A Nash Game Approach to Mixed H_2/H_∞ Control

D. J. N. Limebeer, *Fellow, IEEE*, B. D. O. Anderson, *Fellow, IEEE*, and B. Hendel

***Abstract*—The established theory of nonzero sum games is used to solve a mixed H_2/H_∞ control problem. Our idea is to use the two pay-off functions associated with a two-player Nash game to represent the H_2 and H_∞ criteria separately. We treat the state-feedback problem and we find necessary and sufficient conditions for the existence of a solution. Both the finite and infinite time problems are considered. In the infinite horizon case we present a full stability analysis. The resulting controller is a constant state-feedback law, characterized by the solution to a pair of cross-coupled Riccati equations, which may be solved using a standard numerical integration procedure.**

We begin our development by considering strategy sets containing linear controllers only. At the end of the paper we broaden the strategy sets to include a class of nonlinear controls. It turns out that this extension has no effect on the necessary and sufficient conditions for the existence of a solution or on the nature of the controllers.

I. Introduction

It is well known that the solution to multivariable H_∞ control problems is hardly ever unique. If the solution is not optimal, it is never unique, even in the scalar case. With these comments in mind, the question arises as to what one can sensibly do with the remaining degrees of freedom. Some authors have suggested recovering uniqueness by strengthening the optimality criterion and solving a "super-optimal" problem [14], [16], [19], [26], [29]. Another possibility is entropy minimization, which was introduced into the literature by Arov and Krein [2], with other contributions coming from [11]–[13], [20], [24]. Entropy minimization is of particular interest in the present context because entropy provides an upper bound on the H_2 norm of the input–output operator. One may therefore think of entropy minimization as minimizing an upper bound on the H_2 cost. This gives entropy minimization an H_2/H_∞ interpretation. As an additional bonus we mention that the solution to the minimum entropy problem is particularly simple. In the case of most state-space representation formulae for all solutions, all one need do is set the free parameter to zero [13], [20], [24].

Another approach to mixed H_2/H_∞ problems is provided by Bernstein and Haddad [6]. Their approach is to minimize an auxiliary cost subject to an H_∞ norm constraint; the auxiliary cost is an upper bound on the H_2 norm of the closed-loop function. It turns out that the entropy minimization and the minimization of the auxiliary cost are equivalent in the single exogenous input case [22].

Manuscript received May 6, 1991; revised June 12, 1992, and Dec. 21, 1992.

D. J. N. Limebeer and B. Hendel are with the Department of Electrical Engineering, Imperial College, Exhibition Rd., London.

B. D. O. Anderson is with the Department of System Engineering, Australian National University, Canberra, Australia.

IEEE Log Number 9213558.

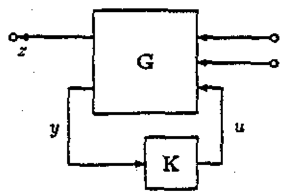

Fig. 1. An archetypal mixed H_2/H_∞ configuration.

The work of Bernstein and Haddad is extended in [10], [30], where another mixed H_2/H_∞ problem is addressed. The system considered is dual to the Bernstein–Haddad setup and is illustrated in Fig. 1. The signal w_0 is assumed to be white noise, while w_1 is a signal of bounded power; the space of bounded power signals will be denoted by $\mathcal{P}$. Given some prescribed γ, the aim is to minimize

$$J = \sup_{w_1 \in \mathcal{P}} \{\|z\|_p^2 - \gamma^2 \|w_1\|_p^2\}$$

in which $\|\cdot\|_p$ is the power semi-norm defined by [10], [30]

$$\|z\|_p^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} z'(t) z(t)\, dt.$$

The outcome of the work in [6], [10], [30] is a formula for a mixed H_2/H_∞ controller, which is parameterized in terms of a pair of cross-coupled Riccati equations and a standard H_∞ Riccati equation. A recent paper has proved that the results of Bernstein and Haddad [6], and the results of Doyle, Zhou, Bodenheimer, and Glover [10], [30], are in fact dual [28]. Put another way, the work in [10], [30] also minimizes a Bernstein–Haddad type auxiliary cost while ensuring that $\|\mathbf{R}_{zw_1}\|_\infty < \gamma$ where $\mathbf{R}_{zw_1}$ maps w_1 to z in Fig. 1. The central drawback of these approaches is concerned with the solution procedures for the Riccati equations: currently a computationally expensive method based on homotopy algorithms is all that is on offer.

Khargonekar and Rotea examine multiple-objective control problems, which include mixed H_2/H_∞ type problems. In contrast to the other work in this area, they give an algorithmic solution based on convex optimization [15].

In the present paper we seek to solve a state-feedback mixed H_2/H_∞ control problem via the solution of an associated Nash game. As is well known [5], [25], two-player nonzero sum games have two performance criteria, and the idea is to use one performance index to reflect an H_∞ constraint, while the second reflects an H_2 optimality requirement. The solution turns out to be a state-feedback law that is specified by the problem data and the solution of a pair of cross-coupled Riccati equations. In contrast to the work in [10], [30], the Riccati

0018-9286/94$04.00

equations may be solved by standard numerical integration. The solution of the algebraic equations is the limiting solution of the cross-coupled differential equations.

Following the precise problem statement given in Section II-A, we find necessary and sufficient conditions for the existence of a Nash equilibrium. In the first instance we restrict the set of admissible controllers to be linear and memoryless. A proof of the necessary and sufficient conditions appears in Section II-B. The aim of Section II-C is to provide our previous work with a stochastic interpretation that resembles classical LQG control. Section II-D provides a reconciliation with pure H_2 and H_∞ control. In particular, we show that H_2, H_∞, and H_2/H_∞ control may all be captured as special cases of a two-player Nash game. Section III deals with the infinite horizon case and the associated stability theory. Given some detectability conditions, we give necessary and sufficient conditions for the existence of an infinite horizon Nash equilibrium, and we show that the associated equilibrium strategies are stabilizing in some meaningful sense. In Section IV we prove that any nonlinear analytic equilibrium controller is necessarily linear. We decided to treat this more general case at the end of the paper for purely didactic reasons. An example is given in Section V, and brief conclusions appear in Section VI.

Our notation and conventions are standard. For $A \in C^{n\times m}$, A' denotes the complex conjugate transpose. $\mathcal{E}\{\cdot\}$ is the expectation operator. $\lambda_i(\cdot)$ denotes the set of eigenvalues. $\mathcal{RH}_\infty^+$ is the space of real rational matrices analytic in the closed right half plane.

$$\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$$

in which $\overline{\sigma}$ is the largest singular value.

II. The H_2/H_∞ Control Problem

A. Problem Statement

We begin by considering an H_2/H_∞ problem in which there is a single exogenous input and a single output. A second input driven by white noise will be introduced at a later stage of our development.

Suppose we are given a linear time-varying system described by

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)$$
$$x(0) = x_0 \tag{2.1}$$
$$z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix} \tag{2.2}$$

in which the entries of $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$, and $D(t)$ are continuous functions of time. There is no need to explicitly note the dimensions of the various matrixes that appear in (2.1) and (2.2). We suppose also that $D'(t)D(t) = I$, which is always achievable by scaling provided that $D(t)$ has full column rank for all t in some optimization interval (which may be finite or infinite). We have used (2.2) rather than the more natural

$$z(t) = C(t)x(t) + D(t)u(t)$$

to avoid the appearance of cross-terms. (For the remainder of this section we will take it as read that the problem data are time varying.)

We wish to:

1) Find a feedback control law $u^*(t, x)$ such that
$$\|z\|^2_{2,[0,T]} < \gamma^2\|w\|^2_{2,[0,T]} \qquad \forall w(t) \neq 0 \in \mathcal{L}_2[0, T]. \tag{2.3}$$
The notation $\|\cdot\|_{2,[0,T]}$ is used to denote the 2-norm on the time support $[0, T]$. When the time interval is $[0, \infty]$, we revert to the usual notation $\|\cdot\|_2$. This condition can be interpreted as an $\mathcal{L}_\infty$ norm constraint of the form
$$\|\mathbf{R}_{zw}\|_{\infty,[0,T]} < \gamma \tag{2.4}$$
where the operator $\mathbf{R}_{zw}$ maps the disturbance signal $w(t)$ to the output $z(t)$ when the optimal control law $u^*(t, x)$ is invoked. The notation $\|\cdot\|_{\infty,[0,T]}$ flags the fact that the infinity norm is induced by $\|\cdot\|_{2,[0,T]}$; we use $\|\cdot\|_\infty$ for the operator norm induced by $\|\cdot\|_2$.
2) We require the control $u^*(t, x)$ to regulate the state $x(t)$ in such a way as to minimize the output energy when the worst-case disturbance $w^*(t, x)$ is applied to the system.[1] One can think of this second condition as an alternative to the minimization of an auxiliary cost associated with the system in Fig. 1.

As we will now show, this problem may be formulated as a LQ nonzero sum game. The two cost functions we will use are

$$J_1(u, w) = \int_0^T (\gamma^2 w'(t)w(t) - z'(t)z(t))\, dt \tag{2.5}$$

and

$$J_2(u, w) = \int_0^T z'(t)z(t)\, dt. \tag{2.6}$$

The first is associated with an H_∞ criterion, while the second is used for the H_2 optimization part of the problem. The aim is to find equilibrium strategies u^* and w^* that satisfy the Nash equilibria defined by

$$J_1(u^*, w^*) \leq J_1(u^*, w) \tag{2.7}$$
$$J_2(u^*, w^*) \leq J_2(u, w^*). \tag{2.8}$$

If $J_1(u^*, w^*) \geq 0$, we must have $\|z\|^2_{2,[0,T]} \leq \gamma^2\|w\|^2_{2,[0,T]}$ for all $w \in \mathcal{L}_2[0, T]$, which ensures that $\|\mathbf{R}_{zw}\|_{\infty,[0,T]} \leq \gamma$. The second Nash inequality shows that u^* regulates the state to zero with minimum output energy when the input disturbance is at its worst and given by w^*. If the input disturbance is given by any other signal, the controller will be suboptimal in this H_2 sense.

Before continuing, we have to decide on the strategy set for each player. To avoid difficulties with nonunique global Nash solutions with possibly differing Nash costs, we force both players to use linear, memoryless feedback controls; the difficulties associated with feedback strategies involving memory are beautifully illustrated by example in Basar [3]. To choose the control law $u^*(t, x)$ from a linear, memoryless feedback

[1]The signal $w^*(t, x)$ is worst case in the sense that it achieves the maximum possible energy gain from the disturbance input to the output.

strategy set is standard. To impose the same condition on $w^*(t, x)$ is slightly restrictive but necessary for a unique Nash equilibrium. We point out that if there exists a memoryless feedback strategy $w^*(t, x)$ such that $J_1(u^*, w^*) \geq 0$, then the H_∞ criterion

$$\|z\|^2_{2,[0,T]} \leq \gamma^2 \|w\|^2_{2,[0,T]}$$

is satisfied *for all finite energy signals* $w(t) \in \mathcal{L}_2[0, T]$.

This choice of strategy sets also results in a simple controller implementation, which is easily compared with the corresponding LQ and H_∞ controllers and with the mixed H_2/H_∞ theories described in [6], [10], [13], [22], [28], [30].

In Section IV we remove the linearity assumption on the two controllers. In particular, we extend both strategy sets to include control laws that are only required to be analytic in the state along the equilibrium trajectories.

B. The Necessary and Sufficient Conditions For the Existence of Linear Controls

The aim of this section is to give necessary and sufficient conditions for the existence of linear, memoryless Nash equilibrium controls. When controllers exist, we show that they are unique and parameterized by a pair of cross-coupled Riccati differential equations. We begin our presentation with a statement of our first main result. The proof will be presented in Sections II-B(1) and II-B(2). We use Ω to denote the set of all locally bounded (in x and t)[2], linear and memoryless state-feedback controls on $[0, T]$. We refer the reader to a paper by Lukes, which contains results of a similar flavor [21]. The performance criteria considered in [21] are a little less general than the ones used here and the theoretical development is quite different.

Theorem 2.1: Given the system described by:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad x(0) = x_0 \tag{2.9}$$

$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix} \qquad D'D = I \tag{2.10}$$

there exist Nash equilibrium strategies

$$u^*(t, x) \in \Omega$$
$$w^*(t, x) \in \Omega$$

such that

$$J_1(u^*, w^*) \leq J_1(u^*, w)\ \forall w(t) \in \Omega$$
$$J_2(u^*, w^*) \leq J_2(u, w^*)\ \forall u(t) \in \Omega$$

where

$$J_1(u, w) = \int_0^T [\gamma^2 w'(t)w(t) - z'(t)z(t)]\, dt \tag{2.11}$$

$$J_2(u, w) = \int_0^T z'(t)z(t)\, dt \tag{2.12}$$

if and only if the coupled Riccati differential equations

$$\begin{aligned} -\dot{P}_1(t) &= A'P_1(t) + P_1(t)A - C'C \\ &\quad - [P_1(t) \quad P_2(t)] \begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}; \\ P_1(T) &= 0 \end{aligned} \tag{2.13}$$

$$\begin{aligned} -\dot{P}_2(t) &= A'P_2(t) + P_2(t)A + C'C \\ &\quad - [P_1(t) \quad P_2(t)] \begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}; \\ P_2(T) &= 0 \end{aligned} \tag{2.14}$$

have solutions $P_1(t) \leq 0$ and $P_2(t) \geq 0$ on $[0, T]$. If solutions exist, we have that:

i) The Nash equilibrium strategies are uniquely specified by

$$u^*(t, x) = -B_2'P_2(t)x(t) \tag{2.15}$$
$$w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t). \tag{2.16}$$

ii)

$$J_1(u^*, w^*) = x_0'P_1(0)x_0 \tag{2.17}$$
$$J_2(u^*, w^*) = x_0'P_2(0)x_0. \tag{2.18}$$

iii) In the case that $u(t) = u^*(t, x)$ with $x_0 = 0$,

$$\|\mathbf{R}_{zw}\|_{\infty,[0,T]} < \gamma\ \forall w \in \mathcal{L}_2[0, T] \tag{2.19}$$

where the operator $\mathbf{R}_{zw}$ is defined by

$$\dot{x}(t) = (A - B_2B_2'P_2(t))x(t) + B_1 w(t) \tag{2.20}$$

$$z(t) = \begin{bmatrix} C \\ DB_2'P_2(t) \end{bmatrix} x(t). \tag{2.21}$$

1) The Sufficient Conditions: To show that the mixed H_2/H_∞ control problem has solutions $u^*(t, x)$ and $w^*(t, x)$ if the Riccati equations (2.13) and (2.14) have solutions on $[0, T]$, we use a standard completion of squares argument.

Let us consider the cost function $J_1(u, w)$ first. Since $P_1(T) = 0$, it is easy to show that

$$\begin{aligned} J_1(u, w) &- x_0'P_1(0)x_0 \\ &= \int_0^T \Big[\gamma^2 w'(t)w(t) - z'(t)z(t) \\ &\quad + \frac{d}{dt}(x'(t)P_1(t)x(t))\Big]\, dt \\ &= \int_0^T [\gamma^2 w'(t)w(t) - z'(t)z(t) \\ &\quad + \dot{x}'(t)P_1(t)x(t) + x'(t)\dot{P}_1x(t) \\ &\quad + x'(t)P_1(t)\dot{x}(t)]\, dt. \end{aligned}$$

Substituting for $\dot{P}_1(t)$ and $\dot{x}(t)$ from equations (2.13) and (2.9) and rearranging gives

$$\begin{aligned} J_1(u, w) &- x_0'P_1(0)x_0 \\ &= \int_0^T [\gamma^2(w(t) - w^*(t, x))'(w(t) - w^*(t, x)) \\ &\quad - u'(t)u(t) + u^{*\prime}(t, x)u^*(t, x) \\ &\quad + 2x'(t)P_1(t)B_2(u(t) - u^*(t, x))]\, dt \end{aligned}$$

[2] We do not allow unbounded controls, because in this case the existence of Nash equilibria is not equivalent to the existence of solutions to certain Riccati equations. This point is nicely illustrated in Example 5.1 [4].

where $u^*(t, x)$ and $w^*(t, x)$ are defined in (2.15) and (2.16). Setting $u(t) = u^*(t, x)$ gives

$$J_1(u^*, w) - x_0'P_1(0)x_0 = \int_0^T \gamma^2(w(t) - w^*(t, x))'(w(t) - w^*(t, x))\,dt \quad (2.22)$$

and, therefore, that

$$J_1(u^*, w^*) \le J_1(u^*, w)$$

with

$$J_1(u^*, w^*) = x_0'P_1(0)x_0.$$

Similarly,

$$\begin{aligned} J_2(u, w) &- x_0'P_2(0)x_0 \\ &= \int_0^T [(u(t) - u^*(t, x))'(u(t) - u^*(t, x)) \\ &\quad + 2x'(t)P_2(t)B_1(w(t) - w^*(t, x))]\,dt \end{aligned} \quad (2.23)$$

and setting $w(t) = w^*(t, x)$ results in

$$J_2(u^*, w^*) \le J_2(u, w^*)$$

and

$$J_2(u^*, w^*) = x_0'P_2(0)x_0.$$

We demonstrate that $\|\mathbf{R}_{zw}\|_{\infty, [0, T]} < \gamma$ for $x(0) = 0$ and $u(t) = u^*(t, x)$ by considering $J_1(u^*, w)$. It follows from (2.22) that

$$\begin{aligned} \gamma^2\|w(t)\|^2_{2, [0, T]} &- \|z(t)\|^2_{2, [0, T]} \\ &= \gamma^2\|w(t) - w^*(t, x)\|^2_{2, [0, T]} \\ &= \gamma^2\|\mathbf{L}w(t)\|^2_{2, [0, T]} \\ &\ge \epsilon\|w(t)\|^2_{2, [0, T]} \end{aligned}$$

in which ϵ is some positive number and $\mathbf{L}$ is an operator mapping $w \to w - w^*$. The above argument relies on the fact that $\mathbf{L}$ given by

$$\begin{aligned} \dot{x}(t) &= (A - B_2B_2'P_2(t))x(t) + B_1w(t) \quad x(0) = 0 \\ w(t) - w^*(t, x) &= -\gamma^{-2}B_1'P_1(t)x(t) + Iw(t) \end{aligned}$$

is causal with a causal inverse.

We establish the signs of $P_1(t)$ and $P_2(t)$ as follows:

i) A completion of squares argument similar to that which led to (2.23) gives

$$\begin{aligned} \int_t^T &(x'C'Cx + u'u)\,d\tau \\ &= x'(t)P_2(t)x(t) + \int_t^T (u - u^*)'(u - u^*)\,d\tau \\ &\quad + 2\int_\tau^T x'(t)P_2(t)B_1(w - w^*)\,d\tau. \end{aligned}$$

Setting $u = u^*$ and $w = w^*$ yields

$$x'(t)P_2(t)x(t) = \int_t^T (x'C'Cx + u^{*'}u^*)\,d\tau \ge 0$$

for arbitrary $x(t)$. Consequently $P_2(t) \ge 0$ for all times for which the solution exists.

ii) A similar calculation using $J_1(\cdot, \cdot)$ with $u = u^*$ and $w \equiv 0$ gives

$$x'(t)P_1(t)x(t) = -\int_t^T z'(t)z(t)\,d\tau - \gamma^{-2}\int_t^T w^{*'}w^*\,d\tau \le 0.$$

Thus $P_1(t) \le 0$ for all $t \in [0, T]$. This completes the proof of sufficiency. ∎

Remark: Since

$$x'(t)(P_1(t) + P_2(t))x(t) = \gamma^2\int_t^T w^{*'}w^*\,d\tau$$

we see that $(P_1(t) + P_2(t)) \ge 0$ for all $t \in [0, T]$ for which a solution exists. Notice that by adding $J_1(\cdot, \cdot)$ and $J_2(\cdot, \cdot)$, we can prove that the energy in the worst-case feedback disturbance is given by

$$\|w^*\|^2_{2, [0, T]} = \gamma^{-2}x_0'(P_1(0) + P_2(0))x_0.$$

2) The Necessary Conditions: Here we assume that an admissible Nash equilibrium strategy pair exists, and we will show that the coupled Riccati differential equations (2.13) and (2.14) have solutions $P_1(t) \le 0$ and $P_2(t) \ge 0$ on $[0, T]$. We need the following result on properties of contractive operators to establish the necessary conditions.

Lemma 2.2: Suppose

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(0) = 0 \\ y(t) &= Cx(t) \end{aligned}$$

describes a linear operator $\mathbf{R}$. Then the following two statements are equivalent:

i) $\|\mathbf{R}\|_{\infty, [0, T]} < \gamma$.

ii) The Riccati equation

$$-\dot{P} = A'P + PA - \gamma^{-2}PBB'P - C'C \quad (2.24)$$

with terminal condition $P(T) = 0$ has no finite escape time on $[0, T]$.

In the event that either of these conditions is satisfied, the optimal control problem

$$\min_u \left\{ J = \int_0^T (\gamma^2u'u - y'y)(t)\,dt \right\}$$

has a solution and the minimizing control is given by

$$u^*(t, x) = -B'P(t, T)x(t) \quad (2.25)$$

where $P(t, T)$ is the nonpositive solution to the Riccati equation (2.24).

Proof: If (2.24) has a solution, completing the square gives

$$J(u) - x_t'P(t, T)x_t = \int_t^T (u + B'Px)'(u + B'Px)\,dt$$

for all $t \in [0, T]$ and initial condition x_t.

Setting $u(t) = 0$ and defining $\Phi(\sigma, t)$ as the transition matrix associated with A, we get

$$x_t' P(t, T) x_t = -\int_t^T x_t' \Phi'(\sigma, t) \cdot (P(\sigma, T) B B' P(\sigma, T) + C'C) \Phi(\sigma, t) x_t \, d\sigma$$

for all x_t. Thus $P(t, T) \le 0$.

If $t = 0$ and $x_0 = 0$, we obtain

$$\gamma^2 \|u\|^2_{2, [0, T]} - \|y\|^2_{2, [0, T]} = \|u - u^*\|^2_{2, [0, T]}$$

where $u^*(t) = -B'P(t, T)x(t)$. Now the operator $\mathbf{L}$ mapping $u \to u - u^*$ has realization

$$\dot{x} = Ax + Bu$$
$$u - u^* = B'Px + Iu$$

which shows that it is invertible. Thus there exists an $\epsilon > 0$ such that

$$\gamma^2 \|u\|^2_{2, [0, T]} - \|y\|^2_{2, [0, T]} = \|\mathbf{L}u\|^2_{2, [0, T]} \ge \epsilon^2 \|u\|^2_{2, [0, T]}$$

and so $(\gamma^2 - \epsilon^2)\|u\|^2_{2, [0, T]} \ge \|y\|^2_{2, [0, T]}$, which proves that $\|\mathbf{R}\|_\infty < \gamma$.

We will now show that $\|\mathbf{R}\|_{\infty, [0, T]} < \gamma$ guarantees that (2.24) has a solution on $[0, T]$. By assumption, there exists an $\epsilon > 0$ such that

$$\int_0^T (\gamma^2 u'u - y'y)(t) \ge \epsilon \int_0^T (u'u)(t) \, dt \qquad (2.26)$$

for all $u \in \mathcal{L}_2[0, T]$. Now consider the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \begin{bmatrix} x(t^*) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2.27)$$

for any $t^* \in [0, T]$. If $t^* = T$ it is clear that $\lambda \equiv 0$ and $x \equiv 0$. Now consider $0 \le t^* < T$ and let x and λ be any solution to (2.27).

Define

$$\overline{u}(t) = \begin{cases} \gamma^{-2}B'\lambda & t^* < t < T \\ 0 & t \le t^* \end{cases}.$$

Since $x(0) = 0$ and $\overline{u}(t) = 0$ for $t \le t^*$, this gives

$$\begin{aligned} \int_0^T (\gamma^2 u'u - y'y)(t) \, dt &= \int_{t^*}^T (\gamma^{-2}\lambda' BB'\lambda - x'C'Cx)(t) \, dt \\ &= \int_{t^*}^T (\lambda'(\dot{x} - Ax) + x'(\dot{\lambda} + A'\lambda))(t) \, dt \\ &= \int_{t^*}^T \frac{d}{dt}(\lambda' x)(t) \, dt \\ &= 0 \end{aligned}$$

since $x(t^*) = 0$ and $\lambda(T) = 0$.

It now follows from (2.26) that $\overline{u}(t) \equiv 0$. Thus the two-point boundary value problem (2.27) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \begin{bmatrix} x(t^*) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has $x(t) \equiv 0$ and $\lambda(t) \equiv 0$ as its only solution. This shows that (2.27) has no conjugate points in $[0, T]$ [18]. The existence of a solution to (2.24) now follows from Lemma 2.2 in [18]. ■

We are now in a position to prove necessity. Since we are only considering linear controls at this point, we assume the existence of Nash controls of the form

$$w^*(t, x) = K_1(t)x(t)$$
$$u^*(t, x) = K_2(t)x(t).$$

i) Implementing $u^*(t, x)$, we obtain the operator $\mathbf{R}_{zw}$ defined by

$$\dot{x}(t) = (A + B_2K_2(t))x(t) + B_1 w(t) \qquad (2.28)$$
$$z(t) = \begin{bmatrix} C \\ DK_2(t) \end{bmatrix} x(t). \qquad (2.29)$$

Defining $\Phi(t, 0)$ to be the transition matrix associated with

$$\dot{x}(t) = (A + B_1K_1(t) + B_2K_2(t))x(t) \quad x(0) = x_0$$

we get

$$\begin{aligned} J_1(u^*, w^*) &= x_0' \left[\int_0^T \Phi'(t, 0)(\gamma^2 K_1' K_1 - C'C - K_2'K_2)(t)\Phi(t, 0) \, dt \right] x_0 \\ &= 0 \text{ if } x_0 = 0. \end{aligned}$$

Since $u^*(t, x)$ is an equilibrium strategy, $J_1(u^*, w) \ge \eta \|w\|^2_{2, [0, T]}$ for all w and some $\eta > 0$ and hence $\|\mathbf{R}_{zw}\|_{\infty, [0, T]} < \gamma$. It now follows from Lemma 2.2 that the control problem

$$\min_{w \in \Omega} J_1(u^*, w)$$

has a unique solution $w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t)$ where $P_1(t) \le 0$ solves

$$\begin{aligned} -\dot{P}_1(t) &= (A + B_2K_2(t))'P_1(t) + P_1(t)(A + B_2K_2(t)) \\ &\quad - \gamma^{-2}P_1(t)B_1B_1'P_1(t) - C'C - K_2'(t)K_2(t) \qquad (2.30) \end{aligned}$$

with $P_1(T) = 0$. Hence $K_1(t) = -\gamma^{-2}B_1'P_1(t)$.

ii) Implementing $w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t)$ gives

$$\dot{x}(t) = (A - \gamma^{-2}B_1B_1'P_1(t))x(t) + B_2u(t)$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}.$$

Since

$$\min_{u \in \Omega} J_2(w^*, u)$$

is a standard optimal control problem,

$$u^*(t, x) = -B_2'P_2(t)x(t)$$

where $P_2(t)$ is the nonnegative solution of

$$\begin{aligned}-\dot{P}_2 &= (A-\gamma^{-2}B_1B_1'P_1(t))'P_2(t)\\ &+P_2(t)(A-\gamma^{-2}B_1B_1'P_1(t))\\ &-P_2(t)B_2B_2'P_2(t)+C'C \quad P_2(T)=0.\end{aligned}$$

It is well known that $P_2(t)$ exists on $[0, T]$. By uniqueness, $K_2(t) = -B_2'P_2(t)$. Substituting for $K_2(t)$ in (2.30) gives the required Riccati differential equation (2.13).

C. A Stochastic Interpretation

In the previous section we found a control law $u^*(t, x)$ that achieves $\|\mathbf{R}_{zw}\|_{\infty, [0, T]} < \gamma$, and at the same time solves the deterministic regulator problem

$$\min_u \left\{ J_2(u, w^*) = \int_0^T z'(t)z(t)\,dt \right\}.$$

In this section we extend the analysis to the case of a second white noise disturbance input. In particular, we show that the control law $u^*(t, x)$ solves the stochastic linear regulator problem

$$\min_u \left\{ \overline{J}_2(u, w^*) = \mathcal{E}\left\{ \int_0^T z'(t)z(t)\,dt \right\} \right\}$$

when the equation for the state dynamics is replaced by

$$\dot{x}(t) = Ax(t) + B_0w_0(t) + B_1w(t) + B_2u(t)$$

in which $w_0(t)$ is a realization of a white noise process. This is precisely the situation depicted in Fig. 1.

Theorem 2.3: Suppose

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_0w_0(t) + B_1w(t) + B_2u(t)\\ x(0) &= x_0 &(2.31)\\ z(t) &= \begin{bmatrix} Cx(t)\\ Du(t)\end{bmatrix} \quad D'D = I &(2.32)\end{aligned}$$

with $\mathcal{E}(x_0x_0') = Q_0$ and $\mathcal{E}(w_0(\tau)w_0(t)) = I\delta(t-\tau)$.

Suppose the coupled Riccati equations (2.13) and (2.14) have solutions, then the control law $u^*(t, x) = -B_2'P_2(t)x(t)$

i) results in $\|\mathbf{R}_{zw}\|_{\infty, [0, T]} < \gamma$, where the operator $\mathbf{R}_{zw}$ is described by:

$$\begin{aligned}\dot{x}(t) &= (A - B_2B_2'P_2(t))x(t) + B_1w(t) \quad x(0) = 0\\ z(t) &= \begin{bmatrix} C\\ -DB_2'P_2(t)\end{bmatrix}x(t)\end{aligned}$$

and

ii) solves the stochastic linear regulator problem

$$\min_u \left\{ \overline{J}_2(u, w^*) = \mathcal{E}\left\{ \int_0^T z'(t)z(t)\,dt \right\} \right\} \tag{2.33}$$

with $w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t)$. In addition, we get

$$\overline{J}_2(u^*, w^*) = tr\left[P_2(0)Q_0 + \int_0^T B_0'(t)P_2(t)B_0(t)\,dt\right]. \tag{2.34}$$

Proof: We know from previous results that $u^*(t, x) = -B_2'P_2(t)x(t)$ gives $\|\mathbf{R}_{zw}\|_{\infty, [0, T]} < \gamma$, $\forall w \in \mathcal{L}_2[0, T]$. Implementing the worst-case disturbance $w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t)$ gives

$$\begin{aligned}\dot{x}(t) &= (A - \gamma^{-2}B_1B_1'P_1(t))x(t) + B_0w_0(t) + B_2u(t)\\ z(t) &= \begin{bmatrix} Cx(t)\\ Du(t)\end{bmatrix}.\end{aligned}$$

We wish to find the control law $u^*(t, x)$ that minimizes

$$\overline{J}_2(u, w^*) = \mathcal{E}\left\{ \int_0^T z'(t)z(t)\,dt \right\}.$$

Since this is a standard stochastic linear regulator problem solved by

$$u^*(t, x) = -B_2'P(t)x(t)$$

where $P(t)$ satisfies

$$\begin{aligned}-\dot{P}(t) &= (A-\gamma^{-2}B_1B_1'P_1(t))'P(t)\\ &+P(t)(A-\gamma^{-2}B_1B_1'P_1(t)) - P(t)B_2B_2'P(t) + C'C\end{aligned}$$

with $P(T) = 0$, all we need to do is set $P(t) \equiv P_2(t)$. It is also known that the minimal value of the cost criterion is given by

$$\overline{J}_2(u^*, w^*) = tr\left[P(0)Q_0 + \int_0^T B_0'(t)P(t)B_0(t)\,dt\right].$$

∎

Remark: Implementing $u^*(t, x)$ and $w^*(t, x)$ gives

$$\begin{aligned}\dot{x} &= (A - \gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2)x + B_0w_0 \quad x(0) = 0\\ w^* &= -\gamma^{-2}B_1'P_1x.\end{aligned}$$

It is immediate from (2.13) and (2.14) that

$$\begin{aligned}-(\dot{P}_1 + \dot{P}_2) &= (A - \gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2)'(P_1 + P_2)\\ &+ (P_1 + P_2)(A - \gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2)\\ &+ \gamma^{-2}P_1B_1B_1'P_1\end{aligned}$$

in which $(P_1 + P_2)(T) = 0$. It now follows from a standard result on systems driven by stochastic processes [17, Theorem 1.54], that the energy in the worst-case disturbance is given by:

$$\begin{aligned}&\mathcal{E}\left\{ \int_0^T w^{*'}(t, x)w^*(t, x)\,dt \right\}\\ &= \mathcal{E}\left\{ \int_0^T x'(\gamma^{-2}P_1B_1B_1'P_1)x\,dt \right\}\\ &= \gamma^{-2}tr\left\{ \int_0^T B_0'(t)(P_1 + P_2)(t)B_0(t)\,dt \right\}.\end{aligned}$$

D. Reconciliation of H_2, H_∞, and Mixed H_2/H_∞ Theories

In this section we establish a link between LQ control, H_∞ control and our mixed H_2/H_∞ control problem. Each of the three problems may be generated as special cases of the following nonzero sum, two-player Nash differential game:

Given the system described by:

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \quad x(0) = x_0$$
$$z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix} \quad D'(t)D(t) = I$$

find Nash equilibrium strategies $u^*(t, x)$ and $w^*(t, x)$ in Ω (the set of linear memoryless feedback laws) which satisfy

$$J_1(u^*, w^*) \le J_1(u^*, w)$$
$$J_2(u^*, w^*) \le J_2(u, w^*),$$

where

$$J_1(u, w) = \int_0^T [\gamma^2 w'(t)w(t) - z'(t)z(t)]\, dt$$
$$J_2(u, w) = \int_0^T [z'(t)z(t) - \rho^2 w'(t)w(t)]\, dt.$$

The solution to this game is given by

$$u^*(t, x) = -B_2' S_2(t)x(t)$$

and

$$w^*(t, x) = -\gamma^{-2} B_1' S_1(t)x(t)$$

where $S_1(t)$ and $S_2(t)$ satisfy the coupled Riccati differential equations

$$-\dot{S}_1(t) = A'S_1(t) + S_1(t)A - C'C - [S_1(t) \;\; S_2(t)] \begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix} \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix}$$
$$-\dot{S}_2(t) = A'S_2(t) + S_2(t)A + C'C - [S_1(t) \;\; S_2(t)] \begin{bmatrix} \rho^2\gamma^{-4}B_1B_1' & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix} \cdot \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix}$$

with $S_1(T) = S_2(T) = 0$ on $[0, T]$, [25].

i) The standard optimal control problem

$$\min_u \left\{ J_2(u) = \int_0^T z'(t)z(t)\, dt \right\}$$

where

$$\dot{x}(t) = Ax(t) + B_2u(t) \quad x(0) = x_0$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix} \qquad D'D = I$$

is obtained by setting $\rho = 0$ and $\gamma \to \infty$, in which case

$$S_1(t) \to -P(t)$$
$$S_2(t) \to P(t)$$

where $P(t)$ satisfies the LQ Riccati equation

$$-\dot{P}(t) = A'P(t) + P(t)A - P(t)B_2B_2'P(t) + C'C$$
$$P(T) = 0$$

with $u^*(t, x) = -B_2'P(t)x(t)$ and $w^*(t, x) \equiv 0$.

ii) To generate the H_∞ control problem, we set $\rho = \gamma$. Here we get

$$P_\infty(t) = -S_1(t) = S_2(t)$$

where $P_\infty(t)$ satisfies

$$-\dot{P}_\infty = A'P_\infty(t) + P_\infty(t)A - P_\infty(t)(B_2B_2' - \gamma^{-2}B_1B_1')P_\infty(t) + C'C$$

with $P_\infty(T) = 0$. The Nash equilibria are given by $u^*(t, x) = -B_2'P_\infty(t)x(t)$ and $w^*(t, x) = \gamma^{-2}B_1'P_\infty(t)x(t)$. Taken together, the two Nash inequalities form the usual saddle point conditions associated with zero sum games.

iii) The mixed H_2/H_∞ control problem is retrieved by setting $\rho = 0$. This gives

$$P_1(t) = S_1(t)$$
$$P_2(t) = S_2(t)$$

where $P_1(t)$ and $P_2(t)$ satisfy the coupled Riccati equations (2.13) and (2.14) and the equilibrium solutions are $u^*(t, x) = -B_2'P_2(t)x(t)$ and $w^*(t, x) = -\gamma^{-2}B_1'P_1(t)x(t)$.

III. Infinite Horizon Case

The main result of this section gives necessary and sufficient conditions for the existence of an H_2/H_∞ optimal controller in the infinite horizon case. For the purpose of this section the data are assumed constant.

Theorem 3.1: Given the system:

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t) \quad x(0) = 0$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix} \tag{3.35}$$

with $D'D = I$ and (A, C) detectable, suppose there exist solutions $P_1 \le 0$ and $P_2 \ge 0$ that satisfy:

$$0 = A'P_1 + P_1A - C'C - [P_1 \;\; P_2] \begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{3.36}$$

$$0 = A'P_2 + P_2A + C'C - [P_1 \;\; P_2] \begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \tag{3.37}$$

Then:

i) $\operatorname{Re} \lambda_i(A - B_2B_2'P_2) < 0$, so that (A, B_2) is stabilizable

ii) If $((A - \gamma^{-2}B_1B_1'P_1), C)$ is detectable, then

a) $\operatorname{Re} \lambda_i(A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1) < 0$ and

b) the strategies

$$u^*(t, x)(t) = -B_2'P_2x(t) \tag{3.38}$$
$$w^*(t, x) = -\gamma^{-2}B_1'P_1x(t) \tag{3.39}$$

result in

i) $\mathbf{R}_{zw} \in \mathcal{RH}_\infty^+$ such that $\|\mathbf{R}_{zw}\|_\infty < \gamma$ when $u(t) = u^*(t, x)$

ii) $J_2(u^*, w^*) \le J_2(u, w^*)$.

Conversely, suppose that (A, B_2) is stabilizable and that there exist time-invariant feedback strategies

$$u^*(t, x) = K_2 x(t)$$
$$w^*(t, x) = K_1 x(t)$$

such that

i) $\|\mathbf{R}_{zw}\|_\infty < \gamma$ when $u(t) = u^*(t, x)$
ii) $J_2(u^*, w^*) \le J_2(u, w^*)$
iii) $\mathrm{Re}\,\lambda_i(A + B_2K_2) < 0$
iv) $((A + B_1K_1), C)$ detectable.

then there exist solutions $P_1 \le 0$ and $P_2 \ge 0$ which satisfy (3.36) and (3.37).

Proof: We begin by rearranging (3.36) and (3.37) as

$$0 = (A - B_2B_2'P_2)'P_1 + P_1(A - B_2B_2'P_2) - C'C - \gamma^{-2}P_1B_1B_1'P_1 - P_2B_2B_2'P_2 \quad (3.40)$$

$$0 = (A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1)'P_2 + P_2(A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1) + C'C + P_2B_2B_2'P_2 \quad (3.41)$$

in which $P_1 \le 0$ and $P_2 \ge 0$ is assumed.

Note that

$$(A, C) \text{ detectable} \Rightarrow \left(A, \begin{bmatrix} C \\ B_2'P_2 \end{bmatrix}\right) \text{ detectable}$$
$$\Rightarrow \left((A - B_2B_2'P_2), \begin{bmatrix} C \\ B_2'P_2 \end{bmatrix}\right) \text{ detectable.}$$

It now follows that

$$\mathrm{Re}\,\lambda_i(A - B_2'B_2P_2) < 0$$

by the standard properties of Lyapunov equations [27]. This means that $\mathbf{R}_{zw} \in RH_\infty^+$ as required.

The assumed detectability of $((A - \gamma^{-2}B_1B_1'P_1), C)$ implies the detectability of

$$\left((A - \gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2), \begin{bmatrix} C \\ B_2'P_2 \end{bmatrix}\right).$$

That $\mathrm{Re}\,\lambda_i(A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1) < 0$ again follows from the standard properties of Lyapunov equations.

Setting $u(t) = u^*(t, x)$ gives

$$\dot{x}(t) = (A - B_2B_2'P_2)x(t) + B_1w(t)$$
$$z(t) = \begin{bmatrix} C \\ DB_2'P_2 \end{bmatrix} x(t).$$

Since $\mathrm{Re}\,\lambda_i(A - B_2B_2'P_2) < 0$, $w \in \mathcal{L}_{2+} \Rightarrow x(t) \in \mathcal{L}_{2+}$. Completing the square gives

$$\int_0^\infty (\gamma^2 w'w - z'z)\,dt = \gamma^2 \int_0^\infty (w - w^*)'(w - w^*)\,dt \ge 0$$

in which

$$w^*(t) = -\gamma^{-2}B_1'P_1x(t) = -\gamma^{-2}B_1'P_1 \exp\{(A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1)t\} \cdot x(0) \in \mathcal{L}_{2+}.$$

Defining the invertible operator $\mathbf{L}$ from $w(t) \to w(t) - w^*(t, x)$ by

$$\dot{x}(t) = (A - B_2B_2'P_2)x(t) + B_1w(t)$$
$$w(t) - w^*(t, x) = -\gamma^{-2}B_1'P_1x(t) + Iw(t)$$

gives

$$\gamma^2\|w\|_2^2 - \|z\|_2^2 = \gamma^2\|\mathbf{L}w\|_2^2 \ge \epsilon\|w\|_2^2$$

for some positive ϵ. Consequently, $\|\mathbf{R}_{zw}\|_\infty < \gamma$.

Setting $w(t) = w^*(t, x)$ gives

$$\dot{x}(t) = (A - \gamma^{-2}B_1B_1'P_1)x(t) + B_2u(t)$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}.$$

Observing that

$$\min_u \left\{ J_2(u, w^*) = \int_0^\infty z'(t)z(t)\,dt \right\}$$

is a standard optimal control problem with $((A - \gamma^{-2}B_1B_1'P_1), B_2)$ stabilizable (since $\mathrm{Re}\,\lambda_i(A - B_2B_2'P_2 - \gamma^{-2}B_1B_1'P_1) < 0$ and $((A - \gamma^{-2}B_1B_1'P_1), C)$ detectable, gives $u^*(t, x) = -B_2'P_2x(t)$. This results in $J_2(u^*, w^*) \le J_2(u, w^*)$ as required.

We now establish the opposite implications.

a) Implementing $u^*(t, x) = K_2x(t)$ gives

$$\dot{x}(t) = (A + B_2K_2)x(t) + B_1w(t)$$
$$z(t) = \begin{bmatrix} C \\ DK_2 \end{bmatrix} x(t)$$

where $\mathrm{Re}\,\lambda_i(A + B_2K_2) < 0$ and $\|\mathbf{R}_{zw}\|_\infty < \gamma$ by assumption. It now follows from [9, Lemma 4] that there exists a $P_1 \le 0$ that satisfies

$$0 = (A + B_2K_2)'P_1 + P_1(A + B_2K_2) - \gamma^{-2}P_1B_1B_1'P_1 - C'C - K_2'K_2 \quad (3.42)$$

such that

$$\mathrm{Re}\,\lambda_i(A + B_2K_2 - \gamma^{-2}B_1B_1'P_1) < 0.$$

As a consequence $((A - \gamma^{-2}B_1B_1'P_1), B_2)$ is stabilizable.

Since $\mathrm{Re}\,\lambda_i(A + B_2K_2) < 0$, $w(t) \in \mathcal{L}_{2+} \Rightarrow x(t) \in \mathcal{L}_{2+}$. Completing the square yields

$$\int_0^\infty (\gamma^2 w'w - z'z)\,dt = \gamma^2 \int_0^\infty (w + \gamma^{-2}B_1'P_1x)'(w + \gamma^{-2}B_1'P_1x)\,dt.$$

This gives

$$w^*(t) = -\gamma^{-2}B_1'P_1x(t) = -\gamma^{-2}B_1'P_1 \exp\{(A + B_2K_2 - \gamma^{-2}B_1B_1'P_1)t\} \cdot x(0) \in \mathcal{L}_2$$
$$\Rightarrow K_1 = -\gamma^{-2}B_1'P_1.$$

b) Implementing $w^*(t, x) = -\gamma^{-2}B_1'P_1x(t)$ gives

$$\dot{x}(t) = (A - \gamma^{-2}B_1B_1'P_1)x(t) + B_2u(t)$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}$$

with $((A - \gamma^{-2}B_1B_1'P_1), B_2)$ stabilizable and $((A - \gamma^{-2}B_1B_1'P_1), C)$ detectable. Since

$$\min_u\left\{J_2(u, w^*) = \int_0^\infty x'C'Cx + u'u\,dt\right\}$$

is a standard optimal control problem, it may be solved by $u^*(t, x) = -B_2'P_2x(t)$, where P_2 is the nonnegative solution of

$$0 = (A - \gamma^{-2}B_1B_1'P_1)'P_2 + P_2(A - \gamma^{-2}B_1B_1'P_1) - P_2B_2B_2'P_2 + C'C.$$

Substituting for $K_2 = -B_2'P_2$ in (3.42) gives the Riccati equation (3.36) as required. ■

Remark (Connections with the Power Semi-Norm): If we consider:

$$\dot{x}(t) = Ax(t) + B_0w_0(t) + B_1w_1(t) + B_2u(t) \quad (3.43)$$
$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix} \quad (3.44)$$

where w_0 is white noise with unit spectral density and

$$u^* = -B_2'P_2x$$

then

$$\sup_{w_1\in\mathcal{P}}\{\|z\|_p^2 - \gamma^2\|w_1\|_p^2\} = \text{trace}\{-B_0'P_1B_0\} \quad (3.45)$$

follows from [10, Theorem 1]. *As opposed to [10], [30], this quantity is not minimized in our work.*

As we will show, however, $\|z\|_p^2$ is minimized by u^* for a class of bounded power input signals w_1. It follows by direct calculation using (2.14) that

$$\frac{d}{dt}(x'P_2x) + z'z = 2w_0'B_0'P_2x + (u - u^*)'(u - u^*) + 2x'P_2B_1(w_1 - \gamma^{-2}B_1'Px).$$

Integrating from $-T$ to T and taking averages gives

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{d}{dt}(x'P_2x)\,dt = -\|z\|_p^2 + \|u - u^*\|_p^2 + 2\,\text{trace}(B_0'P_2R_{xw_0}(0)) + \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}x'P_2B_1(w_1 - \gamma^{-2}B_1'P_1)\,dt \quad (3.46)$$

in which $R_{xw_0}(\tau)$ is the cross-correlation between x and w_0. Noting that the left-hand side of (3.46) is zero for any bounded power signal w_1 and exploiting the fact that $R_{xw_0}(0) = \frac{1}{2}B_0$ [30] gives

$$\|z\|_p^2 = \|u - u^*\|_p^2 + \text{trace}\{B_0'P_2B_0\} + \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}x'P_2B_1(w_1 - \gamma^{-2}B_1'P_1)\,dt.$$

It now follows that

$$\|z\|_p^2 = \text{trace}\{B_0'P_2B_0\}$$

is optimal for $u = u^*$ and any bounded power input w_1 resulting in

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}x'P_2B_1(w_1 - \gamma^{-2}B_1'P_1)\,dt = 0.$$

IV. Necessary Conditions For a Class of Nonlinear Controls

In this section we extend the strategy set Ω of admissible equilibrium controls to include all functions that are continuous in time for $t \in [0, T]$ and analytic at $x = 0$. We begin with a definition.

Definition 4.1: An m-vector valued function $u(t, x)$ is analytic at $x = 0$, for each fixed t, if in some positive neighborhood $\|x\| < \delta$, each component $u_j(t, x)$, $j = 1, 2, \cdots, m$, of $u(t, x)$ may be represented by a convergent power series in x of the form

$$u_j(t, x) = \sum_{k=0}^{\infty}\alpha_{jk}'(t)\phi_k \quad (4.47)$$

in which $\alpha_{jk}(t)$ is a column vector and ϕ_k is a column vector consisting of all possible forms

$$(x_1)^{i_1}(x_2)^{i_2}\cdots(x_n)^{i_n}$$

such that

$$\sum_{l=1}^{n}i_l = k \quad i_l \geq 0 \quad l = 1, \cdots, n.$$

The rest of the work in this section is concerned with the Hamilton–Jacobi equations for Nash games with closed-loop controls. These equations are derived in [25] using the Hamiltonian functions

$$H_1(t, x, u, w, p_1) = \gamma^2w'w - x'C'Cx - u'u + p_1'(Ax + B_1w + B_2u) \quad (4.48)$$

and

$$H_2(t, x, u, w, p_2) = x'C'Cx + u'u + p_2'(Ax + B_1w + B_2u). \quad (4.49)$$

The equilibrium controls must satisfy

$$\frac{\partial H_1}{\partial w} = 0 \Rightarrow w^*(t, x) = -\frac{1}{2}\gamma^{-2}B_1'(t)p_1(t) \quad (4.50)$$

and

$$\frac{\partial H_2}{\partial u} = 0 \Rightarrow u^*(t, x) = -\frac{1}{2}B_2'(t)p_2(t) \quad (4.51)$$

and the value functions V_i obey the partial differential equations

$$\frac{\partial V_i}{\partial t} = -H_i(t, x, u^*, w^*, p_i) \quad (4.52)$$

and

$$\frac{\partial V_i}{\partial x} = p_i. \quad (4.53)$$

Our first result establishes analyticity of the value functions in the neighborhood of $x = 0$.

Lemma 4.2: Suppose

$$\dot{x}(t) = A(t)x(t) + B_1(t)w^*(t, x) + B_2(t)u^*(t, x)$$
$$x(0) = 0$$

in which $u^*(t, x)$ and $w^*(t, x)$ have power series expansions about $x = 0$ which are convergent in the neighborhood $\|x\| < \delta$.

Then the value functions

$$V_1(t, x) = \int_0^t (\gamma^2 w^{*'} w^* - x'C'Cx - u^{*'}u^*)(t)\, dt$$
$$V_2(t, x) = \int_0^t (x'C'Cx + u^{*'}u^*)(t)\, dt$$

are analytic in the neighborhood $\|x\| < \sigma \leq \delta$ for each fixed $t \in [0, T]$ and may be expressed locally as convergent power series of the form

$$V_j(t, x) = \sum_{k=0}^{\infty} \beta'_{kj}(t)\phi_k \quad j = 1, 2 \tag{4.54}$$

where ϕ_k is defined in Definition 4.1 and $\beta_{kj}(t)$ is a column vector of the same dimension as ϕ_k.

Proof: The analyticity and continuity properties of $u^*(t, x)$ and $w^*(t, x)$ are inherited by the function $f(t, x) = Ax(t) + B_1w^*(t, x) + B_2u^*(t, x)$. Hence, $f(t, x)$ is a continuous function of time on $[0, T]$ with a convergent power series expansion about $x = 0$ in the neighborhood $\|x\| < \delta$. By [8, Theorem 8.4], there then exists a $\sigma \leq \delta$ such that for any $x \in X$, where

$$X := \{x(t) : \|x(t)\| < \sigma\}$$

there exists a unique solution $\varphi(t, x)$ to $\dot{x}(t) = f(t, x)$, which is continuous in time and for each fixed t is an analytic function of x for all $x \in X$. It is now immediate that

$$V_1(t, x) = \int_0^t (\gamma^2 w^{*'} w^* - x'C'Cx - u^{*'}u^*)\, dt$$
$$V_2(t, x) = \int_0^t (x'C'Cx + u^{*'}u^*)\, dt$$

have power series expansions about the origin for each fixed $t \in [0, T]$. ■

Our next result concerns a property of the solutions to the Hamilton–Jacobi equations.

Lemma 4.3: Let $V_1(t, x)$ and $V_2(t, x)$ solve the Hamilton–Jacobi equations associated with the game. Then, $\overline{V}_1(t, x)$ and $\overline{V}_2(t, x)$, defined as

$$\overline{V}_i(t, x) = \lambda^{-2} V_i(t, \lambda x), \quad i = 1, 2 \tag{4.55}$$

for an arbitrary real scalar λ, are also solutions to the Hamiltonian-Jacobi equations.

Proof: Substituting (4.50)–(4.53) into (4.48) and (4.49) gives the following Hamilton–Jacobi equations:

$$\left(\frac{\partial V_1}{\partial t}\right) = \frac{1}{4}\left(\frac{\partial V_1}{\partial x}\right)' B_1\gamma^{-2}B_1'\left(\frac{\partial V_1}{\partial x}\right) + x'C'Cx + \frac{1}{2}\left(\frac{\partial V_1}{\partial x}\right)' B_2B_2'\left(\frac{\partial V_2}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial V_2}{\partial x}\right)' B_2B_2'\left(\frac{\partial V_2}{\partial x}\right) - \left(\frac{\partial V_1}{\partial x}\right)' Ax \tag{4.56}$$

$$\left(\frac{\partial V_2}{\partial t}\right) = \frac{1}{4}\left(\frac{\partial V_2}{\partial x}\right)' B_2B_2'\left(\frac{\partial V_2}{\partial x}\right) - x'C'Cx + \frac{1}{2}\left(\frac{\partial V_2}{\partial x}\right)' B_1\gamma^{-2}B_1'\left(\frac{\partial V_1}{\partial x}\right) - \left(\frac{\partial V_2}{\partial x}\right)' Ax \tag{4.57}$$

with terminal conditions

$$V_i(T, x(T)) = 0 \quad \forall x \qquad i = 1, 2.$$

Using the definition of $\overline{V}_i$ in (4.55), we now observe that

$$\frac{\partial \overline{V}_i}{\partial t} = \frac{1}{\lambda^2}\frac{\partial V_i(t, y)}{\partial t}\bigg|_{y=\lambda x} \tag{4.58}$$

and

$$\frac{\partial \overline{V}_i}{\partial x} = \frac{1}{\lambda}\frac{\partial V_i(t, y)}{\partial y}\bigg|_{y=\lambda x}. \tag{4.59}$$

Using (4.58) and (4.56), we establish that

$$\begin{aligned}\frac{\partial \overline{V}_1}{\partial t} &= \frac{1}{\lambda^2}\left\{\frac{1}{4}\left(\frac{\partial V_1}{\partial y}\right)' B_1\gamma^{-2}B_1'\left(\frac{\partial V_1}{\partial y}\right) + y'C'Cy + \frac{1}{2}\left(\frac{\partial V_1}{\partial y}\right)' B_2B_2'\left(\frac{\partial V_2}{\partial y}\right) + \frac{1}{4}\left(\frac{\partial V_2}{\partial y}\right)' B_2B_2'\left(\frac{\partial V_2}{\partial y}\right)\left(\frac{\partial V_1}{\partial y}\right)' Ay\right\} \\ &= \frac{1}{4}\left(\frac{\partial \overline{V}_1}{\partial x}\right)' B_1\gamma^{-2}B_1'\left(\frac{\partial \overline{V}_1}{\partial x}\right) + x'C'Cx + \frac{1}{2}\left(\frac{\partial \overline{V}_1}{\partial x}\right)' B_2B_2'\left(\frac{\partial \overline{V}_2}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial \overline{V}_2}{\partial x}\right)' B_2B_2'\left(\frac{\partial \overline{V}_2}{\partial x}\right) - \left(\frac{\partial \overline{V}_1}{\partial x}\right)' Ax\end{aligned}$$

by setting $y = \lambda x$ and using (4.59).

This shows that $\overline{V}_1$ does indeed satisfy the Hamilton–Jacobi equation (4.56). A similar calculation establishes that $\overline{V}_2$ is a solution to (4.57). ■

Corollary 4.4: If there exists a solution pair $V_1(t, x)$ and $V_2(t, x)$ to the Hamilton–Jacobi equations (4.56) and (4.57), which is analytic in the neighborhood $\|x\| < \sigma$, then there exists a second solution pair $\overline{V}_1(t, x)$ and $\overline{V}_2(t, x)$, which is analytic in an arbitrarily large neighborhood of $x = 0$.

Proof: The proof follows immediately by comparing the power series expansions of $V_i(t, x)$ and $\overline{V}_i(t, x)$ and noting that if the expansion of $V_i(t, x)$ converges for $\|x\| < \sigma$ then the expansion of $\overline{V}_i(t, x)$ must converge in the neighborhood of $\|x\| < \sigma/|\lambda|$ for arbitrarily small λ. ∎

Remark 1: A necessary condition for a function $V(t, x)$ to be quadratic in x is that it satisfies [1]

$$\lambda^2 V(t, x) = V(\lambda x, t).$$

Remark 2: If (as will be shown in the next theorem) the $V_i(t, x)$ are quadratic in x, it follows that

$$\overline{V}_i(tx) \equiv V_i(t, x), \quad i = 1, 2, \quad \forall t \in [0, T].$$

We are now ready to show that any analytic, memoryless and possibly nonlinear equilibrium solutions to the mixed H_2/H_∞ problem are in fact linear.

Theorem 4.5: Suppose equilibrium feedback controls $u^*(t, x)$ and $w^*(t, x)$ exist that satisfy the Nash conditions (2.7) and (2.8). Suppose also that $u^*(t, x)$ and $w^*(t, x)$ are continuous in time on $[0, T]$ and have power series expansions in the neighborhood of $x = 0$. Then the coupled Riccati equations

$$\begin{aligned} -\dot{P}_1(t) &= \{A'P_1 + P_1A - \gamma^{-2}P_1B_1B_1'P_1 - P_2B_2B_2'P_2 \\ &\quad - P_1B_2B_2'P_2 - P_2B_2B_2'P_1 - C'C\}(t) \\ P_1(T) &= 0 \end{aligned} \tag{4.60}$$

$$\begin{aligned} -\dot{P}_2(t) &= \{A'P_2 + P_2A - P_2B_2B_2'P_2 - \gamma^{-2}P_1B_1B_2'P_2 \\ &\quad - \gamma^{-2}P_2B_1B_1'P_1 + C'C\}(t) \\ P_2(T) &= 0 \end{aligned} \tag{4.61}$$

have solutions $P_1(t)$ and $P_2(t)$ on $[0, T]$ and the equilibrium controls are given by

$$w^*(t, x) = -\gamma^{-2}B_1'(t)P_1(t)x(t) \tag{4.62}$$

$$u^*(t, x) = -B_2'(t)P_2(t)x(t). \tag{4.63}$$

Proof: Since equilibrium controls are assumed to exist, their associated value functions must satisfy the Hamilton–Jacobi equations (4.56) and (4.57).

We know from Lemma 4.2 and Corollary 4.4 that there exist solutions $\overline{V}_i$ to the Hamilton–Jacobi equations, $\overline{V}_i$ which may be expanded as

$$\overline{V}_i(t, x) = \sum_{k=0}^{\infty} \beta_{ki}'(t)\phi_k \tag{4.64}$$

in an arbitrarily large neighborhood of $x = 0$. As before, ϕ_k is a vector consisting of all possible forms

$$(x_1(t))^{i_1}(x_2(t))^{i_2}\cdots(x_n(t))^{i_n}$$

where

$$\sum_l i_l = k \quad i_l \geq 0 \quad l = 1, 2, \cdots, n.$$

We may now evaluate the following partial derivatives

$$\frac{\partial \overline{V}_i}{\partial t} = \sum_{k=0}^{\infty} \dot{\beta}_{ki}'(t)\phi_k \tag{4.65}$$

and

$$\frac{\partial \overline{V}_i}{\partial x} = \begin{bmatrix} \sum_{k=0}^{\infty} \beta_{ki}'(t)\frac{\partial}{\partial x_1}(\phi_k) \\ \vdots \\ \sum_{k=0}^{\infty} \beta_{ki}'(t)\frac{\partial}{\partial x_n}(\phi_k) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{\infty} \beta_{ki}'(t)\Theta_k^1\phi_{k-1} \\ \vdots \\ \sum_{k=1}^{\infty} \beta_{ki}'(t)\Theta_k^n\phi_{k-1} \end{bmatrix}. \tag{4.66}$$

If ϕ_k and β_{ki} are vectors of dimension p_k, Θ_k^j, $j = 1, \cdots, n$ are matrixes of dimensions $p_k \times p_{k-1}$.[3]

Exploiting the fact that $\phi_0 = 1$, $\phi_1 = x$, and $\frac{\partial}{\partial x}(\phi_1) = 1$, (4.64) and (4.66) may be rewritten as

$$\overline{V}_i(t, x) = \beta_{0i}' + \beta_{1i}'x + \sum_{k=2}^{\infty} \beta_{ki}'(t)\phi_k \tag{4.67}$$

and

$$\frac{\partial \overline{V}_i}{\partial x} = \beta_{1i} + \begin{bmatrix} \sum_{k=2}^{\infty} \beta_{ki}'(t)\Theta_k^1\phi_{k-1} \\ \vdots \\ \sum_{k=2}^{\infty} \beta_{ki}'(t)\Theta_k^n\phi_{k-1} \end{bmatrix}. \tag{4.68}$$

It is now possible to show that the value functions $\overline{V}_i$, $i = 1, 2$, are necessarily quadratic forms in $x(t)$ by substituting (4.65) and (4.66) into the Hamilton–Jacobi equations (4.56) and (4.57), and equating terms with equal powers of x. We consider the constant term first.

$$(4.56) \Rightarrow \dot{\beta}_{01}' = \left[F_{01}(\beta_{11}) + F_{02}(\beta_{12}) + F_{03}\left(\begin{bmatrix}\beta_{11} \\ \beta_{12}\end{bmatrix}\right)\right] \tag{4.69}$$

$$(4.57) \Rightarrow \dot{\beta}_{02}' = \left[G_{01}(\beta_{12}) + G_{02}\left(\begin{bmatrix}\beta_{11} \\ \beta_{12}\end{bmatrix}\right)\right] \tag{4.70}$$

where, for example, $F_{01}(\beta_{11})$, $G_{01}(\beta_{12})$, and $G_{02}\left(\begin{bmatrix}\beta_{11} \\ \beta_{12}\end{bmatrix}\right)$ are given by

$$\begin{aligned} F_{01}(\beta_{11}) &= \frac{1}{4}\gamma^{-2}\beta_{11}'(t)B_1B_1'\beta_{11}(t) \\ G_{01}(\beta_{12}) &= \frac{1}{4}\beta_{12}'(t)B_2B_2'\beta_{12}(t) \\ G_{02}\left(\begin{bmatrix}\beta_{11} \\ \beta_{12}\end{bmatrix}\right) &= \frac{1}{2}\gamma^{-2}\beta_{12}'(t)B_1B_1'\beta_{11}(t). \end{aligned}$$

Equations (4.69) and (4.70) clearly have terminal conditions $\beta_{01}(T) = 0$ and $\beta_{02}(T) = 0$, respectively. The matrix-valued functions $F_{0i}(\cdot)$ and $G_{0i}(\cdot)$ have the important property

$$\begin{aligned} F_{0i}(\beta) &= 0 \text{ if } \beta = 0 \\ G_{0i}(\beta) &= 0 \text{ if } \beta = 0. \end{aligned}$$

As yet, (4.69) and (4.70) do not tell us anything about $\beta_{01}(t)$ and $\beta_{02}(t)$. Evaluating the linear terms gives

$$(4.56) \Rightarrow \dot{\beta}_{11}'\phi_1 = [F_{11}(\beta_{11}) + F_{12}(\beta_{12})]\phi_1 \tag{4.71}$$

$$(4.57) \Rightarrow \dot{\beta}_{12}'\phi_1 = [G_{11}(\beta_{11}) + G_{12}(\beta_{12})]\phi_1 \tag{4.72}$$

[3] For the purpose of illustration, consider the case $n = 2$ and $k = 3$. Then, for example,

$$\begin{aligned} \frac{\partial}{\partial x_1}(\phi_3) &= \frac{\partial}{\partial x_1}[x_1^3 \;\; x_2^3 \;\; x_1^2x_2 \;\; x_1x_2^2]' \\ &= [3x_1^2 \;\; 0 \;\; 2x_1x_2 \;\; x_2^2]' \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix} = \Theta_3^1\phi_2. \end{aligned}$$

where, for example, the function $F_{11}(\beta_{11})$ is given by

$$F_{11}(\beta_{11}) = \frac{1}{2}\gamma^{-2}[\Theta_2^{1'}\beta_{21}(t) \ \cdots \ \Theta_2^{n'}\beta_{21}(t)]B_1B_1'\beta_{11}$$
$$= \frac{1}{2}[\Theta_2^{1'}\beta_{22}(t) \ \cdots \ \Theta_2^{n'}\beta_{22}(t)]B_2B_2'\beta_{11} - \beta_{11}'A.$$

The terminal conditions of the differential equations are $\beta_{11}(T) = 0$ and $\beta_{12}(T) = 0$.

Equations (4.71) and (4.72) are coupled *homogeneous* ordinary differential equations in $\beta_{11}(t)$ and $\beta_{12}(t)$ with zero terminal conditions. We again have the property that

$$F_{1i}(\beta_{1i}) = 0 \text{ if } \beta_{1i} = 0, i = 1, 2$$
$$G_{1i}(\beta_{1i}) = 0 \text{ if } \beta_{1i} = 0$$

and it therefore follows that

$$\beta_{11}(t) \equiv 0$$
$$\beta_{12}(t) \equiv 0.$$

Returning to (4.69) and (4.70) gives

$$\dot\beta_{01}(t) = 0 \qquad \beta_{01}(T) = 0$$
$$\dot\beta_{02}(t) = 0 \qquad \beta_{02}(T) = 0$$

and so

$$\beta_{01}(t) \equiv 0$$
$$\beta_{02}(t) \equiv 0.$$

The power series expansions of the value functions therefore contain no constant or linear terms. Expanding the quadratic terms gives

$$(4.56) \Rightarrow \dot\beta_{21}'\phi_2 = \left[F_{21}(\beta_{21}) + F_{22}(\beta_{22}) + F_{23}\left(\begin{bmatrix}\beta_{11}\\ \beta_{22}\end{bmatrix}\right)\right] \cdot \phi_2 + \overline{F}\phi_2 \qquad (4.73)$$

$$(4.57) \Rightarrow \dot\beta_{22}'\phi_2 = \left[G_{21}(\beta_{22}) + G_{22}\left(\begin{bmatrix}\beta_{21}\\ \beta_{22}\end{bmatrix}\right)\right]\phi_2 + \overline{G}\phi_2 \qquad (4.74)$$

where $\beta_{21}(T) = 0$, $\beta_{22}(T) = 0$ with $\overline{F} > 0$, $\overline{G} > 0$. Since (4.73) and (4.74) are *inhomogeneous* equations in $\beta_{21}(t)$ and $\beta_{22}(t)$, the value functions $\overline{V}_i$ take the form

$$\overline{V}_i = \beta_{2i}'\phi_2 + \sum_{k=3}^{\infty}\beta_{ki}'\phi_k \qquad i = 1, 2. \qquad (4.75)$$

We will now show that the cubic terms in (4.75) vanish. Following that, we use an inductive argument to show that the $\overline{V}_i$'s are quadratic.

Equating the cubic terms in the Hamilton–Jacobi equations gives

$$\dot\beta_{31}'\phi_3 = [F_{31}(\beta_{31}) + F_{32}(\beta_{32})]\phi_3 \qquad (4.76)$$
$$\dot\beta_{32}'\phi_3 = [G_{31}(\beta_{31}) + G_{32}(\beta_{32})]\phi_3 \qquad (4.77)$$

where $\beta_{31}(T) = 0$ and $\beta_{32}(T) = 0$. Since these equations are *homogeneous* in $\beta_{31}(t)$ and $\beta_{32}(t)$ with zero terminal conditions, and since

$$F_{3i}(\beta) = 0 \text{ if } \beta = 0$$
$$G_{3i}(\beta) = 0 \text{ if } \beta = 0$$

we have

$$\beta_{31}(t) \equiv 0$$
$$\beta_{32}(t) \equiv 0.$$

Suppose for the purpose of induction that $\beta_{(N-1)1}(t) \equiv 0$ and $\beta_{(N-1)2}(t) \equiv 0$. Under this assumption, the terms in the Nth powers of x give

$$\dot\beta_{N1}'\phi_N = [F_{N1}(\beta_{N1}) + F_{N2}(\beta_{N2})]\phi_N \qquad (4.78)$$
$$\dot\beta_{N2}'\phi_N = [G_{N1}(\beta_{N1}) + G_{N2}(\beta_{N2})]\phi_N \qquad (4.79)$$

with $\beta_{N1}(T) = \beta_{N2}(T) = 0$. Since

$$F_{Ni}(\beta) = 0 \text{ if } \beta = 0$$
$$G_{Ni}(\beta) = 0 \text{ if } \beta = 0$$

it again follows that

$$\beta_{N1}(t) \equiv 0$$
$$\beta_{N2}(t) \equiv 0.$$

Returning to the fact that $\beta_{31}(t) \equiv 0$ and $\beta_{32}(t) \equiv 0$ it follows that $\beta_{j1}(t) \equiv 0$ and $\beta_{j2}(t) \equiv 0$ for $j = 0, 1$ and $j = 3, 4, \cdots$ yielding

$$\overline{V}_i = \beta_{2i}'(t)\phi_2$$

which can be rewritten as

$$\overline{V}_i = x'P_ix \qquad (4.80)$$

for some $n \times n$ matrixes $P_i(t)$, $i = 1, 2$. It is now immediate that $\overline{V}_i = V_i$, $i = 1, 2$ and so

$$V_i = x'P_ix. \qquad (4.81)$$

Substituting

$$\left(\frac{\partial V_i}{\partial t}\right) = x'\dot P_ix$$
$$\left(\frac{\partial V_i}{\partial x}\right) = 2x'P_i$$

into the Hamilton–Jacobi equations (4.56) and (4.57) shows that $P_1(t)$ and $P_2(t)$ satisfy the coupled Riccati equations (4.60) and (4.61) for all $t \in [0, T]$. Hence, if admissible feedback controls exist on $[0, T]$, the coupled Riccati equations have solutions $P_1(t)$ and $P_2(t)$ for all $t \in [0, T]$. Since

$$p_i = \left(\frac{\partial V_i}{\partial x}\right) = 2P_ix$$

the optimal controls are linear and given by (4.63) and (4.62) as stated. ■

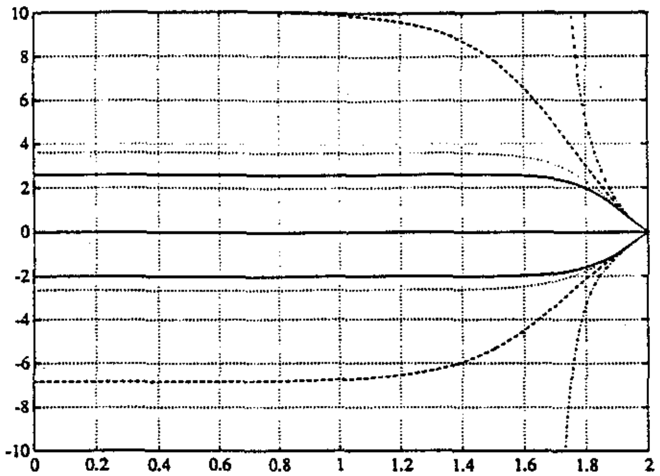

Fig. 2. Solutions to the cross-coupled Riccati equations for the various values of γ given in the table; $p_1(t)$ is always nonpositive while $p_2(t)$ is always nonnegative.

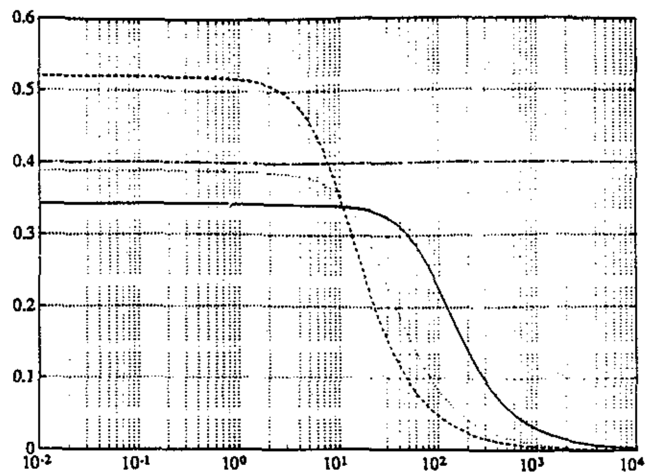

Fig. 3. Closed-loop frequency response plots: $\mathcal{H}_2$ dashed, $\mathcal{H}_\infty$ dotted, and mixed H_2/H_∞ solid.

TABLE I
γ VALUES

$\gamma = 0.50$	solid
$\gamma = 0.45$	dotted
$\gamma = 0.40$	dashed
$\gamma = 0.35$	dot-dash

V. EXAMPLE

The aim of this section is to illustrate some of the ideas of this paper with a simple example. Suppose

$$\dot{x}(t) = 2x(t) + w(t) + 3u(t) \tag{5.82}$$

and that

$$z(t) = \begin{bmatrix} 3x(t) \\ u(t) \end{bmatrix} \tag{5.83}$$

then substituting into (2.13) and (2.14) gives

$$-\dot{p}_1(t) = 4p_1(t) - 9 - \gamma^{-2}p_1^2(t) - 9p_2^2(t) - 18p_1(t)p_2(t) \tag{5.84}$$

and

$$-\dot{p}_2(t) = 4p_2(t) + 9 - 9p_2^2(t) - 2\gamma^{-2}p_1(t)p_2(t) \tag{5.85}$$

respectively, as the cross-coupled H_2/H_∞ Riccati equations. We solve these equations from the terminal conditions $p_1(2) = 0$ and $p_2(2) = 0$ using a standard Runge–Kutta integration procedure. The solution to the algebraic Riccati equations may be obtained by allowing the differential Riccati equation solutions to evolve long enough.

Fig. 2 shows the time evolution of $p_1(t)$ and $p_2(t)$ for the various values of γ given in Table 1. When γ is set at 0.35, we see that there is a finite escape time at $t = 1.68s$. In the case of the $\gamma = 0.4$ solution, (5.84) and (5.85) were solved for 8 s to obtain solutions to the algebraic versions of these equations. We obtained $p_1 = -6.8475$ and $p_2 = 10.0544$. It is also easy to check that these solutions have the stability properties that we established in Theorem 3.1.

Fig. 3 gives the frequency response plots for the closed-loop system corresponding to $u = kx$, where k is either the optimal $\mathcal{H}_2$ state-feedback law, the pure $\mathcal{H}_\infty$ state-feedback law corresponding to $\gamma = 0.4$, or the mixed H_2/H_∞ feedback law also corresponding to $\gamma = 0.4$. In the case of the $\mathcal{H}_2$ controller, the $\|\cdot\|_\infty$ norm of the closed loop is in excess of 0.5, while the other two are below their upper bound of 0.4.

VI. CONCLUSION

We have shown how to solve a mixed H_2/H_∞ problem by formulating it as a two-player Nash game. The necessary and sufficient conditions for the existence of a solution to the mixed H_2/H_∞ problem are given in terms of the existence of solutions to a pair of cross-coupled Riccati differential equations in the finite horizon case, and a pair of cross coupled algebraic equations in the infinite horizon case. In the infinite horizon case we give a full stability analysis. If the controller strategy sets are expanded to include memoryless nonlinear controls that are analytic in the state, the necessary and sufficient conditions for the existence of a solution are unchanged, as are the control laws themselves. We have also established a link between H_2, H_∞ and mixed H_2/H_∞ theories by generating each as a special case of another two-player LQ Nash game.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Control—Linear Quadratic Methods*. Englewood Cliffs, NJ: Prentice-Hall International, 1989.
[2] D. Arov and M. G. Krein, "On the evaluation of entropy functionals and their minima in generalized extension problems," *Acta Scienta Math.*, vol. 45, pp. 33–50, 1983 (in Russian).
[3] T. Basar, "On the uniqueness of the Nash solution in linear-quadratic differential games," *Int. Journal of Game Theory*, vol. 5, pp. 65–90, 1977.
[4] P. Bernhard, "Linear-quadratic, two-person, zero-sum differential games: necessary and sufficient conditions," *JOTA*, vol. 27, pp. 51–69, 1979.
[5] T. Basar and G. J. Olsder, Dynamic Non Co-operative Game Theory. New York: Academic Press, 1982.

[6] D. S. Bernstein and W. M. Haddad, "LQG control with an H_∞ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. AC-34, no. 3, pp. 293–305, 1989.

[7] A. E. Bryson and Y-C. Ho, *Applied Optimal Contr.* MA: Ginn and Company, 1969.

[8] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.* New York: McGraw-Hill, 1955.

[9] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. Francis, "State-space solutions to standard H_2 and H_∞ problems," *IEEE Trans. Auto. Contr.*, vol. AC-34, pp. 831–847, 1989.

[10] J. C. Doyle, K. Zhou, and B. Bodenheimer, "Optimal control with mixed H_2 and H_∞ performance objectives," *Proc. 1989 Amer. Contr. Conf.*, Pittsburgh, PA, 1989, pp. 2065–2070.

[11] H. Dym, "J-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation," *Regional Conf. Series in Mathematics, Am. Math. Soc.*, vol. 71, 1989.

[12] H. Dym and I. Gohberg, "A maximum entropy principle for contractive interpolants," *J. Funct. Analysis*, vol. 65, pp. 83–125, 1986.

[13] K. Glover and D. Mustafa, "Derivation of the maximum entropy H_∞-controller and a state-space formula for its entropy," *Int. J. Contr.*, vol. 50, pp. 899–916, 1989.

[14] I. M. Jaimoukha and D. J. N. Limebeer, "State-space algorithm for the solution of the 2-block super-optimal distance problem," *SIAM J. Contr. Optim.*, vol. 31, no. 5, pp. 1115–1134, 1993.

[15] P. P. Khargonekar and M. A. Rotea, "Mixed H_2/H_∞ control: A convex optimization approach," *IEEE Trans. Auto. Control*, vol. AC-36, no. 7, pp. 824–837, 1991.

[16] H. Kwakernaak, "A polynomial approach to minimax frequency domain optimization of multivariable feedback systems," *Int. J. Contr.*, vol. 44, no. 1, pp. 117–156, 1986.

[17] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems.* J. Wiley&Son, 1972.

[18] D. J. N. Limebeer, B. D. O. Anderson, P. P. Khargonekar, and M. Green, "A game theoretic approach to H_∞ control for time-varying systems," *SIAM J. Contr. Optim.*, vol. 30, no. 2, pp. 262–283, 1992.

[19] D. J. N. Limebeer, G. D. Halikias, and K. Glover, "State-space algorithm for the computation of super-optimal matrix interpolating functions," *Int. J. Contr.*, vol. 50, no. 6, pp. 2431–2466, 1989.

[20] D. J. N. Limebeer and Y. S. Hung, "An analysis of the pole-zero cancellations in H_∞-optimal control problems of the first kind," *SIAM J. Contr. Optim.*, vol. 25, pp. 1457–1493, 1987.

[21] D. L. Lukes, "Equilibrium feedback control in linear games with quadratic costs," *SIAM J. Contr. Optim.*, vol. 9, no. 2, pp. 234–252, 1971.

[22] D. Mustafa, "Relations between maximum entropy H_∞ control and combined H_∞/LQG control," *Syst. Contr. Lett.*, vol. 12, pp. 193–203, 1989.

[23] D. Mustafa and K. Glover, *Minimum Entropy H_∞ Control.* Heidelberg: Springer-Verlag, 1990.

[24] D. Mustafa, K. Glover, and D. J. N. Limebeer, "Solutions to the H_∞ general distance problem which minimize an entropy integral," *Automatica*, vol. 27, pp. 193–199, 1991.

[25] A. W. Starr and Y. C. Ho, "Nonzero-sum differential games," *J. Optim. Theory Appl.*, vol. 3, no. 3, pp. 184–206, 1967.

[26] M. C. Tsai, D. W. Gu, and I. Postlethwaite, "A state space approach to super-optimal H_∞ control problems," *IEEE Trans. Auto. Contr.*, vol. AC-33, no. 9, 1988.

[27] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach.* 3rd ed. New York: Springer-Verlag, 1985.

[28] H. H. Yeh, S. S. Banda, and B. C. Chang, "Necessary and sufficient conditions for mixed H_2 and H_∞ optimal control," *IEEE Trans. Auto. Control*, vol. AC-37, no. 3, pp. 355–358, 1992.

[29] N. J. Young, "The Nevanlinna-Pick problem for matrix-valued functions," *J. Operator Theory*, vol. 15, pp. 239–265, 1986.

[30] K. Zhou, J. C. Doyle, K. Glover, and B. Bodenheimer, "Mixed H_2 and H_∞ control," *Proc. 1990 Amer. Contr. Conf.*, San Diego: 1990, pp. 2502–2507.

David J. N. Limebeer (M'82–SM'87–F'92) was born in Johannesburg, South Africa. He received the B.Sc.(Eng) degree in electrical engineering from the University of the Witwatersrand in 1974, and the M.Sc.(Eng) and Ph.D. degrees in 1977 and 1980, respectively, from the University of Natal in South Africa.

He received the D.Sc.(Eng) degree from London University in 1992. In 1984 he was appointed Lecturer in the Department of Electrical Engineering, Imperial College, London. In 1989 he was promoted to Reader and in 1993 to Professor. He has held summer visiting positions at the University of Southern California, the Australian National University, and Stanford University.

Dr. Limebeer is currently an associate editor of *Automatica*, *Systems and Control Letters*, and the *International Journal of Robust and Nonlinear Control*. He also has served on the British Science and Engineering Research Council's control and instrumentation subcommittee. He was joint recipient of the AACC O. Hugo Schuck award for the best paper presented at the 1983 ACC. His research interests include multivariable system theory, computer aided control system design, system identification, power system stability, and the control of tokamaks.

Brian D. O. Anderson (S'62–M'66–SM'74–F'75) received his education at the University of Sydney and Stanford University.

He has worked in industry and served as an assistant professor in the electrical engineering department at Stanford. He was at the University of Newcastle, Australia from 1967 till 1981 and is now professor of Systems Engineering at the Australian National University. His interests are in control and signal processing.

Dr. Anderson is a Fellow of numerous societies and institutions and holds a doctorate (honoris causa) from the Université Catholique de Louvain, Belgium. He is currently serving a term as President of the International Federation of Automatic Control from 1990 to 1993.

Britta Hendel was born in Akwatia, Ghana, in March 1966. She received the MEng. degree in electrical and electronic engineering from Imperial College, London, in 1989. She completed the Ph.D. degree in the use of game theory in robust control and filtering at the Interdisciplinary Research Centre for Process Systems Engineering, Imperial College, in May 1993. Her research interests lie in linear systems theory, minimax problems and H_∞ control and estimation.

Dr. Hendel was awarded the Sir Leon Bagrit Fellowship in 1989. She is an Associate Member of the Institute of Electrical Engineers (GB).