On the Gain Margin Improvement Using Dynamic Compensation Based on Generalized Sampled-Data Hold Functions

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***Abstract*— This paper shows that the use of dynamic compensation based on generalized sampled-data hold functions (GSHF) can arbitrarily improve the gain margin for continuous-time nonminimum phase linear systems. The GSHF compensator is a particular periodic digital controller much simpler than that used in [4]. The effect of sampling period on the gain margin is analyzed. Furthermore, it is proved that under a mild condition, the gain margin improvement can be achieved without forcing the sampling period small. An important advantage of periodic compensation over LTI compensation is found to be the capability of reducing conflict between gain margin and sensitivity, which always exists for a nonminimum phase plant as far as LTI compensation is concerned [14].**

I. Introduction

Generally speaking, there are two common kinds of periodic digital controllers for continuous-time linear time-invariant (LTI) plants. A controller of one kind (i.e., a dynamical system with periodically time varying elements) is composed of a sampler, a periodic discrete-time dynamic component, and a zero-order hold [7] while a controller of the other kind is composed of a sampler, an LTI compensator, and a periodic control gain known as a generalized sampled-data hold function (GSHF) [5]. We might call the former a conventional periodic digital controller and the latter a GSHF dynamic compensator. Their common feature is the hybrid nature of a continuous-time component and a discrete-time component. The most significant difference between the two configurations is that periodicity of a conventional periodic controller occurs in the dynamic component while periodicity of a GSHF compensator occurs only in the GSHF gain with any dynamic components time-invariant. Evidently, this difference implies that a GSHF dynamic compensator may be more easily implemented in practice than a conventional periodic digital controller.

One of the known advantages of periodic controllers over LTI controllers is their capability of improving arbitrarily the gain margin for an LTI plant in certain cases; see e.g., [7], [9]. Recently, Francis and Georgiou [4] showed that for a discrete-time LTI plant, LTI dynamic pre-compensation with decimation of the plant output (which is equivalent to use of a particular form of linear periodic dynamic compensator) can arbitrarily place nonzero zeros of the resulting system. Using this key idea, they generalized the gain margin result in [7] to the multivariable case. More specifically, the gain margin can be arbitrarily assigned likewise for a multivariable continuous-time LTI plant by way of discretizing the plant with a sufficiently small sampling time and suitable choice of digital periodic controller. According to their method, the design procedure consists of i) discretizing the plant, ii) designing an LTI dynamic forward-compensator for the discretized plant with output decimation (this positions zeros), and iii) designing an LTI feedback compensator (this positions poles). Altogether, this actually leads to a conventional periodic digital controller. From this procedure, it is not hard to observe that the order of such a controller may be very high because of the introduction of precompensation. Other disadvantages, as mentioned in [4], are that the sampling time may have to be very small to permit an increase in the gain margin and the feedback system may become sensitive to variation in parameters other than the gain.

As is well known, the idea of a GSHF function can be powerfully used for many linear control system problems; see [2], [5]. Particularly in [5], Kabamba exhibited some advantages of GSHF nondynamic compensation over LTI compensation. Naturally, the use of GSHF dynamic compensation might be expected to achieve even more profitable objectives. One of the main purposes of this paper is to reveal one of the objectives, the gain margin improvement, and to examine the effect of sampling time on the gain margin.

Even though periodic compensation can arbitrarily improve the gain margin, this compensation cannot improve the minimal sensitivity for an LTI plant, a fact which has been shown by Khargonekar *et al.* [7] as well as by Feintuch and Francis [3]. Recently, in [14], we revealed that there always exists conflict between gain margin and sensitivity for a nonminimum phase LTI plant using LTI compensation. For instance, gain margin maximization will lead to an arbitrarily large sensitivity. Thus, an interesting question naturally arises as to whether periodic compensation can overcome or reduce this compromise associated with LTI compensation. Our partial answer to this question in this paper shows that the use of periodic compensation cannot only arbitrarily improve the gain margin but also keep the sensitivity bounded at the same time.

In the next section, a GSHF dynamic compensator is formulated and a stabilizability condition for a continuous-time LTI plant with the compensator is established. In Section III, we derive an explicit formula for the maximal achievable gain margin of a single-input/single-output (SISO) plant by a GSHF compensator, analyze the effect of sampling time on the maximal gain margin, and show that an arbitrarily large gain margin can be achieved by a GSHF compensator with a sufficiently small sampling period. Section IV discusses the multivariable case. We exhibit the capability of reducing conflict between gain margin and sensitivity by using periodic compensation in Section V. An example is given in Section VI.

II. Preliminary Results

Consider a continuous-time LTI plant $P(s)$ with a minimal state-space model

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control, $y(t) \in \mathbf{R}^r$ is the output, and A, B, C, D are real matrices of appropriate dimensions.

The GSHF dynamic compensator is defined to be the following control law composed of an LTI compensator and a GSHF feedback

$$z_{k+1} = A_c z_k + B_c y(kT) \tag{2.3}$$

$$v_k = C_c z_k + D_c y(kT) \tag{2.4}$$

$$u(t) = F(t)v_k, \quad \text{for } t \in [kT, (k+1)T), \quad k = 0, 1, 2, \cdots \tag{2.5}$$

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where $T > 0$ is the sampling period, A_c, B_c, C_c, and D_c are constant matrices of appropriate dimensions, and $F(t)$ is a T-periodic integrable and bounded hold function matrix of an appropriate dimension.

The following equation

$$\int_0^T \exp[A(T-t)]BF(t)\,dt = G \tag{2.6}$$

for the unknown $F(t)$ with G a given constant matrix plays an important role in the design of the GSHF control law. The properties with respect to this equation are summarized in the following result.

Lemma 2.1: Let (A, B) be controllable, G be given and

$$W(A, B, T) = \int_0^T \exp[A(T-t)]BB^\tau \exp[A^\tau(T-t)]\,dt. \tag{2.7}$$

Then

1)
$$F_0(t) = B^\tau \exp[A^\tau(T-t)]W^{-1}(A, B, T)G \tag{2.8}$$
is the unique solution of (2.6) which minimizes the trace of $\int_0^T F^\tau(t)F(t)\,dt$;
2) for almost all $T > 0$, there exists a piecewise constant solution of (2.6) taking at most n different values in the interval $[0, T]$.

Proof: The proof is omitted. □

Applying the GSHF control law (2.5) to the system (2.1)–(2.2) and sampling the continuous-time state and output, we obtain the following discrete-time system from v_k to $y(kT)$

$$x((k+1)T) = \exp(AT)x(kT) + Gv_k \tag{2.9}$$

$$y(kT) = Cx(k) + DG_0 v_k \tag{2.10}$$

with $G_0 = F(0)$, $F(\cdot)$ and G related as in (2.6), and the associated transfer function

$$\overline{P}(z) = DG_0 + C(zI - \exp(AT))^{-1}G. \tag{2.11}$$

Proposition 2.1: There exists a GSHF dynamic compensator (2.3)–(2.5) stabilizing the continuous-time plant (2.1)–(2.2) if and only if there exist constant matrices G_0 and G such that the induced discrete-time system (2.9)–(2.10) is stabilized by (2.3)–(2.4).

Proof: Since the "only if" part is obvious, it suffices to show that if the closed-loop system composed of (2.9)–(2.10) and (2.3)–(2.4) is stable for some pair (G, G_0), then there exists a GSHF control law (2.5) such that the states of the closed-loop system (2.1)–(2.5) satisfy

$$\|x(t)\| \le \|x(0)\|k_1 e^{-\lambda_1 t}, \quad \forall t \ge 0 \tag{2.12}$$

$$\|z_k\| \le \|z_0\|k_2 e^{-\lambda_2 k}, \quad \forall k \ge 0 \tag{2.13}$$

for some positive constants k_1, k_2, λ_1, and λ_2. To do this, first note from Lemma 2.1 that by solving

$$\int_{t_1}^T \exp[A(T-t)]BF(t)\,dt = G - \int_0^{t_1} \exp[A(T-t)]BG_0\,dt$$

for any fixed $t_1 \in [0, T)$, there are an infinite number of T-periodic integrable and bounded function matrices $F(t)$ satisfying

$$F(t) = G_0, \quad t \in [0, t_1] \tag{2.14}$$

$$F(t+T) = F(t), \quad t \ge 0 \tag{2.15}$$

$$\int_0^T \exp[A(T-t)]BF(t)\,dt = G. \tag{2.16}$$

In particular, we can generically (i.e., for almost all $T > 0$) choose a piecewise constant $F(t)$ satisfying the above conditions and taking at most $n+1$ different values. In this way, it is easily seen that the states $x(kT)$ and z_k of (2.1)–(2.5) are precisely the states of (2.9)–(2.10) and (2.3)–(2.4). Thus by the hypothesis, there exist $k_2 > 0$, $\lambda_2 > 0$ such that (2.13) holds and $x(kT)$ converges exponentially. Further, it can easily be shown that

$$x(t) = M_1(t)x(kT) + M_2(t)v_k, \quad \text{for } t \in [kT, (k+1)T), \quad k = 0, 1, 2, \cdots$$

where $M_1(t)$ and $M_2(t)$ are T-periodic, and

$$M_1(t) = \exp(At), \qquad M_2(t) = \int_0^t \exp(A(t-\tau))BF(\tau)\,d\tau, \quad t \in [0, T).$$

Since $M_1(t)$ and $M_2(t)$ are bounded, and exponential convergence of $x(kT)$ and z_k implies exponential convergence of v_k, it follows that $x(t)$ converges exponentially too, namely, (2.12) holds for some positive constants k_1, λ_1. □

Remark 2.1: Note that for almost all $T > 0$, there exist almost all constant matrices G_0 and G such that the induced system (2.9)–(2.10) is controllable and observable.

Definition 2.1: Let P denote a given multi-input/multi-output (MIMO) LTI plant (continuous time or discrete time). Define the maximal achievable gain margin $\mathbf{K}$ of P with respect to LTI compensation as

$$\mathbf{K} \triangleq \sup\{b/a\colon 0 < a < 1 < b \text{ and there exists an LTI controller stabilizing } kP \text{ for all } k \in [a, b]\}$$

and for a given $T > 0$ define the maximal achievable gain margin $\mathbf{K}_T$ of $P(s)$ with respect to GSHF compensation as

$$\mathbf{K}_T \triangleq \sup\{b/a\colon 0 < a < 1 < b \text{ and there exists a GSHF dynamic controller stabilizing } kP(s) \text{ for all } k \in [a, b]\}.$$

Let us now introduce some notation: $\mathcal{D}$ and $\overline{\mathcal{D}}$ denote the open and closed unit disks, respectively. $\mathbb{C}^+$ and $\overline{\mathbb{C}}^+$ denote the open and closed right-half planes, respectively. $\{z_1, z_2, \cdots, z_{N_1}\}$ denotes the set of unstable zeros (including any infinite zeros) of (2.1)–(2.2) (multiplicities being included) and $\{p_1, p_2, \cdots, p_{N_2}\}$ the set of unstable poles of (2.1)–(2.2) (multiplicities being included). $\mathbf{K}_c$ denote the maximal achievable gain margin of the plant (2.1)–(2.2) with respect to LTI compensation. $\mathbf{K}(G, G_0)$ denotes the maximal achievable gain margin of the discrete-time plant (2.9)–(2.10) with respect to LTI compensation for given (G, G_0). Then the following result is immediately obtained as a consequence of Proposition 2.1.

Corollary 2.1: $\mathbf{K}_T = \max\{\mathbf{K}(G, G_0)\colon G \text{ and } G_0 \text{ are constant matrices}\}$.

The formula for the maximal achievable gain margin of a discrete-time plant can easily be derived by using the same argument as in [6].

Proposition 2.2: Let $P(z)$ be a SISO LTI discrete-time plant with unstable zeros $z_1, z_2, \cdots, z_{N_1}$ and unstable poles $p_1, p_2, \cdots, p_{N_2}$. Assume that $P(z)$ has no unstable pole-zero cancellations. Then the maximal achievable gain margin of $P(z)$ with respect to LTI compensation is equal to $((1+\alpha)/(1-\alpha))^2$ if $\alpha < 1$ and ∞ if $\alpha \geq 1$, where

$$\alpha \triangleq \max\{\gamma > 0;\ \text{there exists an analytic function } f(s): \overline{\mathcal{D}} \to \mathcal{D} \text{ such that } f(a_i) = \gamma b_i,\quad i = 1, 2, \cdots, N\} \tag{2.17}$$

and

$$(a_i, b_i) \triangleq \begin{cases} (\varphi(z_i), 1), & i = 1, 2, \cdots, N_1 \\ (\varphi(p_i), 0), & i = N_1 + 1, N_1 + 2, \cdots, N \end{cases}$$

with $N = N_1 + N_2$ and $\varphi(s) = 1/s$.

Remark 2.2: If z_i and p_j are all simple and lie in $\overline{\mathcal{D}}^c$ (for a discrete-time plant), then the formula for α can be given by the Nevanlinna–Pick interpolation theorem

$$\alpha = \max\left\{\gamma > 0;\ \left[\frac{1-\gamma^2 b_i \bar{b}_j}{1 - a_i \bar{a}_j}\right]_{N\times N} \geq 0\right\}. \tag{2.18}$$

In view of complexity of its closed-form formula, it would be desirable to get some useful bounds for α. An upper bound for α has been obtained in [6]. A lower bound for α is given below, which provides some insight into the effect of the distribution of unstable zeros and unstable poles on α.

Lemma 2.2: Let α be defined as in (2.18) with $|a_i| < 1$, $i = 1, 2, \cdots, N$. Then

$$\begin{aligned} \alpha &\geq f(a_1, \cdots, a_N) \\ &\triangleq \frac{\prod_{1\leq i<j\leq N} |a_j - a_i|}{\sqrt{\left[1 - \left(\prod_{i=1}^{N}|a_i|\right)^{2N}\right]\left[\sum_{i=1}^{N_1}\frac{1}{1-|a_i|^2}\right]}} \cdot \left(\frac{N-1}{\sum_{i=1}^{N}\frac{1}{1-|a_i|^2}}\right)^{N-1/2}. \end{aligned}$$

Proof: The proof is omitted. □

III. SISO CASE

In this section, we only consider the case where $P(s)$ is a SISO strictly proper plant. The discussion of the bicausal case (i.e., $P(\infty)$ is invertible) will be included in the next section.

Theorem 3.1: Suppose that unstable poles p_i of $P(s)$ are simple and in $\mathbb{C}^+$. Let $\mathbf{K}_T$ be defined as in Section II. Then for almost all sampling periods $T > 0$

$$\mathbf{K}_T = \left(\frac{1+\alpha_T}{1-\alpha_T}\right)^2 \tag{3.1}$$

where

$$\alpha_T = \sqrt{1 - [1\cdots 1]L_T^{-1}[1\cdots 1]^{\tau}} \quad \text{and} \quad \cdot\, L_T \triangleq \left[\frac{1}{1-\exp[-(p_i+\bar{p}_j)T]}\right]_{N_2\times N_2}.$$

Proof: Note that G_0 has no effect on $\mathbf{K}(G, G_0)$ because $D = 0$. From Proposition 2.2, it is not hard to see that to maximize $\mathbf{K}(G, G_0)$, G should be chosen such that the system (2.9)–(2.10) has no unstable finite zeros and has an infinite zero with the least multiplicity. Since $(\exp(AT), C)$ is observable for almost all $T > 0$ and (2.9)–(2.10) is SISO if G is a vector, finite zeros of (2.9)–(2.10) can be arbitrarily placed by suitable choice of the vector G, though the poles $\exp(p_i T)$ $(i = 1, \cdots, N_2)$ cannot be changed at all. In this way, there exists a vector G for which (2.9)–(2.10) has no unstable finite zeros and the zero at infinity is simple (i.e., $CG \neq 0$). We denote such a G by G'. Then clearly, $G = G'$ must maximize $\mathbf{K}(G, G_0)$. Now using Corollary 2.1 and Proposition 2.2, we can immediately write down the formula for $\mathbf{K}_T$ as

$$\mathbf{K}_T = \mathbf{K}(G', G_0) = \left(\frac{1+\alpha_T}{1-\alpha_T}\right)^2$$

where

$$\alpha_T = \max\left\{\gamma > 0;\ \begin{bmatrix} 1-\gamma^2 & e \\ e^{\tau} & L_T \end{bmatrix} \geq 0\right\} \quad \text{and} \quad e = [1\cdots 1].$$

But, it is easy to verify that $\alpha_T = \sqrt{1 - [1\cdots 1]L_T^{-1}[1\cdots 1]^T}$; hence, Theorem 3.1 is proved. □

Theorem 3.2: With the same assumptions and notation as in Theorem 3.1, there holds

1) $\mathbf{K}_T$ is strictly decreasing with respect to T.
2)

$$\begin{aligned} &\prod_{1\leq i\leq N_2} \exp(-p_i T) \prod_{1\leq i<j\leq N_2} |\exp(-p_j T) - \exp(-p_i T)| \cdot \left[\frac{N_2}{1 + \sum_{i=1}^{N_2}\frac{1}{1-\exp(-2\mathrm{Re}\,p_i T)}}\right]^{N_2/2} \\ &\leq \alpha_T \leq \sqrt{1 - N_2^2\left(\sum_{1\leq i,j\leq N_2}\frac{1}{1-\exp[-(p_i+\bar{p}_j)T]}\right)^{-1}} \end{aligned}$$

3) $\lim_{T\to 0}\mathbf{K}_T = \infty$ and $\lim_{T\to\infty}\mathbf{K}_T = 1$.

Proof:

1) In fact, it is known from [6] or [14] that the original definition for α_T is

$$\alpha_T = \max\{\gamma > 0;\ \text{there exists an analytic function } f(s): \overline{\mathcal{D}}\to\mathcal{D} \text{ such that } f(0) = \gamma \text{ and } f(\exp(-p_i T)) = 0,\ i = 1, 2, \cdots, N_2\}.$$

Now we claim that $\alpha_{T_1} \geq \alpha_{T_2}$ for $T_2 > T_1 > 0$. In fact, for any fixed $\gamma < \alpha_{T_2}$, there exists an analytic function $f_{T_2}(s): \overline{\mathcal{D}} \to \mathcal{D}$ such that

$$f_{T_2}(0) = \gamma \quad \text{and} \quad f_{T_2}(\exp(-p_i T_2)) = 0, \quad i = 1, 2, \cdots, N_2.$$

Construct

$$f_{T_1}(s) = \left\{\prod_{i=1}^{N_2} f_{T_2}(\exp[(T_1 - T_2)p_i]s)\right\}^{1/N_2}.$$

Then it is routine to verify that $f_{T_1}(s)$ is an analytic function from $\overline{\mathcal{D}}$ to $\mathcal{D}$ such that

$$f_{T_1}(0) = \gamma \quad \text{and} \quad f_{T_1}(\exp(-p_i T_1)) = 0, \quad i = 1, 2, \cdots, N_2.$$

As a consequence, $\alpha_{T_1} \geq \alpha_{T_2}$. From Theorem 3.1, it is easily seen that $(\alpha_T)^2$ can be regarded as an analytic complex function in the finite plane. Thus, $(\alpha_T)^2$ cannot be identical by constant on any real interval of T otherwise it is constant in the whole plane. But, $(\alpha_T)^2$ is nonincreasing; hence, it must be strictly decreasing, which is evidently equivalent to the condition that $\mathbf{K}_T$ is strictly decreasing with respect to T.

2) Obviously, the inequality on the left-hand side is simply a direct application of Lemma 2.2 with $N_1 = 1$, $N = 1 + N_2$, $a_1 = 0$, and $a_i = \exp(-p_{i-1}T)$, $i = 2, 3, \cdots, N$. On the other hand, it can be easily proved by the Schwarz inequality that if Q is an $n \times n$ positive definite matrix, then

$$x^H Q^{-1} x \geq (x^H Q x)^{-1}, \quad \forall x \in \mathbb{C}^n : \|x\|_2 = 1.$$

Using this fact, the right-hand inequality can be readily concluded.

3) Since

$$0 \leq \alpha_T \leq \sqrt{1 - N_2^2 \left(\sum_{1 \leq i,j \leq N_2} \frac{1}{1 - \exp[-(p_i + \overline{p}_j)T]} \right)^{-1}}$$

and the right-hand expression approaches zero as T goes to infinity, it follows that

$$\lim_{T \to \infty} \alpha_T = 0$$

leading to

$$\lim_{T \to \infty} \mathbf{K}_T = 1.$$

To see $\lim_{T \to 0} \mathbf{K}_T = \infty$, let us now prove a stronger result, that is, $\lim_{T \to 0} L^{-1} = 0$. In fact

$$\lim_{T \to 0} TL = \lim_{T \to 0} \left[\frac{T}{1 - \exp[-(p_i + \overline{p}_j)T]} \right]_{N_2 \times N_2} = \left[\frac{1}{p_i + \overline{p}_j} \right]_{N_2 \times N_2} > 0.$$

Consequently

$$\lim_{T \to 0} L^{-1} = \lim_{T \to 0} T(TL)^{-1} = 0$$

which implies that $\lim_{T \to 0} \mathbf{K}_T = \infty$. □

Corollary 3.1: Let $P_T(z)$ be a transfer function obtained by discretizing a continuous-time LTI plant. Assume that $P_T(z)$ has simple unstable poles $\exp(p_i T)$ $(i = 1, \cdots, N)$ but no finite unstable zeros, where T is the sampling time. Further, assume that $P_T(z)$ has an infinite zero of multiplicity one. Then, the maximal achievable gain margin of $P_T(z)$ goes to infinity as T tends to zero.

Remark 3.1: In Corollary 3.1, if $P_T(z)$ has no infinite unstable zeros either, applying Corollary 3.1 to $z^{-1} P_T(z)$ we conclude that for $P_T(z)$, there always exists a strictly proper LTI controller achieving any prescribed gain margin as T is sufficiently small.

Remark 3.2: From [6], it is known that if the plant $P(s)$ has no finite unstable zero, then $\mathbf{K}_c = \infty$. Thus, in this case, GSHF dynamic compensation offers no advantage over LTI compensation in respect of gain margin improvement.

IV. MIMO Case and SISO Bicausal Case

In this section, we turn to a discussion of the multivariable case. But first, let us state a result on the gain margin of an MIMO LTI plant with an LTI compensator (we assume that the definition of gain margin in the SISO case is still applicable to the MIMO case, see [14]).

Lemma 4.1: Let $P(z)$ be an MIMO LTI discrete-time plant with distinct unstable poles $p_1, \cdots, p_N$. Assume that $P(z)$ has a minimal realization (A, B, C, D) with $A \in R^{n \times n}$. If

$$\operatorname{rank} D \geq n - \min_i \operatorname{rank}(A - p_i I) \tag{4.1}$$

$$\operatorname{rank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \geq 2n - \min_i \operatorname{rank}(A - p_i I), \quad \forall z \in \mathcal{D}^c \backslash \{p_1, p_2, \cdots, p_N\} \tag{4.2}$$

then there is no upper bound on the gain margin of $P(z)$ using LTI compensation.

Proof: Let $\mathbf{R}$ denote the ring of proper rational functions which are stable in the discrete-time sense. Then $P(z)$ has a Smith–McMillan form over $\mathbf{R}$ as

$$\begin{bmatrix} \operatorname{diag}(n_1/d_1, n_2/d_2, \cdots, n_l/d_l) & 0 \\ 0 & 0 \end{bmatrix}$$

where $l = \operatorname{rank} P(z)$, 0 represents the zero matrix of appropriate size, n_i divides n_{i+1}, d_{i+1} divides d_i, and (n_i, d_i) are coprime over $\mathbf{R}$. It is clear that the maximal gain margin of $P(z)$ with respect to LTI compensation is greater than or equal to $\max(k_1, \cdots, k_l)$, where k_i represents the maximal gain margin of n_i/d_i with respect to LTI compensation. Thus, to prove Lemma 4.1, it suffices to show that n_i/d_i $(i = 1, \cdots, l)$ has either no zeros in $\mathcal{D}^c$ or no poles in $\mathcal{D}^c$ since this implies that k_i is unbounded. To this end, to secure a contradiction suppose that there exist $1 \leq j \leq l$, $1 \leq \gamma \leq N$, and $z_0 \in \mathcal{D}^c$, $z_0 \neq p_\gamma$ such that

$$n_j(z_0) = 0 \quad \text{and} \quad d_j(p_\gamma) = 0.$$

Obviously, the former implies that $\operatorname{rank} P(z_0) \leq j - 1$, which is equivalent to

$$\operatorname{rank} D \leq j - 1, \quad \text{if } z_0 \text{ is infinite} \tag{4.3}$$

$$\operatorname{rank} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} \leq n + j - 1, \quad \text{if } z_0 \text{ is finite.} \tag{4.4}$$

On the other hand, since

$$P(z) = [(C - DK)(zI - A + BK)^{-1} B + D] \cdot [I - K(zI - A + BK)^{-1} B]^{-1}$$

is a coprime factorization over $\mathbf{R}$ provided that K is an arbitrary matrix such that all the eigenvalues of $A - BK$ lie in $\mathcal{D}$, the nonunit invariant factors (over $\mathbf{R}$) of $[I - K(zI - A + BK)^{-1} B]$ and $\operatorname{diag}(d_1, d_2, \cdots, d_l)$ are the same; see [11]. As a consequence

$$\operatorname{rank} [I - K(p_\gamma I - A + BK)^{-1} B] \leq n - j$$

or equivalently

$$\operatorname{rank} \begin{bmatrix} I & K \\ B & p_\gamma I - A + BK \end{bmatrix} \leq m + n - j$$

from which it follows that

$$\operatorname{rank}(A - p_\gamma I) \leq n - j.$$

This together with (4.3) or (4.4) readily leads to a contradiction to the lemma condition (4.1) or (4.2). Therefore, n_i/d_i has either no zeros in $\mathcal{D}^c$ or no poles in $\mathcal{D}^c$ for all $i = 1, \cdots, l$. □

Remark 4.1: From the above proof, it is easy to see that condition (4.1) is exclusively used to prevent the diagonal elements of the

Smith–McMillan form in $\mathbf{R}$ of $P(z)$ from having both unstable poles and a zero at infinity while condition (4.2) is used to prevent a diagonal element from having both unstable poles and finite unstable zeros.

Remark 4.2: The continuous-time version of Lemma 4.1 can be immediately established without any other modification than the replacement of $\mathcal{D}^c$ by $\overline{\mathbb{C}}^+$ in Lemma 4.1. It should be pointed out that the result thus obtained contains Theorem 3.1 in [6] since the condition of distinct right-half plane poles and no blocking zeros implies the continuous-time version of the conditions (4.1)–(4.2).

The following lemma is instrumental in the proof of Theorem 4.1.

Lemma 4.2 [13]: Suppose $\{D(s), M(s), N(s)\}$ are $p \times q$, $p \times u$, and $p \times v$ polynomial matrices, with $\operatorname{rank} D(s) > p - u - \epsilon$ where ϵ is some nonnegative integer. Then for almost all $v \times u$ constant matrices F

$$\operatorname{rank}[D(s) \;\; M(s) + N(s)F] \geq p - \epsilon, \quad \forall s \in \mathbb{C}$$

if and only if both of the following conditions hold:

1) $\operatorname{rank}[D(s) \;\; M(s) \;\; N(s)] \geq p - \epsilon$, $\forall s \in \mathbb{C}$
2) $\operatorname{rank} D(s) \geq p - u - \epsilon$, $\forall s \in \mathbb{C}$.

Theorem 4.1: Let $P(s)$ be an MIMO LTI continuous-time plant with distinct unstable poles $p_1, \cdots, p_{N_2}$. Let $\mathbf{K}_T$ be defined as in Section II. Assume that $P(s)$ has a minimal realization (2.1)–(2.2). If

$$\operatorname{rank} D \geq n - \min_i \operatorname{rank}(A - p_i I) \tag{4.5}$$

then $\mathbf{K}_T$ is equal to infinity for almost all sampling periods $T > 0$.

Proof: Let $T > 0$ be any fixed sampling period with $T \neq (2k\pi/Im(p_i - p_j))$ for all integers k whenever $Re(p_i - p_j) = 0$. Then, $(\exp(AT), C)$ is observable. It will now be shown that $\mathbf{K}_T = \infty$. But in view of Corollary 2.1 and Lemma 4.1, it obviously suffices to show that there exist $n \times m'$ and $m \times m'$ constant matrices G and G_0 such that

$$\operatorname{rank} DG_0 \geq n - \min_i \operatorname{rank}(\exp(AT) - \exp(p_i T)I) \tag{4.6}$$

$$\operatorname{rank}\begin{bmatrix}\exp(AT) - zI & G \\ C & DG_0\end{bmatrix} \geq 2n - \min_i \operatorname{rank}(\exp(AT) - \exp(p_i T)I), \quad \forall z \in \mathbb{C}. \tag{4.7}$$

To do this, choose $m' \geq n - \min_i \operatorname{rank}(A - p_i I) + 1$. Since

$$\operatorname{rank}(\exp(AT) - \exp(p_i T)I) = \operatorname{rank}(A - p_i I)$$

(4.5) implies that (4.6) holds for almost all $m \times m'$ constant matrices G_0. In addition, from

$$\operatorname{rank} C \geq n - \min_i \operatorname{rank}(A - p_i I)$$

it is easy to see that for an arbitrary constant matrix G_0

$$\operatorname{rank}\begin{bmatrix}\exp(AT) - zI \\ C\end{bmatrix} + m' \geq 2n - \min_i \operatorname{rank}(A - p_i I) + 1, \quad \forall z \in \mathbb{C}$$

$$\operatorname{rank}\begin{bmatrix}\exp(AT) - zI & 0 & I \\ C & DG_0 & 0\end{bmatrix} = n + \operatorname{rank}[C \;\; DG_0] \geq 2n - \min_i \operatorname{rank}(A - p_i I), \quad \forall z \in \mathbb{C}$$

Thus, on applying Lemma 4.2 with

$$\epsilon = r - n + \min_i \operatorname{rank}(A - p_i I),$$

$$D(z) = \begin{bmatrix}\exp(AT) - zI \\ C\end{bmatrix}, \quad M(z) = \begin{bmatrix}0 \\ DG_0\end{bmatrix} \quad \text{and} \quad N(z) = \begin{bmatrix}I \\ 0\end{bmatrix}$$

it immediately follows that (4.7) holds for almost all $n \times m'$ constant matrices G and all $m \times m'$ constant matrices G_0. As a consequence, the theorem is concluded. □

Corollary 4.1: Let $P(s)$ be a SISO LTI bicausal continuous-time plant. Then $\mathbf{K}_T$ is equal to infinity for almost all sampling periods T.

Proof: Let $P(s)$ have a minimal realization (2.1)–(2.2). Then clearly

$$\operatorname{rank}(A - sI) \geq n - 1 \quad \forall s \in \mathbb{C}$$

implying condition (4.5). Hence, Corollary 4.1 is concluded from Theorem 4.1. □

If the plant is bicausal, the controller should be strictly proper to avoid a delay-free loop. Further, as has been shown in [12], stabilization by a nonstrictly proper controller is never robust against singular perturbations in some sense whereas stabilization by a strictly proper controller is robust against singular perturbations. The following result shows the existence of a strictly proper GSHF compensator (i.e., $D_c = 0$) for achieving a prescribed gain margin in case the sampling time is allowed to be small.

Corollary 4.2: Adopt the same hypothesis as in Theorem 4.1. Assume that all the unstable poles $p_1, p_2, \cdots, p_{N_2}$ are simple and $D \neq 0$. Given k_1 and k_2 with $0 < k_1 < 1 < k_2$, there always exists a strictly proper GSHF compensator (2.3)–(2.5) stabilizing $kP(s)$ for all $k \in [k_1, k_2]$.

Proof: From the proofs of Theorem 4.1 and Lemma 4.1, we can find (G, G_0) such that $\overline{P}(z) = DG_0 + C(zI - \exp(AT))^{-1}G$ has a Smith-McMillan form over $\mathbf{R}$ as

$$\begin{bmatrix}\operatorname{diag}(n_1/d_1, n_2/d_2, \cdots, n_l/d_l) & 0 \\ 0 & 0\end{bmatrix}.$$

Moreover, since all poles are simple and d_{i+1} divides d_i, we see that n_1/d_1 has unstable poles $\exp(p_i T)$ $(i = 1, \cdots, N_2)$ but no unstable zeros while $n_2/d_2, \cdots, n_l/d_l$ have no unstable poles. Thus, there is no problem with n_i/d_i $(i = 2, \cdots, l)$ for the construction of a strictly proper controller achieving any prescribed gain margin. Also from Remark 3.1, we can find a strictly proper compensator $c_1(z)$ stabilizing n_1/zd_1 for all $k \in [k_1, k_2]$ provided that T is sufficiently small. Finally, it can be concluded that there exists a strictly proper LTI controller stabilizing $kP(z)$ for all $k \in [k_1, k_2]$. From this, Corollary 4.2 is proved. □

Remark 4.3: From the proof of Theorem 4.1, the following observations can be made. First, G can still be chosen to remove the possible effect of finite unstable zeros on the gain margin even if condition (4.5) fails. Second, if $p_1, p_2, \cdots, p_{N_2}$ are all simple, then (4.5) implies that $P(s)$ does not have a blocking zero at infinity. Third, in the design of a GSHF compensator of the form (2.3)–(2.5), the dimension of v_k can be chosen to be any integer greater than $n - \min_i \operatorname{rank}(A - p_i I)$ and the GSHF gain $F(t)$ can be chosen to have at most $n + 1$ different values.

The following result deals with the case where condition (4.5) fails.

Theorem 4.2: Let $P(s)$ be an MIMO LTI continuous-time plant with a minimal realization (2.1)–(2.2) with $D = 0$. Suppose that unstable poles $p_1, p_2, \cdots, p_{N_2}$ of $P(s)$ are all simple and lie in $\mathbb{C}^+$. Then for almost all sampling periods T

$$\mathbf{K}_T \geq \left(\frac{1+\alpha_T}{1-\alpha_T}\right)^2 \tag{4.8}$$

where α_T is defined as in Theorem 3.1.

Proof: Take arbitrary $T > 0$ such that $T \neq (2k\pi/Im(p_i - p_j))$ for all integers k whenever $Re(p_i - p_j) = 0$. Then

$$\min_i \text{rank}(\exp(AT) - \exp(p_i T)I) = \min_i \text{rank}(A - p_i I) = n - 1$$

and $(\exp(AT), C)$ is observable. Now from the proof of Theorem 4.1, we can generically choose $n \times m'$ constant matrix G with $m' > 1$ such that

$$\text{rank}\, CG \geq 1$$
$$\text{rank}\begin{bmatrix}\exp(AT) - zI & G\\ C & 0\end{bmatrix} \geq n+1, \quad \forall z \in \mathbb{C}.$$

For such a chosen G, the above inequalities evidently imply that the first diagonal element of the Smith–McMillan form in $\mathbf{R}$ of $\overline{P}(z) = C(zI - \exp(AT))^{-1}G$ has a unique unstable zero at infinity of multiplicity one while all the other nonzero diagonal elements have no unstable poles and thus have the maximal gain margin of ∞. Consequently

$$\mathbf{K}_T \geq \mathbf{K}(G, G_0) \geq \left(\frac{1+\alpha_T}{1-\alpha_T}\right)^2.$$

This proves Theorem 4.2. □

As a direct consequence of Theorem 3.2 and Theorem 4.2, it follows that GSHF compensators can still arbitrarily improve the gain margin in the MIMO case.

V. Reduction of Conflict Between Gain Margin and Sensitivity

In [14], it is shown that for SISO nonminimum phase plants, there always exists conflict between gain margin maximization and sensitivity minimization to some extent if LTI compensation is applied. Particularly serious is the fact that gain margin maximization will lead to a closed-loop sensitivity of infinity. The purpose of this section is to show that periodic compensation can reduce this conflict for an MIMO LTI discrete-time plant under some mild condition although it has been indicated in [3], [7] that any time-varying controller cannot improve the minimal sensitivity with respect to LTI compensation. More specifically, an arbitrary gain margin can be achieved without leading to an arbitrarily large sensitivity by using a periodic time-varying controller.

Let $P(z)$ be a $p \times m$ plant and $0 < a < 1 < b$. Define

$$r[a, b] \triangleq \inf\{\|(I+PC)^{-1}\|;\ C(z) \text{ is an LTI controller stabilizing } kP(z) \text{ for each } k \in [a, b]\}$$

$$\tilde{r}[a, b] \triangleq \inf\{\|(I+PC)^{-1}\|;\ C \text{ is a periodic controller stabilizes } kP(z) \text{ for each } k \in [a, b]\}$$

where $\|(I+PC)^{-1}\| \triangleq \sup\{\|(I+PC)^{-1}u\|_2;\ u \text{ in } (h^2)^m,\ \|u\|_2 = 1\}$.

From [14], it is known that $\lim_{b/a \to \beta} r[a, b] = \infty$ if $P(z)$ is SISO and β denotes its maximal achievable gain margin. For $\overline{r}(a, b)$, however, we can establish the following result which ensures that $\overline{r}(a, b)$ is bounded uniformly with respect to any prescribed gain margin.

Theorem 5.1: Let $P(z)$ be a $p \times m$ LTI discrete-time plant with distinct unstable poles $p_1, p_2, \cdots, p_N$. Assume that $P(z)$ has no unstable pole-zero cancellations and has a minimal realization (A, B, C, D) with $A \in R^{n \times n}$. If

$$\text{rank}\, D \geq n - \min_i \text{rank}(A - p_i I) \tag{5.1}$$

then there exists some constant M only dependent upon $P(z)$ such that for any interval $[a, b]$ with $0 < a < 1 < b$

$$\tilde{r}[a, b] \leq M. \tag{5.2}$$

Proof: First, choose an integer $\nu \geq n$ such that $p_i^\nu \neq p_j^\nu$, $i \neq j$. As a consequence

$$\text{rank}(A^\nu - p_i^\nu I) = \text{rank}(A - p_i I). \tag{5.3}$$

Viewing $P(z)$ as a ν-periodic plant, we can write down its LTI transfer matrix representation as follows

$$\tilde{P}(z) = \begin{bmatrix} D & 0 & \cdots & 0\\ CB & D & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ CA^{\nu-2}B & CA^{\nu-3}B & \cdots & D\end{bmatrix} + \begin{bmatrix} C\\ CA\\ \vdots\\ CA^{\nu-1}\end{bmatrix}(zI - A^\nu)^{-1}[A^{\nu-1}B \ \cdots \ AB \ \ B].$$

Then it is known from [7] that

1) $\|(I+PC)^{-1}\| = \|(I+\tilde{P}\tilde{C})^{-1}\|$
2) C stabilizes $kP(z)$ iff $\tilde{C}(z)$ stabilizes $k\tilde{P}(z)$

where C is a ν-periodic controller and $\tilde{C}(z)$ is its LTI transfer matrix representation (note that any periodic system S can be uniquely represented as an LTI system $\tilde{S}(z)$ with $\tilde{S}(\infty)$ lower block triangular, and vice versa). From the above two facts, it can be seen that

$$\tilde{r}[a, b] \triangleq \inf\{\|(I+\tilde{P}\tilde{C})^{-1}\|;\ \tilde{C}(z) \text{ is an LTI controller with } \tilde{C}(\infty) \cdot \text{lower block triangular and stabilizing } k\tilde{P}(z) \cdot \text{for each } k \in [a, b]\}.$$

Let $U(z) = \text{block diag}(z^{-1}I_m \cdots z^{-1}I_m\ I_m)$. Then $\tilde{P}(z)U(z)$ has a stabilizable and detectable realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\tilde{A} = \begin{bmatrix} A^\nu & A^{\nu-1}B & \cdots & AB & 0\\ 0_{\nu m\times n} & & 0_{\nu m\times \nu m} & & \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 & \cdots & 0 & B\\ I_m & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & I_m & 0\\ 0 & \cdots & 0 & I_m\end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C & D & \cdots & 0 & 0\\ CA & CB & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & 0\\ CA^{\nu-1} & CA^{\nu-2}B & \cdots & CB & 0\end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & D \end{bmatrix}.$$

Also, note that $p_1^\nu, \cdots, p_N^\nu$ are all distinct unstable poles of $\tilde{P}(z)U(z)$. Using the following relations

$$\operatorname{rank}[A^{\nu-1}B \ \cdots \ AB \ \ B] = n \quad \text{and} \quad \operatorname{rank}[C^\tau \ \ A^\tau C^\tau \ \cdots \ (A^\tau)^{\nu-1}C^\tau] = n$$

with (5.1) and (5.3), it is trivial to check that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ fulfills conditions (4.1)–(4.2). In this way, it follows from the proof of Lemma 4 that if $\tilde{P}(z)U(z)$ has the following Smith–McMillan form over $\mathbf{R}$

$$V_1(z)\tilde{P}(z)U(z)V_2(z) = \begin{bmatrix} \operatorname{diag}(p_1(z), \cdots, p_l(z)) & 0 \\ 0 & 0 \end{bmatrix}$$

then $\alpha_i \geq 1$, $i = 1, \cdots, l$, where α_i is defined as in Proposition 2.2 associated with p_i, $V_1(z)$ and $V_2(z)$ are unimodular matrices over $\mathbf{R}$. Thus from [14], for any given $\epsilon > 0$ and interval $[a, b]$ with $0 < a < 1 < b$ there exists $c_i(z)$ stabilizing $kp_i(z)$ for each $k \in [a, b]$ such that $\|(1 + p_i c_i)^{-1}\| < 1 + \epsilon$. Let

$$\tilde{C}(z) = U(z)V_2(z)\begin{bmatrix} \operatorname{diag}(c_1(z), \cdots, c_l(z)) & 0 \\ 0 & 0 \end{bmatrix} V_1(z).$$

It is easy to see that $\tilde{C}(z)$ stabilizes $k\tilde{P}(z)$ for each $k \in [a, b]$ and $\tilde{C}(\infty)$ is lower block triangular. In addition

$$\begin{aligned} \|(I + \tilde{P}\tilde{C})^{-1}\| &= \|V_1^{-1}(z)(I + V_1(z)\tilde{P}(z)\tilde{C}(z)V_1^{-1}(z))^{-1}V_1(z)\| \\ &\leq \|V_1^{-1}(z)\| \, \|V_1(z)\| \max_i \|(1 + p_i(z)c_i(z))^{-1}\| \\ &< (1 + \epsilon)\|V_1^{-1}(z)\| \, \|V_1(z)\| \end{aligned}$$

which implies that $\tilde{r}[a, b] < (1 + \epsilon)M$, where $M \triangleq \|V_1^{-1}(z)\| \, \|V_1(z)\|$ only depends on $P(z)$. Since ϵ is arbitrary, (5.2) follows. □

VI. An Example

Consider a SISO LTI continuous-time plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= cx(t) + du(t) \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -12 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [-10 \quad 4], \quad d = 1$$

and the transfer function

$$P(s) = \frac{(s-1)(s-2)}{(s-3)(s-4)}.$$

It has been indicated in [6] that this plant has a maximal gain margin of 1.114 with respect to LTI compensators. But by Corollary 4.1, the plant has a maximal gain margin of ∞ with respect to GSHF compensators. Let us now construct a GSHF compensator which stabilizes $kP(s)$ for all $k \in [k_1, k_2]$ with $k_1 = .5$ and $k_2 = 2$.

To do this, first choose

$$T = 1/4, \quad G_0 = [1 \quad 0], \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} \overline{P}(z) &= dG_0 + c[zI - \exp(AT)]^{-1}G \\ &= \left[1 \quad \tfrac{4z-7.265}{(z-2.117)(z-2.718)}\right] \end{aligned}$$

which has the following Smith–McMillan form in $\mathbf{R}$:

$$\overline{P}(z)U(z) = \left[\tfrac{(z+0.5)^2}{(z-2.117)(z-2.718)} \quad 0\right]$$

with

$$U(z) = \begin{bmatrix} \frac{1.0z+44.004}{z+0.5} & \frac{-4z+7.265}{(z+0.5)^2} \\ \frac{-9.417z+34.835}{z+0.5} & \frac{(z-2.117)(z-2.718)}{(z+0.5)^2} \end{bmatrix}.$$

It is easy to see that

$$p_1(z) = \frac{(z+0.5)^2}{(z-2.117)(z-2.718)}$$

has a maximal gain margin of ∞ with respect to LTI compensators since it has no unstable zeros. Thus, there exists an LTI compensator which stabilizes $kp_1(z)$ for all $k \in [k_1, k_2]$. In [6], it is shown that all solutions to the gain margin problem for an LTI plant with LTI compensation can be found by parameterizing all the corresponding sensitivity functions, which are simply solutions to a certain interpolation problem. Using this idea, one of the required sensitivity functions in our case can be constructed as

$$\begin{aligned} s_1(z) \\ = \frac{(1.404z - 0.6632)(z - 0.3679)(0.4724z - 1)(0.3679z - 1)}{z^4 - 1.502z^3 + 0.7781z^2 - 0.4041z + 0.2341}. \end{aligned} \tag{6.1}$$

Then, the corresponding LTI compensator is

$$\begin{aligned} c_1(z) &= \frac{1 - s_1(z)}{s_1(z)p_1(z)} \\ &= \frac{3.099z^4 - 0.4736z^3 - 6.804z^2 + 4.022z - 0.04085}{(z+0.5)^2(z-0.3679)(z-0.4724)}. \end{aligned}$$

Define

$$\begin{aligned} C(z) \triangleq U(z)\begin{bmatrix} c_1(z) \\ 0 \end{bmatrix} &= \begin{bmatrix} 3.099 \\ -29.17 \end{bmatrix} \\ &+ \begin{bmatrix} \frac{133.9z^4-26.60z^3-294.7z^2-176.8z-1.865}{(z+0.5)^3(z-0.3679)(z-0.4724)} \\ \frac{131.6z^4+37.77z^3-281.9z^2+141.2z-0.7891}{(z+0.5)^3(z-0.3679)(z-0.4724)} \end{bmatrix}. \end{aligned} \tag{6.2}$$

Then $C(z)$ stabilizes $k\overline{P}(z)$ for each $k \in [k_1, k_2]$. On the other hand, one of the periodic GSHF gains associated with G_0 and G by (2.14)–(2.16) can be found as

$$F(t) = \begin{cases} [1 \quad 0], & 0 \leq t < 1/8 \\ [-7.276 \quad -5.102], & 1/8 \leq t < 3/16 \\ [5.602 \quad 20.408], & 3/16 \leq t < 1/4. \end{cases}$$

Thus, the GSHF compensator stabilizing $kP(s)$ for all $k \in [k_1, k_2]$

can be constructed as

$$z_{k+1} = \begin{bmatrix} -0.66 & 0.3368 & 0.2445 & -0.0253 & -0.0217 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} z_k + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} y(kT)$$

$$v_k = \begin{bmatrix} 133.9 & -26.60 & -294.7 & 176.8 & -1.865 \\ 131.6 & 37.77 & -281.9 & 141.2 & -0.7891 \end{bmatrix} z_k + \begin{bmatrix} 3.099 \\ -29.17 \end{bmatrix} y(kT)$$

$$u(t) = F(t)v_k, \quad \text{for } t \in [kT, (k+1)T), \quad k = 0, 1, 2, \cdots.$$

Finally, it is interesting to note that the order of the GSHF compensator does not increase as the required closed-loop gain margin (i.e., k_2/k_1) increases.

VII. Conclusions

In this paper, a GSHF dynamic compensator has been presented and used to improve the gain margin of a continuous-time LTI plant. The significant advantages of this particular digital periodic controller over the periodic controller used in [3] include the greater freedom and simpler structure because no dynamic periodic components are introduced. The main contributions of this note are summarized as follows.

1) It has been shown that for an MIMO LTI continuous-time plant, the GSHF compensator can always achieve an arbitrary closed-loop gain margin by taking the sampling time sufficiently small.
2) More interestingly, under a mild condition on the multiplicity of the possible zero of the plant at infinity, an arbitrary gain margin can be achieved using the GSHF compensator without any constraint on the sampling time. Moreover, for such a plant, there always exists a strictly proper GSHF controller achieving an arbitrarily prescribed gain margin. It is worth mentioning that the condition is satisfied automatically for a bicausal plant.
3) For a SISO strictly proper plant, an explicit formula for the maximal achievable gain margin with respect to GSHF compensators has been derived and the effect of the sampling period on the gain margin has been analyzed in detail. For instance, it has been indicated that the gain margin will decrease to one as the sampling period tends to infinity.
4) It has been discovered that by using periodic compensation, an arbitrary gain margin can be achieved without particularly causing a large sensitivity for a nonminimum phase LTI discrete-time plant. In contrast, LTI compensation does not possess this property.

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Stabilizing Control Law for Hybrid Models

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***Abstract*—This note is concerned with the adaptive stabilizing control problem for jump-Markov systems. A new control law is proposed based on a simple set of algebraic sufficient conditions. It is shown that this approach can be used for a class of stochastic models for which the more restrictive solution proposed by Caines and Zhang [4] does not apply.**

I. Introduction

Physical systems are generally subject to unpredictable disturbances due to their environment, and thus some control must be applied if one wants to maintain the system in a desired state. A good illustration is the regulation of the steam temperature in a solar thermal receiver [19]: to keep the temperature close to a nominal value, the controller must take into account the sudden and random jumps in insolation levels corresponding to sunny and partially cloudy situations.

To study such problems, Jump Linear Systems were introduced as appropriate models. They are hybrid systems: to the continuous state

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