega.$$

Choose a fixed value of ω, say ω_0. If $\omega_0^2 < a$ or $\omega_0^2 > b$, the three Jacobian determinants have the same sign, and a convex polytopic value set results. On the other hand, if $a < \omega_0^2 < b$, this is not the case. However, we can then divide up the parameter space box and consider separately the value sets corresponding to $\gamma_3 \in [a, \sqrt{\omega_0} - \epsilon]$, $\gamma_3 \in [\sqrt{\omega_0} - \epsilon, \sqrt{\omega_0} + \epsilon]$ and $\gamma_3 \in [\sqrt{\omega_0} + \epsilon, b]$ with $\epsilon \to 0$. The first and third lead to convex polytopic sets, and the second, because $\operatorname{Re} f(j\omega) = 0$, to a line $\operatorname{Im} f(j\omega_0) = \gamma_2\omega_0$. Figure 11 depicts the three value sets, the first being

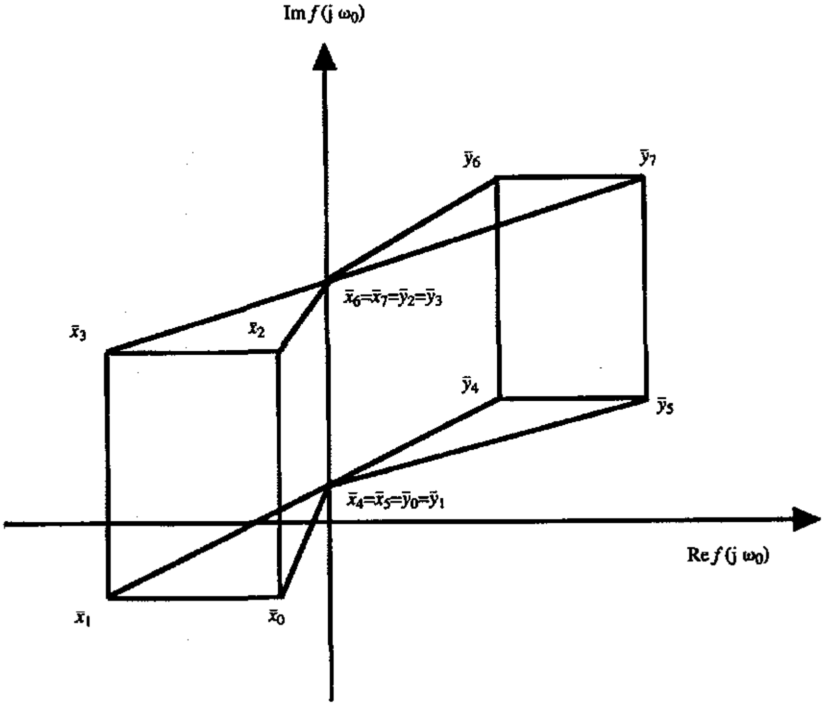

Fig. 11. Value sets obtained from the third-order Hurwitz polynomial.

defined by the $\bar{x}_i$, the third by the $\bar{y}_j$, and the second is the finite segment of the imaginary axis. It is clear then that in this case, intelligent use of Theorem 4.1 establishes the value set very easily.

The following fourth degree polynomial is stable for all $\gamma_i > 0$:

$$f(s) = s^4 + \gamma_4 s^3 + (\gamma_1 + \gamma_2 + \gamma_3)s^2 + \gamma_4(\gamma_1 + \gamma_3)s + \gamma_2\gamma_1.$$

The various Jacobian determinants have the following expressions:

$$J_{12} = -\omega\gamma_4(\gamma_1 - \omega^2), \quad J_{13} = \gamma_2\gamma_4\omega,$$

$$J_{14} = (\gamma_2 - \omega^2)(\gamma_1 + \gamma_3 - \omega^2);$$

and

$$J_{23} = \omega\gamma_4(\gamma_1 - \omega^2), \; J_{24} = (\gamma_1 - \omega^2)(\gamma_1 + \gamma_3 - \omega^2),$$

$$J_{34} = -\omega^3(\gamma_1 + \gamma_3 - \omega^2).$$

As in the previous example, in a general $\Gamma = [a, b]^4$, we can expect the Jacobian determinants to have sign changes. In the previous example, these sign changes occur only along a line parallel to an edge of Γ, and it was this fact that meant that the value set could be simply decomposed, as the union of convex polygons. In this example, however, if ω_0 is such that $2a < \omega_0^2 < 2b$, sign changes of Jacobian determinants occur along the line $\gamma_1 + \gamma_3 = \omega_0^2$ (as well as elsewhere), and this line is not parallel to any edge of Γ.

To fix ideas, let $a = 1$, $b = 2$. For $\omega = 0.5$ and $\omega = 3$, all J_{ij} have constant sign and the value set is convex (see Fig. 12(a) for the $\omega = 0.5$ case). For $\omega = 1.2$, Fig. 12(b) shows the images of the edges. It is clear that the convex hull defined by the corner points does not have a boundary comprising edge images. The value set will go outside the nonconvex polygon of Fig. 12(b), in particular to the lower right. Thus, it will not be possible in this case to express the value set as the union of a finite number of convex sets.

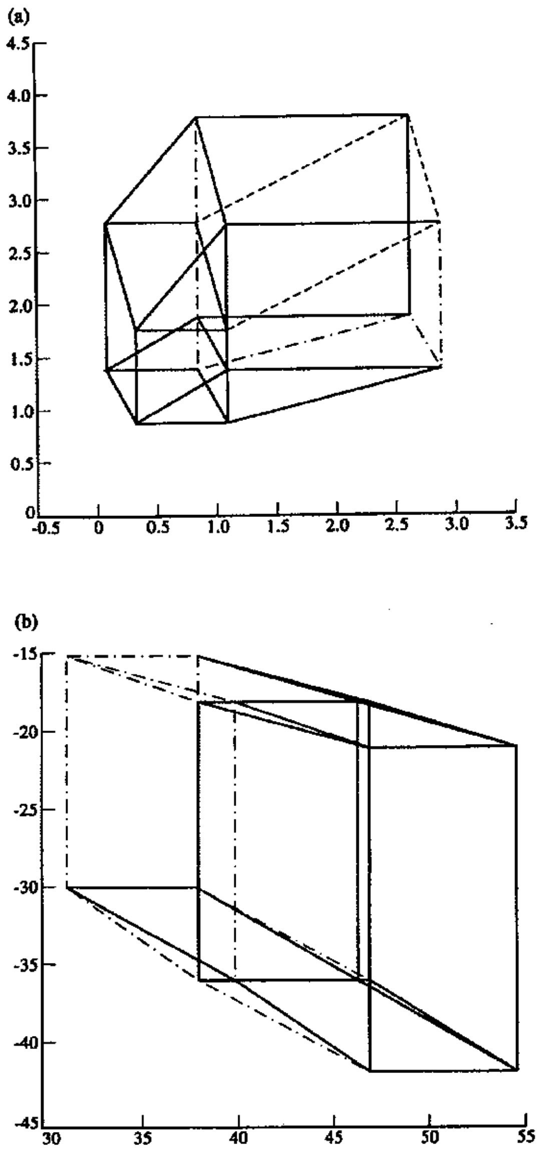

Fig. 12. (a) Value for fourth-order polynomial with $\omega = 0.5$. (b) Value set for fourth-order polynomial with $\omega = 3$.

5.2. *Comparison with the shaping conditions of Djaferis and Hollot*

We will now show how the results of Djaferis (1988) and Dajferis and Hollot (1989) can be phrased in terms of the Jacobian determinants considered here. Observe that with $S = \{1, \ldots, m\}$, a general m-parameter multiaffine polynomial can be expressed as

$$f(s, \gamma) = \sum_{r \subseteq S} \left(\prod_{i \in r} \gamma_i \right) f_r(s). \tag{13}$$

Thus, for example, with $r = \{1, 3, 4\}$, the coefficient of the $\gamma_1\gamma_3\gamma_4$ term in (13) is $f_r(s)$. Changing notation slightly from that used in Section 2, express $f_r(s)$ as a sum of its even and odd parts, i.e.

$$f_r(s) = g_r(s^2) + sh_r(s^2). \tag{14}$$

Now, form the following $1 \leq t \leq m$ disjoint groups, $G_1, \ldots, G_t$ of the coefficient polynomials $f_r(s)$: $f_r(s)$ and $f_\sigma(s)$ must belong to the same group if there exists an i such that $i \in r$ and $i \in \sigma$. Then, the results of Djaferis and Hollot (1989) show that the shaping conditions below ensure the convexity of the value set: for each $j \in \{1, \ldots, t\}$ and each $f_r(s)$, $f_\sigma(s) \in G_j$,

$$g_r(s^2)h_\sigma(s^2) - h_r(s^2)g_\sigma(s^2) = 0. \tag{15}$$

Notice, if for each $i, j \in S$ there exists at least one r obeying:

$$\{i, j\} \subset R \tag{16}$$

and

$$f_r(s) \neq 0, \tag{17}$$

i.e. since each parameter appears in a coupled way with every other parameter, the number of groups must equal one. In this case, it is easy to verify that at every frequency $\omega > 0$, each Jacobian determinant $J_{\alpha\beta}$ is invariant, in fact, identically zero over Γ. Strictly speaking, this case is not covered by the results of this paper because of the requirement of generically nonzero Jacobian determinants. Yet, the techniques presented here easily demonstrate (as do Djaferis and Hollot (1989)) that the value set at each frequency must in this case be a straight line segment.

Now suppose some parameters are uncoupled. Then, in general, equation (15) for multiaffine parametrizations is demonstrably equivalent to the following: (i) J_{ij} is independent of γ_i, γ_j and any parameter they are coupled to; (ii) it is affine in the parameters that are uncoupled to both γ_i, γ_j; and (iii) $J_{ij} = 0$ whenever γ_i and γ_j are uncoupled. In pure geometric terms, these conditions force the value set of any face of Γ that is defined by variations in mutually coupled parameters, to be straight line segments. Of course, the value set of faces defined by variations in uncoupled parameters would necessarily be convex.

Using Fact 2.1, Djaferis and Hollot (1989) demonstrate that under these conditions, barring the case where all parameters are mutually coupled, the resultant value set is a convex parpolygon, i.e. a convex polygon with an even number of sides and with opposite sides parallel to each other. In the two-parameter multiaffine setting, Djaferis (1988) relaxes the conditions of Djaferis and Hollot (1989) to require only (translated to the language of this paper) that J_{12} be independent of one of the two parameters. In this case the value set is a trapezoid.

Observe that neither the conditions of Djaferis and Hollot (1989) nor those of Djaferis (1988) are completely covered by those of this paper. Further, the reverse too is false (our results do not force the value sets to be either parpolygons or trapezoids). Thus, the results of Djaferis (1988) and Djaferis and Hollot (1989), and the results here complement each other.

6. THE CONJECTURE OF HOLLOT AND XU

In Hollot and Xu (1989), the following conjecture was made: $f(j\omega, \Gamma)$ is a convex polygon boundary if and only if all the edges of $\operatorname{conv} f(j\omega, \Gamma_0)$ are images of edges of Γ. The example with value set depicted in Fig. 10 is one which shows the 'if' statement to be false. We can, however, establish a result like the conjecture:

Theorem 6.1. With notation as previously, the outer boundary of $f(j\omega, \Gamma)$ is a convex polygon boundary if and only if all the edges of $\operatorname{conv} f(j\omega, \Gamma_0)$ are images of edges of Γ.

Note, that even if $f(j\omega, \Gamma)$ is not a polygon, if the condition that all edges of $\operatorname{conv} f(j\omega, \Gamma_0)$ are images of edges of Γ is fulfilled at each ω, then the argument set out in the Introduction ensures that to check robust stability of the family of polynomials in question, it suffices to check the edges of this family.

The proof given is inductive and requires certain definitions and lemmas. A k-face of Γ is a k-dimensional subset where all but k of the γ_i take extreme values. Notice that the corners of Γ, the edges of Γ and Γ itself are the 0-faces, 1-faces and the m-face of Γ. Each k-face B of Γ has in turn $2k$ $(k-1)$-faces, all of which are also $(k-1)$-faces of Γ. We also note that the value set of any axis-parallel straight line in Γ is either a straight line or a point. Then, the first lemma is as follows.

Lemma 6.1. Consider an r-face B of Γ and all $(r-1)$-faces, $B_1, \ldots, B_{2r}$ of B. Suppose, for a given ω and some straight line segment S,

$$\bigcup_{i=1}^{2r} f(j\omega, B_i) \subset S.$$

Then for every $P \in B$,

$$\bar{P} = f(j\omega, P) \in \bigcup_{i=1}^{2r} f(j\omega, B_i). \tag{18}$$

Consequently, $\bar{P}$, the image of an arbitrary point of B, coincides with the image in the value set space of at least one point on an $(r-1)$-face of B.

Proof. Through P, there passes at least one axis-parallel line, which intersects two of the B_i at points we will denote by P_1 and P_2. The image of the line segment P_1P_2 is a line segment $\bar{P}_1\bar{P}_2$, which could be degenerate, i.e. a point. Thus, $\bar{P} \in \bar{P}_1\bar{P}_2$. As $\bar{P}_1 \in S$ and $\bar{P}_2 \in S$ and $\bar{P}_1\bar{P}_2$ is a line segment or a point, $\bar{P} \in S$. Equation (18) is established by noting that P_1 and P_2 can be joined by a continuous path comprising the union of axis-parallel line segments lying entirely in $\bigcup_{i=1}^{2r} B_i$, and that the value set of this path is

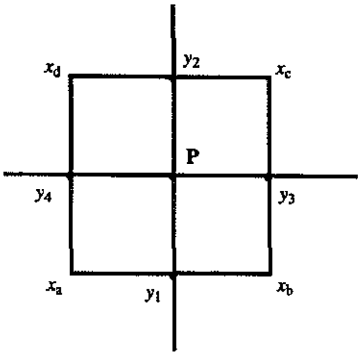

Fig. 13. A 2-face of Γ.

thus continuous. Thus, every point on $\bar{P}_1\bar{P}_2$ has at least one pre-image in one of the B_i. ∎

The next lemma will be used to initiate the inductive proof.

Lemma 6.2. Consider a point P in the strict interior of a 2-face B of Γ. Suppose $\bar{P} = f(j\omega, P)$ belongs to an edge $\bar{x}_i\bar{x}_j$ of $\operatorname{conv} f(j\omega, \Gamma_0)$. Then, there exists at least one Q on an edge of B, such that $\bar{Q} = f(j\omega, P) = \bar{P}$ and the value set of each edge of B is a subset of $\bar{x}_i\bar{x}_j$.

Proof. From the Mapping Theorem, $\bar{P}$ must be on the boundary of $f(j\omega, \Gamma)$, and all points in $f(j\omega, \Gamma)$ must lie on the same side of $\bar{x}_i\bar{x}_j$. (Refer to Fig. 13 which portrays B.) In this figure, y_i belong to appropriate edges and y_1y_2 and y_3y_4 are axis-parallel line segments which intersect at P. Then, the value set of y_1y_2 is either a line segment or a point. In this case

$$\bar{P} = \bar{y}_1\bar{y}_2 \subset \bar{x}_i\bar{x}_j.$$

On the other hand, the interval $\bar{y}_1\bar{y}_2$ still contains $\bar{P}$ and must be a subinterval of $\bar{x}_i\bar{x}_j$. Otherwise $\bar{y}_1$ and $\bar{y}_2$, both of which are members of $f(j\omega, \Gamma)$, lie on opposite sides of $\bar{x}_i\bar{x}_j$, thereby establishing a contradiction. Also, as $\bar{x}_i\bar{x}_j$ is an edge of $\operatorname{conv} f(j\omega, \Gamma_0)$, $\bar{y}_1$ and $\bar{y}_2$ are in $\bar{x}_i\bar{x}_j$. Thus, in either case

$$\bar{P} \in \bar{y}_1\bar{y}_2 \subset \bar{x}_i\bar{x}_j.$$

Likewise

$$\bar{P} \in \bar{y}_3\bar{y}_4 \subset \bar{x}_i\bar{x}_j.$$

As $\bar{y}_1, \bar{y}_2, \bar{y}_3$ and $\bar{y}_4$ belong to $\bar{x}_i\bar{x}_j$, similar arguments establish

$$\bar{y}_1 \in \bar{x}_a\bar{x}_b \subset \bar{x}_i\bar{x}_j,$$

$$\bar{y}_3 \in \bar{x}_b\bar{x}_c \subset \bar{x}_i\bar{x}_j,$$

$$\bar{y}_2 \in \bar{x}_c\bar{x}_d \subset \bar{x}_i\bar{x}_j,$$

$$\bar{y}_4 \in \bar{x}_d\bar{x}_d \subset \bar{x}_i\bar{x}_j.$$

Furthermore,

$$\bar{P} \in \bar{y}_1\bar{y}_2 \subset (\bar{y}_1\bar{x}_b) \cup (\bar{x}_b\bar{x}_c) \cup (\bar{x}_c\bar{y}_2). \quad ∎$$

Hence, the lemma holds.

We can now prove the following proposition which trivially proves Theorem 6.1.

Proposition 6.1. Suppose $\bar{x}_i\bar{x}_j$ is an edge of $\operatorname{conv} f(j\omega, \Gamma_0)$. Suppose for some P in the strict interior of an r-face B, $r \geq 2$, $\bar{P} = f(j\omega, p) \in \bar{x}_i\bar{x}_j$. Then, the value set of every edge of B is a subset of $\bar{x}_i\bar{x}_j$ and there exists at least one Q on an edge of B such that $\bar{Q} = f(j\omega, Q) = \bar{P}$.

Proof. Use induction. By Lemma 6.2, the proposition holds for $r = 2$. Suppose it holds for all $r \leq k < m$. Then, consider P on a $(k+1)$-face of Γ. Call this face $\bar{B}$.

Then there exist $P_1, \ldots, P_{2k}$, one on each k-face of $\bar{B}$, such that the following holds. For each $i \in \{1, \ldots, 2k\}$, there exists $j \neq i$, such that P_i and P_j are on an axis-parallel line containing P in its interior. Call the segment of this line with P_i and P_j as end points, P_iP_j. Then reasoning similar to that in the proof of Lemma 6.2 shows that each P_i is in $\bar{x}_i\bar{x}_j$. The inductive hypothesis proves that the value set of each k-face of $\bar{B}$ is a subset of $\bar{x}_i\bar{x}_j$ and the image in the value set space of every point on each k-face coincides with the image of at least one point on the edges of that k-face. Then, Lemma 6.1 proves the result. ∎

7. CONCLUSIONS

In this paper, we have presented two approaches to the problem of robust multiaffine stability. Firstly, we have presented a condition that is easily checked, on the value of Jacobian determinants at certain corner points; this condition is sufficient to ensure that a value set is a convex polygon with edges which are images of edges of the parameter space box.

Secondly, we have corrected a conjecture of Hollot and Xu, and shown that the only way the outer boundary of a value set can be a convex polygon is if the boundary is obtainable as the image of a collection of parameter space edges. This means that the outer boundary of the value set is polytopic and can be mapped from the parameter set edges if and only if the convex hull of the value set has edges that can be mapped from the edges of Γ.

The greatest difference between the two results is that the first requires simple connectedness of the value set, while the second does not. From the point of view of stability assessment, simple connectedness does not seem vital.

A second difference is that it would seem much easier to determine intervals of frequency over which the Jacobian sign conditions were satisfied than to determine intervals of frequency over which the condition of Theorem 6.1 held. This constitutes an advantage for the practical use of the first result, less comprehensive though it may be.

A third difference is that when all Jacobians have the same sign, e.g. positive, the ordering of the corners for defining the value set boundary is known *ab initio*. In contrast, in applying the second result, one is required first to go through the procedure of identifying the ordering of the points defining the boundary of $\mathrm{conv}\, f(j\omega, \Gamma_0)$.

Finally, we comment that, as already noted, classes of systems can be identified to which the Jacobian theory is applicable (see Poljak (1992)). This reference also discusses the possibility of handling matrix uncertainties. In short, the Jacobian result may be the better starting point for further theoretical developments.

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