# AN ALGEBRAIC SOLUTION TO THE SPECTRAL FACTORIZATION PROBLEM

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# An Algebraic Solution to the Spectral Factorization Problem

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*Abstract*—The problem of giving a spectral factorization of a class of matrices arising in Wiener filtering theory and network synthesis is tackled via an algebraic procedure. A quadratic matrix equation involving only constant matrices is shown to possess solutions which directly define a solution to the spectral factorization problem. A spectral factor with a stable inverse is defined by that unique solution to the quadratic equation which also satisfies a certain eigenvalue inequality. Solution of the quadratic matrix equation and incorporation of the eigenvalue inequality constraint are made possible through determination of a transformation which reduces to Jordan form a matrix formed from the coefficient matrices of the quadratic equation.

## I. Introduction

Multivariable filtering problems [1] and problems of network synthesis [2] have both generated the requirement of giving a certain type of factorization (termed spectral factorization) of a class of matrices whose elements are real rational functions of a complex variable $s$.

Specifically, there is given a square matrix $\Phi(s)$ satisfying

$$\Phi(s) = \Phi'(-s) \tag{1}$$

and

$$\Phi(j\omega) \geq 0 \qquad \text{for all } \omega \tag{2}$$

the notation $\geq 0$ being short for "is non-negative definite." A matrix $W(s)$ is sought which satisfies

$$W'(-s)W(s) = \Phi(s). \tag{3}$$

Normally, it is required that $W(s)$ be analytic in the right half-plane Re $s>0$, and in addition it may be required that $W(s)$ have constant rank there, or equivalently, that $W(s)$ possess a right inverse $W_r^{-1}(s)$ which is analytic in the right half-plane.

Two distinct solutions have been given to the problem as presented in Youla [3] and Davis [4], and minor modifications have been discussed, e.g., by Riddle and Anderson [5]. It has been established [3] that there are many solutions $W(s)$ for (3), but that any two solutions $W_1(s)$ and $W_2(s)$ are related through

$$W_1(s) = V(s)W_2(s) \tag{4}$$

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where $V(s)$ is a *paraunitary* matrix, that is, a matrix satisfying

$$V'(-s)V(s) = I \tag{5}$$

$I$ being the unit matrix. Moreover, from Youla [3], if $W_1$ and $W_2$ both possess right inverses which are analytic in the right half-plane, then $V$ must be constant.

The approach taken here to solve (3) is an algebraic one. Specifically, *a quadratic matrix equation* (involving only constant matrices) is presented, whose solution immediately defines a solution to (3) (Section II). A solution to (3) possessing the additional property of constant right half-plane rank is determined via a solution to the matrix equation which also satisfies an additional matrix eigenvalue inequality (Section III).

Some minor difficulties associated with the situation where $\Phi(\infty)$ is singular are dealt with in Section IV, where it is shown how to reduce the factorization problem of such a $\Phi$ to the problem of factorizing a matrix $\Phi_r$ with the same properties as $\Phi$, save that $\Phi_r(\infty)$ is nonsingular.

In Section V the determination of all solutions of the quadratic equation is discussed, together with a procedure for determining that particular solution satisfying the eigenvalue inequality. The key feature of the algorithm presented is the determination of the eigenvalues of a matrix formed from the coefficient matrices in the quadratic equation [6].

## II. Spectral Factorization for Matrices Nonsingular at Infinity

In this section the problem is considered of factorizing $\Phi(s)$, a matrix of rational functions of $s$, subject to

$$\Phi'(-s) = \Phi(s) \tag{1}$$

$$\Phi(j\omega) \geq 0 \tag{2}$$

and the following condition, to be relaxed later,

$$\det \Phi(\infty) \neq 0. \tag{6}$$

The following assumption will also be made: $\Phi(s)$ has no poles on the $j\omega$ axis.

While it is possible to extend the results to include this case, the extension is awkward; moreover, as pointed out by an anonymous reviewer, an initial excitation with white noise will cause the output power to grow without bound, and this is not properly described by spectral measure.

The factorization proceeds in several steps. First, $\Phi(s)$ is written as

$$\Phi(s) = Z(s) + Z'(-s) \tag{7}$$

where $Z(s)$ is a *positive real matrix* of rational functions; this requires $Z$ to satisfy the following conditions [2]:

1) $Z(s)$ is analytic in Re $s>0$ and has only simple poles on $s=j\omega$. Residue matrices for such poles are nonnegative definite Hermitian.

2) $$Z'(-j\omega) + Z(j\omega) \geq 0. \tag{8}$$

Because of (2) and (7), (8) is automatically satisfied. Using the assumption concerning the lack of $j\omega$-axis poles for $\Phi$, it will be shown that $Z(s)$ can be chosen with no such poles. Then, if $Z(s)$ is chosen to satisfy (7) and to have elements which are analytic in the right half-plane, $Z(s)$ will be positive real.

Second, a minimal realization [7] in the control-systems sense is then formed for $Z$ [see (14)]. This minimal realization is a collection of four constant matrices $\{F, G, H, J\}$.

Third, a matrix equation involving $F$, $G$, $H$, and $J$, and an unknown symmetric matrix $P$ is then constructed by using some recently found results on positive real matrices [8]. Any solution of this equation is shown to directly define a matrix $W(s)$ such that (3) holds.

Consider then (7). Let $\Phi$ be written as

$$\Phi(s) = \frac{\Phi_0(s)}{D(s)} + \Phi(\infty) \tag{9}$$

where $D$ is the least common denominator of the elements of $\Phi$. The degree of the polynomial $D$ is greater than that of any element in the polynomial matrix $\Phi_0(s)$. It is evident that $D(s)$ will consist of a product of terms of the form $(s-s_1)$, $(s+s_1)$, $(s-s_2)$, $(s+s_2)$, $\cdots$, where the $s_i$ may in general be complex and not necessarily distinct. It is also evident that Re $s_i<0$ may be assumed.

A partial fraction expansion of $\Phi$ is then formed so that

$$\Phi(s) = \Phi(\infty) + \frac{\Phi_1}{s-s_1} + \frac{\tilde{\Phi}_1}{s+s_1} + \frac{\Phi_2}{s-s_2} + \cdots \tag{10}$$

where $\Phi_i$ and $\tilde{\Phi}_i$ are constant matrices for simple poles and matrix functions of $s$ for multiple poles. Using (10) to form $\Phi'(-s)$ and equating coefficients of like terms in the partial fraction expansions of $\Phi(s)$ and $\Phi'(-s)$ yields

$$\tilde{\Phi}_i = -\Phi_i'. \tag{11}$$

Consequently (10) becomes

$$\Phi(s) = \Phi(\infty) + \sum_i \frac{\Phi_i}{s-s_i} + \sum_i \frac{-\Phi_i'}{s+s_i}. \tag{12}$$

Defining

$$Z(s) = \frac{\Phi(\infty)}{2} + \sum_i \frac{\Phi_i}{s-s_i} \tag{13}$$

(7) is fulfilled, and $Z(s)$ by some earlier remarks is positive real since the $s_i$ are all in the left half-plane. In the pathological case of $\Phi$ possessing multiple poles, an analogous procedure can be used to define $Z(s)$.

Note that, conceptually, the procedure for obtaining $Z(s)$ from $\Phi(s)$ is simple: a partial fraction expansion of $\Phi(s)$ is made, and that part with right half-plane poles is simply thrown away to leave $Z(s)$.

The next step in the factorization is to write down a minimal realization for $Z$, that is, a set of four matrices $\{F, G, H, J\}$, such that

$$Z(s) = J + H'(sI-F)^{-1}G \tag{14}$$

and such that $F$ has the least possible dimension. Procedures for doing this may be found in, e.g., Kalman [7] and Kalman and Englar [9] and may involve a great amount of complexity, perhaps even as great as that required in subsequent steps of the factorization procedure. Note that in any realization the relation $J=(1/2)\Phi(\infty)$ will hold.

At this stage the following lemma [8] is required:

*Lemma 1*

Let $Z(s)$ be a positive real matrix such that $Z(\infty)$ is finite, with $\{F, G, H, J\}$ constituting a minimal realization for $Z$. Then there exist matrices $P$, $L$, $W_0$ with $P$ positive definite symmetric, such that

$$PF + F'P = -LL' \tag{15a}$$

$$PG = H - LW_0 \tag{15b}$$

$$J + J' = W_0'W_0. \tag{15c}$$

In the proof of the lemma the existence of $P$ is established by a spectral factorization. For the sequel it is sufficient to know that $P$ exists.

Suppose that in (15) $W_0$ is restricted to being nonsingular. Then

$$\begin{aligned} PF + F'P &= -LL' = -(PG-H)W_0^{-1}W_0'^{-1}(PG-H)' \\ &= -(PG-H)(J+J')^{-1}(PG-H)'. \end{aligned} \tag{16}$$

For the $Z(s)$ resulting from $\Phi(s)$, it is true that $J=[\Phi(\infty)/2]$. Further, $J'=J$ since $\Phi'(\infty)=\Phi(\infty)$ by (1). Hence, $J+J'=\Phi(\infty)$ is nonsingular, and

$$PF + F'P = -(PG-H)\Phi^{-1}(\infty)(PG-H)'. \tag{17}$$

The lemma *guarantees* that this equation has a positive definite symmetric solution, but actually any symmetric solution to this equation defines a matrix $W(s)$ satisfying (3), with

$$W(s) = \Phi^{1/2}(\infty) - \Phi^{-1/2}(\infty)(PG-H)'(sI-F)^{-1}G \tag{18}$$

where $\Phi^{1/2}(\infty)$ is the square root of $\Phi(\infty)$, uniquely defined since $\Phi(\infty)$ is positive definite. For by explicit calculation,

$$\begin{aligned} W'(-s)W(s) &= [\Phi^{1/2}(\infty) - G'(-sI - F')^{-1}(PG - H)\Phi^{-1/2}(\infty)] \\ &\quad \cdot [\Phi^{1/2}(\infty) - \Phi^{-1/2}(\infty)(PG - H)'(sI - F)^{-1}G] \\ &= \Phi(\infty) - (PG - H)'(sI - F)^{-1}G \\ &\quad - G'(-sI - F')^{-1}(PG - H) \\ &\quad + G'(-sI - F')^{-1}(PG - H)\Phi^{-1}(\infty) \\ &\quad \cdot (PG - H)'(sI - F)^{-1}G \\ &= \Phi(\infty) - (PG - H)'(sI - F)^{-1}G - G'(-sI - F') \\ &\quad (PG - H) \\ &\quad + G'(-sI - F')^{-1}[P(sI - F) \\ &\quad + (-sI - F')P](sI - F)^{-1}G \qquad \text{using (17)} \\ &= Z(s) + Z'(-s) \qquad \text{using (14)} \\ &= \Phi(s). \qquad (19) \end{aligned}$$

Thus the following theorem has been established:

*Theorem 1*

Let $\Phi(s)$ be a matrix satisfying (1), (2), and (6), and let $Z(s)$ be the positive real matrix such that (7) holds. Moreover, let $Z(s)$ have a minimal realization $\{F, G, H, J\}$. Then any solution of the equation

$$PF + F'P = -(PG - H)\Phi^{-1}(\infty)(PG - H)' \qquad (17)$$

defines a matrix $W(s)$ satisfying (3) through

$$W(s) = \Phi^{1/2}(\infty) - \Phi^{-1/2}(\infty)(PG - H)'(sI - F)^{-1}G. \qquad (18)$$

Note the importance of the lemma in pointing out not merely the form of (17), but also that (17) has at least one solution. Note also that though this solution is known to be positive definite, any symmetric solution of (17) will define a suitable $W(s)$. Such solutions must, however, be at least non-negative definite: this is because the right-hand side of (17) will always be non-positive definite, and this fact, combined with the stability of $F$, guarantees by the well-known lemma of Liapunov (see chapter 3 of Hahn [10]) that $P$ is non-negative definite.

Thus far, no indication has been given of the determination of $W(s)$ such that it has constant rank in the right half-plane. This is the topic of the next section. Neither have means of solving (17) been discussed; such a discussion appears in Section V.

## III. Determination of the Factor with a Stable Inverse

Consider $W(s)$ as in (18). Explicit calculation will show that

$$W^{-1}(s) = \{I + \Phi^{-1}(\infty)(PG - H)' \cdot [sI - F - G\Phi^{-1}(\infty)(PG - H)']^{-1}G\}\Phi^{-1/2}(\infty) \qquad (20)$$

and thus $W^{-1}$ will have no poles in the right half-plane if and only if

$$\text{Re}\,\lambda_i[F + G\Phi^{-1}(\infty)(PG - H)'] < 0 \qquad (21)$$

the notation indicating that the eigenvalues of $F+G\Phi^{-1}(\infty)(PG-H)'$ must have negative real parts.

Equation (21) can be regarded as an additional constraint on $P$, when $P$ is used to define $W(s)$, i.e., in addition to requiring that $P$ satisfy the quadratic equation (17), $P$ must also satisfy (21).

Youla's analysis [3] shows that $W(s)$ is uniquely determined to within an arbitrary left orthogonal matrix multiplier, when $W$ and $W^{-1}$ are both analytic in the right half-plane. This suggests, and it is proved below, that any $P$ satisfying both (17) and (21) is unique.

To establish this uniqueness, suppose $P_1$ and $P_2$ both satisfy

$$PF + F'P = -(PG - H)\Phi^{-1}(\infty)(PG - H)' \qquad (17)$$

subject to

$$\text{Re}\,\lambda_i[F + G\Phi^{-1}(\infty)(PG - H)'] < 0. \qquad (21)$$

Define

$$F_1 = F + G\Phi^{-1}(\infty)(P_1G - H)' \qquad (22)$$

and similarly for $F_2$. Then explicit calculation yields

$$P_1F_1 + F_1'P_1 = -H\Phi^{-1}(\infty)H' + P_1G\Phi^{-1}(\infty)G'P_1 \qquad (23a)$$

$$P_2F_2 + F_2'P_2 = -H\Phi^{-1}(\infty)H' + P_2G\Phi^{-1}(\infty)G'P_2. \qquad (23b)$$

Now observe that

$$\begin{aligned} (P_1 - P_2)F_1 + F_2'(P_1 - P_2) &= P_2(F_2 - F_1) + (F_2' - F_1')P_1 \\ &\quad + P_1F_1 + F_1'P_1 - P_2F_2 - F_2'P_2 \\ &= P_2G\Phi^{-1}(\infty)G'P_1 + P_2G\Phi^{-1}(\infty)G'P_2 \\ &\quad - P_2G\Phi^{-1}(\infty)G'P_1 - P_1G\Phi^{-1}(\infty)G'P_1 \\ &\quad + P_2G\Phi^{-1}(\infty)G'P_1 - P_2G\Phi^{-1}(\infty)G'P_2 \\ &= 0. \qquad (24) \end{aligned}$$

The second line of (24) follows by using (22) and (23). It is known that the equation $AX+XB=0$ has only the solution $X=0$ when $\lambda_i(A)+\lambda_j(B)\neq 0$ for all $i$ and $j$ (see chapter 8 of Gantmacher [11]). But since $F_1$ and $F_2$ have eigenvalues with negative real parts, $\lambda_i(F_1)+\lambda_j(F_2)\neq 0$ for all $i$ and $j$, and from (24) $P_1=P_2$.

Hence, the following theorem has been established:

*Theorem 2*

With the same hypothesis as Theorem 1, there exists a unique solution of (17) which also satisfies

$$\text{Re}\,\lambda_i[F + G\Phi^{-1}(\infty)(PG - H)'] < 0. \qquad (21)$$

This solution defines a matrix $W(s)$ whose inverse is analytic in the right half-plane.

## IV. Extension to the General Case

In this section the problem is considered of factoring $\Phi(s)$ satisfying (1) and (2), and

$$\det \Phi(\infty) = 0. \tag{25}$$

A procedure is described for reducing the problem of factoring such a $\Phi$ to that of factoring a $\Phi_r$, with

$$\det \Phi_r(\infty) \neq 0. \tag{26}$$

The procedure of Section II can be used to define a positive real matrix $Z(s)$ from $\Phi(s)$. Then

$$\det Z(\infty) = 0. \tag{27}$$

Either det $Z(s)$ is zero for all $s$, or just for isolated values of $s$. If $\det Z(s) \equiv 0$, then it is possible to write [2]

$$Z(s) = T_1' \begin{bmatrix} Z_1(s) & 0 \\ 0 & 0 \end{bmatrix} T_1 \tag{28}$$

where $Z_1(s)$ is nonsingular, and $T_1$ is constant and nonsingular. If

$$Z_1(s) + Z_1'(-s) = W_1'(-s)W_1(s) \tag{29}$$

then

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{30}$$

with

$$W(s) = \begin{bmatrix} W_1(s) & 0 \\ 0 & 0 \end{bmatrix} T_1. \tag{31}$$

Hence, it only remains to show how to factorize in the situation where $Z(s)$ is nonsingular except for isolated values of $s$, including $\infty$. In this instance, form

$$Z_1(s) = Z^{-1}(s) \tag{32}$$

and observe that $Z_1(\infty)$ must be nonfinite. Accordingly [2] it is possible to write

$$Z_1(s) = sL + Z_2(s) \tag{33}$$

with $Z_2(\infty)$ finite and $L$ non-negative definite, and *with the degree of a minimal realization of* $Z_2$ (i.e., the dimension of the $F$ matrix) *less than for* $Z(s)$. If

$$Z_2(s) + Z_2'(-s) = W_2'(-s)W_2(s) \tag{34}$$

then explicit calculation shows that

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{35}$$

where

$$W(s) = W_2(s)Z(s). \tag{36}$$

If $Z_2(\infty)$ is nonsingular, a spectral factorization for $Z(s)+Z'(-s)$ results. If not, the process originally applied to $Z(s)$ can be applied to $Z_2(s)$, reducing the spectral factorization problem to that of factorizing a matrix with a minimal realization of lower degree. This process must eventually terminate on reaching some $\Phi_r(s) = Z_r'(-s)+Z_r(s)$ either in an easily factorizable constant matrix (i.e., a matrix with minimal realizations of degree zero) or in a matrix whose value at infinity is nonsingular, and which can then be factorized by the method of Section II. Moreover, through a succession of applications of equations like (31) and (36), a factorization of the original $\Phi$ matrix can be obtained from that of $\Phi_r$.

## V. Solution of the Quadratic Matrix Equation

To solve (17) it is convenient to rewrite it in the form

$$PG\Phi^{-1}(\infty)G'P + P[F - G\Phi^{-1}(\infty)H'] + [F' - H\Phi^{-1}(\infty)G']P + H\Phi^{-1}(\infty)H' = 0 \tag{37}$$

or

$$PCP + PB + B'P + A = 0 \tag{38}$$

where $C = G\Phi^{-1}(\infty)G'$, etc. The solution of such equations is discussed in Potter [6]:

*Lemma 2 (Potter)*

Defining the matrix

$$M = \begin{bmatrix} B' & A \\ -C & -B \end{bmatrix} \tag{39}$$

let $T$ be a matrix such that $T^{-1}MT$ is in Jordan canonical form. Let $a_1, a_2, \cdots, a_n$ be the first $n$ columns of $T$, and suppose $a_i$ is partitioned as $a_i' = [b_i', c_i']$ where $b_i$ and $c_i$ are $n$ vectors. Then if $[c_1, c_2, \cdots, c_n]$ is nonsingular, a solution to (38) is given by

$$P = [b_1, b_2, \cdots, b_n][c_1, c_2, \cdots, c_n]^{-1}. \tag{40}$$

*Proof:* With obvious notation and obvious partitioning

$$\begin{bmatrix} B' & A \\ -C & -B \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{bmatrix} J_1 & J_2 \\ 0 & J_4 \end{bmatrix}. \tag{41}$$

Here $T_1$ is $[b_1, b_2, \cdots, b_n]$ and $T_3$ is $[c_1, c_2, \cdots, c_n]$. Then

$$B'T_1 + AT_3 = T_1J_1 \tag{42a}$$

and

$$-CT_1 - BT_3 = T_3J_1 \tag{42b}$$

from which

$$B'T_1T_3^{-1} + A = T_1J_1T_3^{-1} \tag{43a}$$

and

$$-T_1T_3^{-1}CT_1T_3^{-1} - T_1T_3^{-1}B = T_1J_1T_3^{-1}. \tag{43b}$$

Thus it follows that $T_1T_3^{-1}$ satisfies (38), as required.

Note that there are many matrices $T$ which will bring $M$ into Jordan form, just as there are many Jordan matrices, corresponding to different orderings of the eigenvalues of $M$. It is not possible to guarantee in advance that the submatrix $T_3$ will be nonsingular, and the above lemma in no way claims that $T_3$ will be nonsingular in certain situations.

When $P$ is an $n \times n$ matrix, it is clear that the above procedure will be lengthy; moreover, in itself it does not suggest how to find the particular $P$ of Section IV. This is covered in the following theorem:

*Theorem 3 (Anderson)*

With the same hypothesis as Lemma 2, the matrix $P$ satisfying (37) and the eigenvalue inequality (21) is uniquely determined by ensuring that $J_1$ in (41) is associated with eigenvalues of $M$ with positive real parts, as may always be arranged.

*Proof:* Before giving the proof it should be noted that Potter [6] contains a similar theorem, valid when the matrices $A$ and $-C$ of (38) are non-negative definite. Here $A$ and $+C$ are non-negative definite. What is also necessary here is that $F-G\Phi^{-1}(\infty)H'$ be stable, a requirement absent from Potter's theorems.

With $A$ and $C$ symmetric, $M$ is a Hamiltonian matrix, and thus for each eigenvalue $\lambda$ there is also an eigenvalue $-\lambda$. This means that if $M$ has no imaginary eigenvalues (which we assume to be the case), $J_1$ in (41) may be taken to have all positive diagonal entries. Then from (42b),

$$-CT_1 - BT = T_3 J_1$$

so that

$$-CT_1T_3^{-1} - B = T_3J_1T_3^{-1}$$

or

$$-CP - B = T_3J_1T_3^{-1}. \tag{44}$$

Thus, the matrix $CP+B$ has all eigenvalues with negative real parts. Substituting for $C$ and $B$ their equivalences in terms of $F$, $G$, etc., leads to (21), the eigenvalue inequality.

## VI. Conclusions

A statement of the spectral factorization problem has been given in algebraic form. To solve the problem, a quadratic matrix equation must be solved, and to obtain a factor with the additional property of having a stable right half-plane inverse (i.e., an inverse analytic in the right half-plane) a particular solution satisfying an eigenvalue inequality must be found. One way of solving this equation is to determine a transformation which reduces to Jordan form a matrix whose dimension is twice that of the unknown matrix. For large dimension matrices it is thus clear that an iterative computer solution might present a reasonable alternative, though it would appear difficult to incorporate the eigenvalue inequality constraint in such an iterative procedure.

It also seems that the quadratic matrix equation could be viewed as the limit of a differential matrix Riccati equation. The limit of a solution to this Riccati equation, obtained by straightforward iteration, is a solution of the quadratic equation. The principal difficulty involved in using such an iterative process is to clarify the question of which particular solution of the quadratic equation results by choosing a particular initial condition for the Riccati equation. The procedure is the subject of current investigations.

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