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# Gain Margin Improvement Using Generalized Sampled-data Hold Function Based Multirate Output Compensator*†

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*A GSHF-based multirate output compensator for a SISO strictly proper, nonminimum-phase continuous-time plant is proposed. It is strictly proper and achieves the same level of gain margin obtained via a GSHF-based single-rate nonstrictly proper controller.*

**Key Words**—Gain margin; generalized sampled-data hold function; multirate output sampling; strictly proper compensator.

**Abstract**—For a SISO, strictly proper, nonminimum-phase, continuous-time, linear time-invariant (LTI) plant, it was shown in Yan, W., B. D. O. Anderson and R. R. Bitmead (1993). *IEEE Trans. Automat. Contr.*, that the closed-loop gain margin obtained via a generalized sampled-data hold function (GSHF) based dynamic compensator is significantly improved over that achieved via a conventional periodic controller used in Francis, B. A. and T. T. Georgiou (1988). *IEEE Trans. Automat. Contr.*, **AC-33**, 820–832. Nevertheless, the compensator so designed is not necessarily strictly proper. It is well-known that it is practically difficult and sometimes impossible to implement a nonstrictly proper compensator due to the requirement of zero computation time. Further, as has been shown in Vidyasagar, M. (1985). *Systems and Control Letters*, **5**, 413–418, stabilization by a nonstrictly proper controller is never robust against singular perturbations. In this paper, we propose a new type of GSHF based compensator which employs multirate sampling of the plant output. Using the proposed compensator, not only the same level of gain margin as in Yan, W., B. D. O. Anderson and R. R. Bitmead (1993). *IEEE Trans. Automat. Contr.*, can be achieved, but also, more importantly, the compensator is strictly proper.

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## 1. INTRODUCTION

IN THE LAST DECADE and particularly the last few years, a number of results on periodically time-varying (PTV) digital controllers have been reported Anderson and Moore (1981), Araki and Hagiwara (1986), Chammas and Leondes (1978, 1979b), Francis and Georgiou (1988), Greschak and Verghese (1982), Hagiwara and Araki (1988), Khargonekar *et al.* (1985) and Mita *et al.* (1987). It is now well-known that PTV digital controllers used in conjunction with LTI plants can offer a new dimension of flexibility in the design process. To recap, they have been used to achieve equivalent state feedback without observers, pole assignment, zero assignment, gain margin improvement, strong and simultaneous stabilization and the removal of decentralized fixed modes in decentralized control. A survey of PTV digital controllers is reported in Er (1992). One of these results which is both theoretically interesting and practically significant corresponds to the problem of gain margin improvement. The advantage of periodic controllers over LTI controllers in improving the gain margin for a nonminimum-phase FDLTI plant seems to have been first indicated in Khargonekar *et al.* (1985). Nevertheless, the result of Khargonekar *et al.* (1985) is only relevant to SISO discrete-time FDLTI bicausal plants with periodic discrete-time dynamic compensators. Some years later, a similar gain margin result for SISO continuous-time FDLTI plants with periodic continuous-time dynamic compensators of a particular form was reported in Lee *et al.* (1987). At this juncture, it is important to mention that these two results do not involve the implementation of a digital controller with a continuous-time plant (through A/D elements, etc).

Recently, it was shown in Francis and Georgiou (1988) that for a discrete-time FDLTI plant, LTI dynamic pre-compensation with decimation of the plant output (which is equivalent to the use of a particular form of linear periodic dynamic compensator) can arbitrarily position the finite zeros of the resulting closed-loop system. Using this idea, the gain margin result of Khargonekar *et al.* (1985) was generalized to the multivariable case in Francis and Georgiou (1988). To be precise, it was shown that the gain margin can be arbitrarily assigned likewise for a multivariable continuous-time FDLTI plant by way of discretizing the plant with a sufficiently small sampling time and suitable choice of a digital periodic controller. This kind of controller is commonly called a conventional periodic controller. The design procedure consists of:

- Discretizing the plant.
- Designing a LTI dynamic forward-compensator with decimation of the plant output (which is equivalent to the use of a particular form of linear periodic dynamic compensator) to position the zeros of the discretized plant.
- Designing a LTI feedback compensator which positions the poles of the discretized plant.

From this procedure, it is not difficult to see that the order of such a controller may be very high due to the introduction of pre-compensation. Another disadvantage, as mentioned in Francis and Georgiou (1988), is that the sampling time may have to be very small to permit an increase in the gain margin.

Another kind of digital periodic controller which possesses the capability of gain margin improvement is a GSHF based dynamic compensator proposed in Yan *et al.* (1993). In Yan *et al.* (1993), further improvement of gain margin using a GSHF dynamic compensator over a conventional periodic controller was revealed. The approach therein is different from that of Francis and Georgiou (1988). The key idea revolves around positioning the finite zeros of the discretized plant via the use of a GSHF developed in Kabamba (1987). One of the interesting results is that for a SISO, strictly proper, continuous-time, FDLTI plant, significant gain margin improvement using a GSHF based dynamic compensator over a conventional periodic digital controller is achieved. Unfortunately, the compensator so designed is not necessarily strictly proper. The disadvantages of a nonstrictly proper compensator are two-fold: first, it is well-known that it is practically difficult or sometimes impossible to implement a nonstrictly proper compensator due to the implied assumption of zero computation time. Second, as has been shown in Vidyasagar (1985), stabilization by a nonstrictly proper controller is never robust against singular perturbations whereas stabilization by a strictly proper controller is always robust against singular perturbations. In this paper, we explore the concept of multirate sampling of the plant output developed in Hagiwara and Araki (1988) and propose a new type of GSHF based compensator which employs multirate sampling of the plant output with output-rate multiplicity, $N^O = 2$. Using the proposed compensator, not only the maximal level of gain margin obtained in Yan *et al.* (1993) can be achieved, but also, more importantly, the new compensator is strictly proper. As a consequence, it is more easily realizable and guaranteed to be robust against singular perturbations.

The paper is organized as follows: a review of the GSHF dynamic compensator developed in Yan *et al.* (1993) appears in Section 2. In particular, the formula for the maximal level of achievable gain margin derived in Yan *et al.* (1993) is simplified. In the next section, the GSHF based multirate output compensator is proposed. An explicit formula for the maximal gain margin achievable by the new compensator for the SISO plant considered in Section 2 is derived. A design procedure for the construction of the proposed compensator is also outlined. It is further shown that the maximal level of gain margin obtained in Yan *et al.* (1993) can be achieved by the new compensator. An example appears in Section 4 to illustrate the ideas and methods presented. Section 5 contains concluding remarks.

## 2. REVIEW OF GSHF BASED DYNAMIC COMPENSATOR

Consider a SISO, strictly proper, continuous-time, FDLTI plant $P(s)$ with a minimal state-space model given by

$$\dot{x}(t) = Ax(t) + bu(t) \quad x(0) = x_0, \tag{1}$$

$$y(t) = cx(t). \tag{2}$$

For simplicity, the following assumption is made.

*Assumption* 1. All the unstable poles of $P(s)$ are assumed to be simple and in the open right half-plane, denoted by $H$.

The GSHF dynamic compensator proposed in Yan *et al.* (1993) has the form

$$z_d(k+1) = A_c z_d(k) + b_c y_d(k), \quad (3)$$

$$v_d(k) = c_c z_d(k) + d_c y_d(k), \quad (4)$$

$$u(t) = f(t) v_d(k) \quad (5)$$

$$t \in [kT_0, (k+1)T_0)$$

$$k = 0, 1, 2, \ldots,$$

where $z_d(k) \in \mathbb{R}^{n_c}$, $y_d(k) \triangleq y(kT_0)$, $v_d(k) \in \mathbb{R}^1$, $T_0 > 0$ is the frame period (which is also the sampling period for single-rate sampling), $A_c$, $b_c$, $c_c$ and $d_c$ are constant matrices of appropriate dimensions and $f(t)$ is a $T_0$-periodic integrable and bounded matrix function of an appropriate dimension. Note that the frame period $T_0$ is chosen according to the recommendations given in Åström and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990) and such that $T_0 \neq 2k\pi/\mathrm{Im}\,(p_i - p_j)$ whenever $\mathrm{Re}\,(p_i - p_j) = 0$, $k = 0, 1, 2, \ldots$. The latter condition ensures that the discrete-time unstable poles are also simple. Note also that in general $d_c \neq 0$ so that the compensator is not strictly proper.

The following equation

$$\int_0^{T_0} \exp\,[A(T_0 - t)] b f(t)\,\mathrm{d}t = g_d, \quad (6)$$

for the unknown $f(t)$ with $g_d$ a given constant vector, plays an important role in the design of the GSHF based compensator. The properties with respect to (6) are summarized in the following lemma.

*Lemma* 2.1 (*Yan* et al., 1993). Let $(A, B)$ be controllable, $G_d$ be given and

$$W(A, B, T_0) = \int_0^{T_0} \exp\,[A(T_0 - t)] \times BB' \exp\,[A'(T_0 - t)]\,\mathrm{d}t. \quad (7)$$

Then

(i) $F_0(t) = B' \exp\,[A'(T_0 - t)] W^{-1}(A, B, T_0) G_d$ is the unique optimal solution of (6) in the sense of minimizing $tr \int_0^{T_0} F'(t)F(t)\,\mathrm{d}t$;

(ii) for almost all $T_0 > 0$, there exists a piecewise constant solution of (6) taking at most $n$ different values in the interval $[0, T_0]$;

(iii) for almost all $T_0 > 0$, there exists a sequence of piecewise constant solutions $F_k(t)$ of (6) which uniformly converges to $F_0(t)$ in the interval $[0, T_0]$ under a usual matrix norm.

Applying the GSHF control law of (3)–(5) to the continuous-time plant (1) and (2) and sampling the continuous-time state and output with single-rate $T_0$, we obtain the following discrete-time system from $v_d(k)$ to $y_d(k)$.

$$x_d(k+1) = A_d x_d(k) + g_d v_d(k), \quad (8)$$

$$y_d(k) = c x_d(k), \quad (9)$$

where

$$A_d = \exp\,(AT_0) \quad (10)$$

and $g_d$ is related to the GSHF gain $f(t)$ as in (6).

The associated transfer function is

$$P(z) = c(zI - A_d)^{-1} g_d. \quad (11)$$

Note that the discretized system is strictly proper.

The following definition is needed in the subsequent development.

*Definition* 2.1. Let $P(s)$ denote the transfer function of the SISO continuous-time FDLTI plant given by (1) and (2). For a given $T_0 > 0$, define the maximal achievable gain margin $K_1(T_0)$ of $P(s)$ with respect to GSHF compensation as

$$K_1(T_0) \triangleq \sup\,\{k_2/k_1 : 0 < k_1 < 1 < k_2 \text{ and there exists a controller (3)–(5) stabilizing } kP(s) \text{ for all } k \in [k_1, k_2]\}.$$

*Remark* 2.1. The significance of the maximal gain margin $K_1(T_0)$ is that for any given pair $(k_1, k_2)$ with $0 < k_1 < k_2$ and $k_2/k_1 < K_1(T_0)$, there exists a controller (3)–(5) which stabilizes $kP(s)$ for all $k \in [k_1, k_2]$. In fact, it follows from the above definition that there exists a pair $(k_1^*, k_2^*)$ with $0 < k_1^* < 1 < k_2^*$ and $k_2/k_1 < k_2^*/k_1^* \leq K_1(T_0)$ such that $kP(s)$ is stabilized by a controller $C$ for all $k \in [k_1^*, k_2^*]$. It turns out that for any given $\alpha > 0$, $kP(s)$ is stabilized by a controller $C/\alpha$ for all $k \in [\alpha k_1^*, \alpha k_2^*]$. In particular, choose $\alpha_0 \in (k_2/k_2^*, k_1/k_1^*)$, which is possible due to $k_2/k_1 < k_2^*/k_1^*$. Since there holds $[k_1, k_2] \subset [\alpha_0 k_1^*, \alpha_0 k_2^*]$, it is borne out that the controller $C/\alpha_0$ stabilizes $kP(s)$ for all $k \in [k_1, k_2]$.

Concerning the maximal achievable gain margin for (1) and (2), we have the following theorem.

*Theorem* 2.1 (*Yan* et al., 1993). Adopt Assumption 1. Let $K_1(T)$ be as defined in Definition 2.1; then, for almost all sampling periods $T > 0$,

$$K_1(T) = \left(\frac{1 + \alpha_T}{1 - \alpha_T}\right)^2, \quad (12)$$

where

$$\alpha_T = \sqrt{1 - eL_T^{-1}e'} \quad (13)$$

and

$$e \triangleq [1 \;\; 1 \;\; \cdots \;\; 1] \in \mathbb{R}^{1\times N_2}, \tag{14}$$

$$L_T \triangleq \left[\frac{1}{1-\exp\left[-(p_i+\bar{p}_j)T\right]}\right]_{i,j=1,2,\ldots,N_2} \in \mathbb{R}^{N_2\times N_2}, \tag{15}$$

where $p_i$, $i = 1, 2, \ldots, N_2$ are the unstable poles of the continuous-time plant $P(s)$.

Now, we shall show that the above expression can be simplified to a form which does not involve matrix inversion.

*Theorem* 2.2. Adopt Assumption 1. The maximal level of achievable gain margin, $K_1(T)$ given in Theorem 2.1 can be rewritten as

$$K_1(T) = \left(\frac{1+\exp\left[\left(-\sum_{i=1}^{N_2} p_i\right)T\right]}{1-\exp\left[\left(-\sum_{i=1}^{N_2} p_i\right)T\right]}\right)^2, \tag{16}$$

where the $p_i$, $i = 1, 2, \ldots, N_2$ denote the unstable poles of the continuous-time system $P(s)$.

*Proof.* Define $\lambda_i \triangleq \exp(-p_i T)$. Observe that

$$L_T = [1/(1-\lambda_i\bar{\lambda}_j)]_{i,j=1,2,\ldots,N_2}. \tag{17}$$

Further,

$$\det\begin{bmatrix} L_T & e' \\ e & 1\end{bmatrix} = (1 - eL_T^{-1}e')\det(L_T) = \det(L_T - e'e). \tag{18}$$

Since $|\lambda_i| < 1$, it follows that

$$L_T = \left[\sum_{k=0}^{\infty} (\lambda_i\bar{\lambda}_j)^k\right]_{i,j=1,2,\ldots,N_2}$$

$$= \sum_{k=0}^{\infty}\left(\begin{bmatrix}(\lambda_1)^k \\ (\lambda_2)^k \\ \vdots \\ (\lambda_{N_2})^k\end{bmatrix}[(\bar{\lambda}_1)^k \;\; (\bar{\lambda}_2)^k \;\; \cdots \;\; (\bar{\lambda}_{N_2})^k]\right)$$

$$= \sum_{k=1}^{\infty}\left(\begin{bmatrix}(\lambda_1)^k \\ (\lambda_2)^k \\ \vdots \\ (\lambda_{N_2})^k\end{bmatrix}[(\bar{\lambda}_1)^k \;\; (\bar{\lambda}_2)^k \;\; \cdots \;\; (\bar{\lambda}_{N_2})^k]\right) + e'e. \tag{19}$$

So,

$$L_T - e'e = \sum_{k=1}^{\infty}\left(\begin{bmatrix}(\lambda_1)^k \\ (\lambda_2)^k \\ \vdots \\ (\lambda_{N_2})^k\end{bmatrix}[(\bar{\lambda}_1)^k \;\; (\bar{\lambda}_2)^k \;\; \cdots \;\; (\bar{\lambda}_{N_2})^k]\right)$$

$$= \operatorname{diag}\{\lambda_i\} L_T \operatorname{diag}\{\bar{\lambda}_i\} \quad (i = 1, 2, \ldots, N_2). \tag{20}$$

It follows from (18) and (20) that

$$(1 - eL_T^{-1}e')\det(L_T) = \det(\operatorname{diag}\{\lambda_i\})\det(L_T)\det(\operatorname{diag}\{\bar{\lambda}_i\}) \quad (i = 1, 2, \ldots, N_2), \tag{21}$$

i.e.

$$1 - eL_T^{-1}e' = \prod_{i=1}^{N_2} |\lambda_i|^2$$

or

$$\alpha_T = \prod_{i=1}^{N_2} |\lambda_i|.$$

Since complex conjugates always occur in pairs and $|\lambda_i| = \lambda_i$ for each real $\lambda_i$, we have

$$\alpha_T = \prod_{i=1}^{N_2} \lambda_i = \exp\left[\left(-\sum_{i=1}^{N_2} p_i\right)T\right].$$

Hence, the result of the theorem is established. □

## 3. GSHF BASED MULTIRATE OUTPUT COMPENSATOR

The proposed GSHF based dynamic compensator employing multirate sampling of the plant output with output-rate multiplicity, $N^O = 2$ consists of an LTI compensator and a GSHF control law as follows:

$$z_d(k+1) = \bar{A}_c z_d(k) + \bar{B}_c \bar{y}_d(k), \tag{22}$$

$$v_d(k) = \bar{C}_c z_d(k), \tag{23}$$

$$u(t) = f(t)v_d(k), \tag{24}$$

$$t \in [kT_0, (k+1)T_0) \quad k = 0, 1, 2, \ldots,$$

where $z_d(k) \in \mathbb{R}^{n_c}$, $v_d(k) \in \mathbb{R}^1$ and the collection of the output measurements in the time interval $[kT_0, \overline{k+1}T_0)$, $k = 0, 1, 2, \ldots$ are given by

$$\bar{y}_d(k) = \begin{bmatrix} y(kT_0) \\ y(kT_0 + T^O)\end{bmatrix} \in \mathbb{R}^2. \tag{25}$$

Here, $T_0 > 0$ is the frame period, $T^O = T_0/N^O$ is the sampling time, $N^O = 2$ is the output-rate multiplicity, $\bar{A}_c$, $\bar{B}_c$ and $\bar{C}_c$ are constant matrices of appropriate dimensions and $f(t)$ is a $T_0$-periodic integrable and bounded hold function matrix of appropriate dimension.

Applying the proposed compensator (22)–(24) and sampling the continuous-time state by single-rate $T_0$, and the output with multirate, $T^O$, the following discrete-time system from $v_d(k)$ to $\bar{y}_d(k)$ is obtained:

$$x_d(k+1) = A_d x_d(k) + g_d v_d(k), \tag{26}$$

$$\bar{y}_d(k) = \bar{C}_d x_d(k) + \bar{d}_d v_d(k), \tag{27}$$

where $A_d$ is as defined in (10) and

$$g_d = \int_0^{T_0} \exp[A(T_0 - t)]bf(t)\,dt, \quad (28)$$

$$\bar{C}_d = \begin{bmatrix} c \\ c\exp(AT^{\text{O}}) \end{bmatrix}, \quad (29)$$

$$\bar{d}_d = \begin{bmatrix} 0 \\ \int_0^{T^{\text{O}}} \exp[A(T^{\text{O}} - t)]bf(t)\,dt \end{bmatrix}. \quad (30)$$

The associated transfer function is

$$\bar{P}(z) = \bar{d}_d + \bar{C}_d(zI - A_d)^{-1}g_d. \quad (31)$$

By suitable choice of $g_d$, the discretized plant can have no unstable zeros. A procedure for choosing $g_d$ for this purpose is given in Appendix A of Er and Anderson (1994). An alternative procedure to avoid the unstable zeros can be found in Mita *et al.* (1990). For a given constant matrix $g_d$, the relation of $g_d$ to the GSHF gain, $f(t)$ is as defined in (6). Further, Lemma 2.1 applies here. A block diagram showing the closed-loop configuration with the proposed compensator is shown in Fig. 1.

Note that the discretized plant has a direct feedthrough term, $\bar{d}_d$. We remark that the notation $\bar{y}_d(k)$, though perhaps standard, is a little misleading because the two entries of $\bar{y}_d(k)$ are not both available at time $kT_0$, but one is available only at time $kT_0 + T^{\text{O}}$. A non-strictly proper compensator using $\bar{y}_d(k)$ as input at time $kT_0$ could not then actually operate. In (23), $v_d(k)$, assumed to be available at time $kT_0$, depends on $\bar{y}_d(l)$ for $l < k$ and thus the time for complete receipt of $\bar{y}_d(k)$, which is $(k - 1/2)T_0$ until $kT_0$, is available for computation.

The following definition is needed for the subsequent development.

*Definition* 3.1. Let $P(s)$ denote the transfer function of the SISO continuous-time FDLTI plant given by (1) and (2). For a given $T_0 > 0$, define the maximal achievable gain margin $K_2(T_0)$ of $P(s)$ with respect to the GSHF based multirate output compensation as

$$K_2(T_0) \triangleq \sup\{k_2/k_1 : 0 < k_1 < 1 < k_2 \text{ and there exists a strictly proper controller (22)–(24) stabilizing } kP(s) \text{ for all } k \in [k_1, k_2]\}.$$

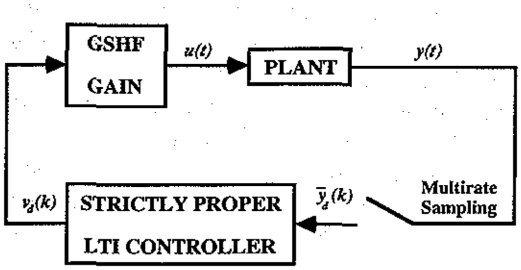


FIG. 1. Closed-loop configuration with the proposed compensator.

As foreshadowed in the introduction, the proposed compensator is strictly proper rather than just proper. Now, we state and prove the theorem concerning the existence of a strictly proper GSHF based multirate output compensator.

*Theorem* 3.1. Adopt Assumption 1. Consider the SISO, strictly proper, continuous-time, LTI minimal plant $P(s)$ given by (1) and (2). Let $K_1(T_0)$ be as defined in Definition 2.1. For a prescribed level of gain margin $k_2/k_1 < K_1(T_0)$ with $0 < k_1 < 1 < k_2$, there always exists a strictly proper GSHF based multirate output compensator (22)–(24) stabilizing $kP(s)$ for all $k \in [k_1, k_2]$.

*Proof.* Let $R$ denote the ring of proper rational functions which are stable in the discrete-time sense. Then $\bar{P}(z)$ in (31) has a Smith–McMillan form (see Kailath (1980) for details on Smith–McMillan form) over $R$ as follows:

$$\begin{bmatrix} n_1/d_1 \\ 0 \end{bmatrix},$$

where 0 represents the zero matrix of appropriate size, $(n_1, d_1)$ are coprime over $R$ and $n_1/d_1$, which is proper but not strictly proper, will have unstable poles $\exp(p_iT_0)$, $(i = 1, 2, \ldots, N_2)$ but stable finite zeros (owing to suitable choice of $g_d$ described in Appendix A of Er and Anderson, 1994.) As shown in the procedure for constructing a strictly proper GSHF based multirate output compensator to be given subsequently, the existence of a strictly proper GSHF based multirate output compensator stabilizing $kP(s)$ for all $k \in [k_1, k_2]$, where $0 < k_1 < 1 < k_2$ amounts to constructing a sensitivity function $s_1(z)$ which has the following properties:

(i) $s_1(z)$ is real rational and analytic for $D^c$ and $G$ where $D^c$ and $G$ are the complement of open unit disk and $\mathbb{C}\backslash\{(-\infty, -a'] \cup [b', \infty)\}$, where $a' = k_1/(1 - k_1)$ and $b' = k_2/(k_2 - 1)$ respectively.

(ii) The zeros of $s_1(z)$ contain $\exp(p_1T_0)$, $\exp(p_2T_0), \ldots, \exp(p_{N_2}T_0)$ multiplicities included.

The construction of the sensitivity function with the above properties in turn amounts to a special form of the Nevanlinna–Pick interpolation problem, namely finding an analytic function $F(z)$ from $\bar{D}$ to $D$ satisfying

$$F\left(\frac{1}{\exp(p_i T_0)}\right)=0 \quad i=1,2,\ldots,N_2,$$

where $\bar{D}$ and $D$ are the closed unit disk and the open unit disk respectively. The constructed sensitivity function, $s_1(z)$ is guaranteed to be stable by virtue of the assumption $k_2/k_1 < K_1(T_0)$. (The details of this will be explained in Step (v) of the procedure.) Hence, the theorem is established. □

*Remark* 3.1. In Yan *et al.* (1993), it was shown that given $k_1$ and $k_2$, a strictly proper GSHF compensator stabilizing $kP(s)$ for all $k \in [k_1, k_2]$ can be constructed for a SISO continuous-time FDLTI plant. Nevertheless, the plant considered is bicausal rather than strictly proper.

*Corollary* 3.1. Let $K_1(T_0)$ and $K_2(T_0)$ be as defined in Definition 2.1 and 3.1 respectively. Then,

$$K_2(T_0) \geq K_1(T_0). \tag{32}$$

*Proof.* The result follows from the proof of Theorem 3.1 and Definition 3.1. □

*Remark* 3.2. At this stage, it is not clear whether the strict inequality in (32) will hold.

A systematic procedure towards constructing a strictly proper GSHF based multirate output compensator is now outlined:

(i) Select $T_0$ according to the recommendations given in Åström and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990) and $T_0 \neq 2k\pi/\mathrm{Im}(p_i - p_j)$ whenever $\mathrm{Re}(p_i - p_j) = 0$, $k = 0, 1, 2, \ldots$. The latter condition ensures that the discrete-time unstable poles are also simple under Assumption 1.

(ii) Choose a $g_d$ so that the plant has no unstable zeros. A procedure for this purpose is given in Appendix A of Er and Anderson (1994). The corresponding $f(t)$ can be computed via Lemma 2.1.

(iii) Choose an output-rate multiplicity $N^O = 2$ and discretize the continuous-time plant with $T_0$ obtained in (i). Let $\bar{P}(z)$ denote the transfer function matrix of the discretized plant.

(iv) With a suitable unimodular matrix, $U(z)$, transform $\bar{P}(z)$ to the following Smith–McMillan form in $R$ i.e.

$$U(z)\bar{P}(z)=\begin{bmatrix} p_1(z) \\ 0 \end{bmatrix},$$

where the zeros of $p_1(z)$, denoted by $z_{d,i}$, $i=1,2,\ldots,N_2$, are finite and stable.

(v) Given $k_1$ and $k_2$ with $k_2/k_1 < K_1(T_0)$, compute $\gamma$ from

$$k_2/k_1=\left(\frac{\gamma+\exp\left[-\left(\sum_{i=1}^{N_2} p_i\right)T_0\right]}{\gamma-\exp\left[-\left(\sum_{i=1}^{N_2} p_i\right)T_0\right]}\right)^2. \tag{33}$$

By (16), there always exists $\gamma > 1$. As will be shown later, this form of $k_2/k_1$ is related to obtaining a strictly proper compensator. Further, the value $\gamma$ has implication on the stability of the sensitivity function, $s_1(z)$.

(vi) Construct a sensitivity function $s_1(z)$ according to the procedure outlined in the Appendix. One such sensitivity function $s_1(z)$, is

$$s_1(z)=\frac{4k_1k_2\gamma\prod_{i=1}^{N_2}[z-\exp(p_iT_0)][\exp(p_iT_0)z-1]}{A(z)},$$

where

$$\begin{aligned} A(z) &= (k_2-k_1)\gamma^2\prod_{i=1}^{N_2}[\exp(p_iT_0)z-1]^2 \\ &\quad -2(k_2+k_1-2k_1k_2)\gamma\prod_{i=1}^{N_2} \\ &\quad \times[z-\exp(p_iT_0)][\exp(p_iT_0)z-1] \\ &\quad +(k_2-k_1)\prod_{i=1}^{N_2}[z-\exp(p_iT_0)]^2 \end{aligned}$$

with $\gamma > 1$.

(vii) The corresponding LTI compensator is given by

$$\begin{aligned} c_1(z) &= \frac{1-s_1(z)}{s_1(z)p_1(z)} = \frac{A(z)-4k_1k_2\gamma\prod_{i=1}^{N_2}[z-\exp(p_iT_0)][\exp(p_iT_0)z-1]}{4k_1k_2\prod_{i=1}^{N_2}[\exp(p_iT_0)z-1][z-z_{d,i}]} \\ &= \frac{\left[\left(\gamma\prod_{i=1}^{N_2}\exp(p_iT_0)-1\right)^2 k_2-\left(\gamma\prod_{i=1}^{N_2}\exp(p_iT_0)+1\right)^2 k_1\right]}{4k_1k_2\gamma\prod_{i=1}^{N_2}[\exp(p_iT_0)z-1](z-z_{d,i})} z^{2N_2} + \frac{\text{terms of degrees lower than } 2N_2}{4k_1k_2\gamma\prod_{i=1}^{N_2}[\exp(p_iT_0)z-1](z-z_{d,i})}, \end{aligned}$$

where $z_{d,i}$s are the stable and finite zeros of $p_1(z)$.

(viii) Define

$$C(z) \triangleq [c_1(z) \quad 0]U(z) = [c_{1,1}(z) \quad c_{1,2}(z)],$$

where each $c_{1,j}(z)$, $j=1,2$ is of the form

$$c_{1,j}(z) = d_j + \frac{b_{j,1}z^{n-1} + b_{j,2}z^{n-2} + \cdots + b_{j,n}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

and

$$d_j = \left[\left(\gamma \prod_{i=1}^{N_2} \exp(p_i T_0) - 1\right)^2 k_2 - \left(\gamma \prod_{i=1}^{N_2} \exp(p_i T_0) + 1\right)^2 k_1\right] \Big/ A_j,$$

where $A_j$s, $a_i$s and $b_{j,i}$s are constants. Note that with the choice of $k_2/k_1$ in Step (v), $d_j = 0$, $j = 1, 2$.

(ix) The proposed GSHF based multirate output compensator can then be constructed as

$$z_d(k+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} z_d(k) + \begin{bmatrix} b_{1,n} & b_{2,n} \\ b_{1,n-1} & b_{2,n-1} \\ \vdots & \vdots \\ b_{1,2} & b_{2,2} \\ b_{1,1} & b_{2,1} \end{bmatrix} [y_d(k) \quad y_d(k+\tfrac{1}{2})]$$

$$v_d(k) = [0 \; 0 \; \cdots \; 0 \; 1] z_d(k).$$

We remark that from Step (vi) of the design procedure, it can be seen that the ratio of the constant term of $A(z)$ and the coefficient of $z^{2N_2}$ is given by

$$M = \frac{(k_2 - k_1)\left(\gamma^2 \exp\left[\left(\sum_{i=1}^{N_2} 2p_i\right)T_0\right] + 1\right) - 2(k_2 + k_1 - 2k_1k_2)\gamma \exp\left[\left(\sum_{i=1}^{N_2} p_i\right)T_0\right]}{(k_2 - k_1)\left(\gamma^2 + \exp\left[\left(\sum_{i=1}^{N_2} 2p_i\right)T_0\right]\right) - 2(k_2 + k_1 - 2k_1k_2)\gamma \exp\left[\left(\sum_{i=1}^{N_2} p_i\right)T_0\right]}.$$

It is easy to see that $M \to 1$ as $\gamma \to 1$. As a consequence, the product of the poles of $s_1(z) \to 1$. This implies that the $H^\infty$ norm of the sensitivity function $s_1(z)$ tends to infinity, which coincides with one of the results in Yan and Anderson (1990).

## 4. AN EXAMPLE

For the following example

$$P(s) = \frac{s-1}{(s+1)(s-2)},$$

it has been calculated in Yan *et al.* (1993) that the maximal gain margin of the plant using a proper GSHF compensator is given by

$$K_1(T_0) = \left(\frac{1 + \exp(-2T_0)}{1 - \exp(-2T_0)}\right)^2.$$

We shall now construct a GSHF based multirate output compensator and show that the same level of gain margin is obtained.

Consider the following state-space model of the plant

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \quad c = [2 \quad 1].$$

Choose $T_0 = 0.1$, $N^O = 2$ and $g_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that the plant has finite and stable zeros. One of the GSHF gains associated with $g_d$ can be found as follows:

$$f(t) = \begin{cases} 370.6 & 0 \le t < 0.05 \\ -352.5 & 0.05 \le t \le 0.1 \end{cases}.$$

Then,

$$\bar{P}(z) = \bar{d}_d + \bar{C}_d[zI - \exp(AT_0)]^{-1} g_d = \frac{1}{z - 1.2214} \begin{bmatrix} 1 \\ 18.6409(z - 1.1621) \end{bmatrix}.$$

With the following unimodular matrix

$$U(z) = \begin{bmatrix} \frac{z + 1.6005}{z + 0.5} & \frac{0.05365z + 0.0623}{z + 0.5} \\ -\frac{18.6409(z - 1.1621)}{z + 0.5} & \frac{1}{z + 0.5} \end{bmatrix},$$

$\bar{P}(z)$ has the following Smith–McMillan form in $R$

$$U(z)\bar{P}(z) = \begin{bmatrix} \frac{z + 0.5}{z - 1.2214} \\ 0 \end{bmatrix} = \begin{bmatrix} p_1(z) \\ 0 \end{bmatrix}.$$

Using the procedure of constructing a sensitivity function, $s_1(z)$ outlined in the Appendix, the required sensitivity function is

$$s_1(z) = \frac{4k_1k_2\gamma(z - 1.2214)(1.2214z - 1)}{A(z)},$$

where

$$\begin{aligned}A(z) &= [4.8856\gamma k_1k_2 + (1.2214\gamma - 1)^2 k_2 \\ &\quad - (1.2214\gamma + 1)^2 k_1]z^2 \\ &\quad - [9.9673\gamma k_1 k_2 - 2(2.4918\gamma \\ &\quad - 1.2214(\gamma^2+1))k_2 - 2(2.4918\gamma \\ &\quad + 1.2214(\gamma^2+1))k_1]z + [4.8856\gamma k_1k_2 \\ &\quad + (\gamma - 1.2214)^2 k_2 - (\gamma + 1.2214)^2 k_1].\end{aligned}$$

For $\gamma = 1.01$,

$$\begin{aligned}A(z) &= (4.9345k_1k_2 + 0.0546k_2 - 4.989k_1)z^2 \\ &\quad - [10.067k_1k_2 - 0.0988k_2 - 9.9682k_1]z \\ &\quad + [4.9345k_1k_2 + 0.0477k_2 - 4.9792k_1].\end{aligned}$$

The corresponding LTI compensator is

$$\begin{aligned}c_1(z) &= \frac{1 - s_1(z)}{s_1(z)p_1(z)} \\ &= \frac{(0.0546k_2 - 4.989k_1)z^2 + (0.0988k_2 + 9.9682k_1)z + (0.0477k_2 - 4.9792k_1)}{4.04k_1k_2(1.2214z - 1)(z + 0.5)}\end{aligned}$$

Define

$$\begin{aligned}C(z) &\triangleq [c_1(z) \quad 0]U(z) \\ &= [c_{1,1}(z) \quad c_{1,2}(z)],\end{aligned}$$

where

$$c_{1,1}(z) = \left(\frac{0.0111}{k_1} - \frac{1.0111}{k_2}\right) + \frac{\left(\frac{0.0377}{k_1} + \frac{1.9833}{k_2}\right)z^2 + \left(\frac{0.0417}{k_1} + \frac{2.2242}{k_2}\right)z + \left(\frac{0.0155}{k_1} - \frac{1.615}{k_2}\right)}{z^3 + 0.1813z^2 - 0.5687z - 0.2046}$$

$$c_{1,2}(z) = \left(\frac{0.0006}{k_1} - \frac{0.0542}{k_2}\right) + \frac{\left(\frac{0.0018}{k_1} + \frac{0.2249}{k_2}\right)z^2 + \left(\frac{0.0018}{k_1} + \frac{0.0717}{k_2}\right)z + \left(\frac{0.0006}{k_1} - \frac{0.0629}{k_2}\right)}{z^3 + 0.1813z^2 - 0.5687z - 0.2046}.$$

It follows that the compensator is strictly proper if and only if $k_2/k_1 = 91.37$. This is the gain margin achieved by the proposed compensator for $T_0 = 0.1$. For comparison, it has been calculated in Francis and Georgiou (1988) that the maximal gain margins achieved by conventional LTI compensator and conventional periodic controller with the same $T_0$ are 4 and 25.67 respectively. Further, with respect to proper GSHF compensator with the same $T_0$, the maximal gain margin is 100.63. Note that our compensator is strictly proper, rather than just proper. As a consequence, it is more easily realizable and guaranteed to be robust against singular perturbations whereas the compensator in Yan *et al.* (1993) may not be.

Observe also that with $k_2/k_1 = 91.37$, the ratio of the constant term and the coefficient of $z^2$ of the sensitivity function $s_1(z)$ is <1 for $k_2 > 1 > k_1 > 0$. Hence, the sensitivity function is stable. It can be shown that by taking $\gamma$ even closer to one, the gain margin achieved by the proposed compensator tends to 100.63. Nevertheless, as mentioned earlier, the $H^\infty$ norm of the sensitivity function $s_1(z)$ tends to infinity, which coincides with one of the results in Yan and Anderson (1990).

The proposed GSHF compensator with multirate output sampling can be constructed as

$$\begin{aligned}z_d(k+1) &= \begin{bmatrix} 0 & 0 & 0.2046 \\ 1 & 0 & 0.5687 \\ 0 & 1 & -0.1813 \end{bmatrix} z_d(k) \\ &\quad + \begin{bmatrix} \frac{0.0155}{k_1} - \frac{1.615}{k_2} & \frac{0.0006}{k_2} - \frac{0.0629}{k_1} \\ \frac{0.0417}{k_1} + \frac{2.2242}{k_2} & \frac{0.0018}{k_1} + \frac{0.0717}{k_2} \\ \frac{0.0377}{k_1} + \frac{1.9833}{k_2} & \frac{0.0018}{k_1} + \frac{0.2249}{k_2} \end{bmatrix} \\ &\quad \times [y_d(k) \quad y_d(k+\tfrac{1}{2})] \\ v_d(k) &= [0 \quad 0 \quad \cdots \quad 0 \quad 1]z_d(k) \\ u(t) &= f(t)v_d(k) \quad t \in [kT_0, \overline{k+1}T_0), \\ &\qquad k = 0, 1, 2, \ldots.\end{aligned}$$

## 5. CONCLUSION

In this paper, we have shown that for a SISO, strictly proper, nonminimum phase, continuous-time, FDLTI plant, the closed-loop gain margin obtained via GSHF based dynamic compensator employing multirate sampling of the plant output is significantly improved over that achieved via a conventional periodic controller used in Francis and Georgiou (1988). It turns out that the maximal level of gain margin improvement obtained in Yan *et al.* (1993) can be achieved by the proposed compensator. More significantly, our compensator is strictly proper rather than just proper. As a consequence, our compensator could be implemented in practice and is guaranteed to be robust against singular perturbations whereas the compensator proposed in Yan *et al.* (1993) may not be.

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## APPENDIX: PROCEDURE FOR CONSTRUCTING A SENSITIVITY FUNCTION

This appendix only serves as a guide for the construction of a sensitivity function for our problem. More details on the construction of a sensitivity function can be found in Khargonekar and Tannenbaum (1985).

Consider the following commutative diagram of mappings:

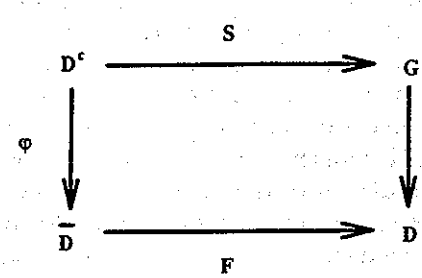


FIG. A.1. Commutative diagram of of mappings.

Here, $D$, $\bar{D}$ and $D^c$ denote the open unit disk, closed unit disk and complement of open unit disk respectively. The region $G$ is defined as

$$G \triangleq \mathbb{C}\backslash\{(-\infty, -a'] \cup [b', \infty)\}, \tag{A.1}$$

where $a' = k_1/(1-k_1)$ and $b' = k_2/(k_2-1)$.

Let $\varphi$ be the conformal mapping from $D^c$ to $\bar{D}$ i.e.

$$\varphi(z) = 1/z. \tag{A.2}$$

Further, let $\theta(z)$ be a conformal mapping from $G$ to $D$. The existence of $\theta(z)$ under standard assumptions on $G$ is in general the consequence of the well-known Riemann Mapping Theorem in Curtiss (1978), and there exist dictionaries, e.g. Kober (1952), where such mappings may be found. For the region of interest, $G$, the conformal mapping from $G$ to $D$ found in Khargonekar and Tannenbaum (1985) is given by

$$\theta(z)=\frac{1-\left(\frac{1-z/b'}{1+z/a'}\right)^{1/2}}{1+\left(\frac{1-z/b'}{1+z/a'}\right)^{1/2}} \quad (A.3)$$

and $\theta(0)=0$. From (A.3), it can be shown that the inverse mapping from $D$ to $G$ is given by

$$\theta^{-1}(z)=\frac{4k_1k_2z}{(k_2-k_1)-2(k_2+k_1-2k_1k_2)z+(k_2-k_1)z^2}, \quad (A.4)$$

which is rational.

Note that in our case, the plant $p_1(z)$ has no unstable zeros and its unstable poles are $\exp(p_iT_0)$, $i=1,2,\ldots,N_2$. Therefore, finding a sensitivity function $S(z)$ from $D^c$ to $G$ with

$$S(\exp(p_iT_0))=0 \quad i=1,2,\ldots,N_2, \quad (A.5)$$

amounts to a special form of the Nevanlinna–Pick interpolation problem, namely, finding an analytic function $F(z)$ from $\bar{D}$ to $D$ with

$$F\left(\frac{1}{\exp(p_iT_0)}\right)=0 \quad i=1,2,\ldots,N_2. \quad (A.6)$$

One such mapping is

$$F(z)=\frac{\prod_{i=1}^{N_2}\left(z-\frac{1}{\exp(p_iT_0)}\right)}{\gamma\prod_{i=1}^{N_2}\left(\frac{z}{\exp(p_iT_0)}-1\right)}, \quad (A.7)$$

where $p_1, p_2, \ldots, p_{N_2}$ being the unstable poles of the continuous-time system (1) and (2) and $\gamma>1$. Note that other possible mappings from $\bar{D}$ to $D$ can be constructed using a procedure described in Khargonekar and Tannenbaum (1985).

As a consequence, one sensitivity function $S(z)$ can be constructed as follows:

$$S(z)=\theta^{-1}(F(\varphi(z)))$$

$$=\frac{4k_1k_2\phi(z)}{(k_2-k_1)-2(k_2+k_1-2k_1k_2)\phi(z)+(k_2-k_1)(\phi(z))^2}, \quad (A.8)$$

where

$$\phi(z)\triangleq F(\varphi(z))$$

$$=\frac{\prod_{i=1}^{N_2}(z-\exp(p_iT_0))}{\gamma\prod_{i=1}^{N_2}(\exp(p_iT_0)z-1)}. \quad (A.9)$$

The features of the so constructed sensitivity function are as follows:

(i) $S(z)$ is real rational and analytic from $D^c$ to $G$.

(ii) The zeros of $S(z)$ contain $\{\exp(p_1T_0), \exp(p_2T_0), \ldots, \exp(p_{N_2}T_0)\}$ multiplicities included.