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DISCRETE-TIME LOOP TRANSFER RECOVERY VIA GENERALIZED SAMPLED-DATA HOLD FUNCTIONS BASED COMPENSATOR

M. J. ER

School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 2263

AND

BRIAN D. O. ANDERSON

Department of Systems Engineering and Cooperative Research Centre for Robust and Adaptive Systems, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia

SUMMARY

Loop transfer recovery (LTR) techniques are known to enhance the input or output robustness properties of linear quadratic gaussian (LQG) designs. One restriction of the existing discrete-time LQG/LTR methods is that they can obtain arbitrarily good recovery only for minimum-phase plants. A number of researchers have attempted to devise new techniques to cope with non-minimum-phase plants and have achieved some degrees of success.[6–9] Nevertheless, their methods only work for a restricted class of non-minimum-phase systems. Here, we explore the zero placement capability of generalized sampled-data hold functions (GSHF) developed in Reference 14 and show that using the arbitrary zero placement capability of GSHF, the discretized plant can always be made minimum-phase. As a consequence, we are able to achieve discrete-time perfect recovery using a GSHF-based compensator irrespective of whether the underlying continuous-time plant is minimum-phase or not.

KEY WORDS Linear quadratic gaussian designs Loop transfer recovery
Generalized sampled-data hold functions Non-minimum-phase systems

1. INTRODUCTION

It is well-known that the plant input robustness properties of a state feedback design, such as measured by phase margins, for example, can evaporate with a state estimate feedback design.[1–5] An important class of state feedback design is linear quadratic (LQ) design, with the associated state estimate feedback design being the linear quadratic gaussian (LQG) design.

Recently, extension of the linear quadratic gaussian with loop transfer recovery (LQG/LTR) design techniques to discrete-time systems has been studied by a number of researchers.[6–9] The motivation for such an interest is due to the following two reasons: firstly, guaranteed feedback properties for the discrete-time LQ optimal regulator or Kalman filter do exist,[11–13] although they are not as attractive as in the continuous-time case. Naturally, it is desirable to have a method of recovering these properties. Secondly, as in continuous-time systems, the

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LTR procedure significantly simplifies the use of the LQG methodology. Knowing that it will be recovered in the LTR procedure, the designer can mainly concentrate on the design of the state feedback loop.

In Reference 6, the problem of recovering properties at the plant output in discrete-time is studied. Two types of Kalman filters, namely current-estimation type and one-step-ahead prediction type, are considered for the compensator design, which is synthesized as the series connection of the Kalman filter and the optimal state estimate feedback law. One of the major and interesting results is that if the plant is minimum-phase with all its infinite zeros of order one and if the cheap regulator is applied to the current-estimation type Kalman filter, then the state feedback loop of the observer can be perfectly recovered. It is also observed that, although it is generally impossible to have perfect recovery when the plant is non-minimum-phase or when the prediction-type Kalman filter has to be used, a useful degree of recovery is often obtained. An interpretation of this phenomenon in terms of asymptotic eigenvalue locations is also provided in Reference 6.

In Reference 7, the mechanism of loop recovery in Reference 6 is studied. In order to achieve perfect loop recovery and avoid obtaining an unstable controller simultaneously, the poles of the compensator are selected so that only the minimum-phase zeros of the plant are cancelled. As a consequence, the output feedback loop transfer function is still non-minimum-phase. This in turn limits the achievable performance, and 'good' loop shapes for the target loop function are difficult to achieve.

The LTR procedure using prediction estimators for square discrete-time minimum-phase systems is considered in Reference 8. It is shown that, although perfect recovery is impossible, the feedback properties obtained by the recovery techniques are those that can be recovered best in the presence of the delay in the controller.

The problem of applying existing LQG/LTR method to discrete-time non-minimum-phase plants is highlighted by Reference 9 which generalizes the results of Reference 10 to discrete-time. Adopting an approach similar to Reference 10, the authors consider an all-pass/minimum-phase factored model for the discrete-time plants and employ the LTR procedure proposed by Reference 6. The results that they obtain are not surprising considering that the approach is similar to Reference 10; similar conclusions to those appearing in the continuous-time case are derived.

In this paper, we explore the zero placement capability of generalized sampled-data hold functions (GSHF) developed in Reference 14 and show that any continuous-time plant can be discretized to a minimum-phase one with a simple zero at infinity, irrespective of whether the underlying continuous-time plant is minimum-phase or not. As a consequence, employing the discrete-time LQG/LTR procedure for minimum-phase systems of Reference 6 in conjunction with the proposed GSHF-based compensator, we are able to achieve perfect loop recovery irrespective of whether the underlying continuous-time plant is minimum-phase or not.

This paper is organized as follows. In Section 2, we present the proposed method of using GSHF to position the finite zeros of the continuous-time plant and show that the discretized plant can always be made minimum-phase with zero at infinity of order one regardless of whether the original continuous-time plant is minimum-phase or not. Results for perfect loop recovery are then developed. An example illustrating the idea is given in Section 3. Conclusions are drawn in Section 4.

2. PERFECT LOOP RECOVERY VIA GSHF

In this section, we exploit the power of GSHF and show that the finite zeros of the discrete-time system formed by cascade of GSHF, plant and sampler can be arbitrarily placed so that

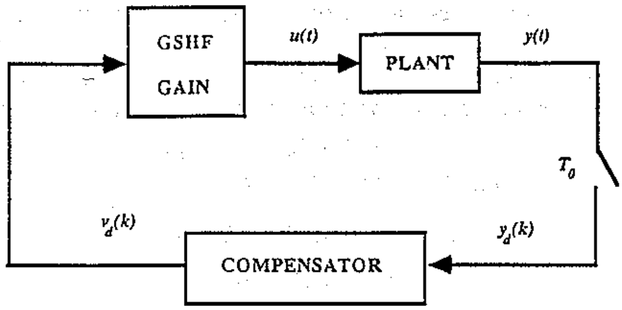

Figure 1. Connection of plant and GSHF-based compensator

the resulting discretized plant is minimum-phase. A block diagram illustrating the interconnection of the continuous-time plant and the GSHF-based compensator is shown in Figure 1; the subsystem GSHF is defined further below.

To fix idea, we begin with a continuous-time, strictly proper, FDLTI, square plant with a minimal state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are the input and output respectively, $x \in \mathbb{R}^n$ is the state vector, and A, B, C are constant matrices.

Using GSHF, the input to the plant, $u(t)$ becomes

$$u(t) = F(t) v_d(k) \qquad \text{for} \qquad t \in [kT_0, \overline{k+1}T_0) \tag{2}$$

where $T_0 > 0$ is the sampling period, $F(t)$ is a T_0-periodic integrable and bounded hold function matrix of appropriate dimension (selectable by the designer), and $v_d(k)$, whose dimension is the same as that of $u(t)$, is the output of the Kalman filter-based compensator.

The following equation

$$\int_0^{T_0} \exp[A(T_0 - t)] BF(t) \, dt = G \tag{3}$$

for the unknown $F(t)$ with G a given constant matrix, plays an important role in the design of the GSHF-based compensator. The properties with respect to (3) are summarized in the following lemma.

Lemma 2.1[15]

Let (A, B) be controllable, G be given and

$$W(A, B, T_0) = \int_0^{T_0} \exp[A(T_0 - t)] BB' \exp[A'(T_0 - t)] \, dt \tag{4}$$

Then

1. $F_0(t) = B' \exp[A'(T_0 - t)] W^{-1}(A, B, T_0) G$ is the unique optimal solution of (3) in the sense of minimizing $\operatorname{tr} \int_0^{T_0} F'(t)F(t) \, dt$;

2. for almost all $T_0 > 0$, there exists a piecewise constant solution of (3) taking at most n different values in the interval $[0, T_0]$;
3. for almost all $T_0 > 0$, there exists a sequence of piecewise constant solutions $F_k(t)$ of (3) which uniformly converges to $F_0(t)$ in the interval $[0, T_0]$ under a usual matrix norm.

Applying (2) to the system (1) and sampling the continuous-time state and output, we obtain the following discrete-time system from $v_d(k)$ to $y_d(k)$:

$$x_d(k+1) = A_d x_d(k) + G v_d(k) \tag{5a}$$

$$y_d(k) = C_d x_d(x) \tag{5b}$$

where A_d and C_d are related to A and C as follows:

$$A_d = \exp(A T_0) \qquad C_d = C \tag{6}$$

and G is related to the GSHF gain $F(t)$ as in (3). The associated transfer function is

$$P_{gshf}(z) \triangleq C_d(zI - A_d)^{-1} G \tag{7}$$

Concerning the discrete-time plant (7), we have the following important lemma.

Lemma 2.2

The discrete-time plant $P_{gshf}(z)$ can always be made minimum-phase with zero at infinity of order one via choice of a suitable GSHF gain, $F(t)$ irrespective of whether the underlying continuous-time system (1) is minimum-phase or non-minimum-phase.

Proof. Take arbitrary $T_0 > 0$ such that

$$T_0 \neq \frac{2k\pi}{\operatorname{Im}(p_i - p_j)} \qquad \text{for all integers } k$$

whenever $\operatorname{Re}(p_i - p_j) = 0$ with p_i, p_j being the poles of the plant. Then (A_d, C_d) is observable and (A_d, G) is stabilizable,[16] we can generically choose an $n \times m$ constant matrix G such that

$$\det(C_d G) \neq 0$$

$$\operatorname{rank} \begin{bmatrix} zI - A_d & G \\ C_d & 0 \end{bmatrix} = n + m \;\; \forall z \in \bar{D}$$

where $\bar{D} = \{z \in \mathbb{C}: |z| \geqslant 1\}$ with $\mathbb{C} = \{\text{complex numbers}\}$. For such a G, the above inequalities evidently imply that all finite zeros are minimum-phase and $\lim_{z \to 0} z P_{gshf}(z)$ is nonsingular. A systematic procedure for choosing G is given in the appendix. □

Remark 2.0

We comment that the procedure in the appendix uses the observer canonical form, which is not numerically attractive. On the other hand, since the only constraint on the zeros is that they have to be minimum-phase, i.e. their precise positions are not crucial, the numerical issues will not be so bothersome. A deeper question is what choice of G is optimal. We cannot yet deal with this question.

Next, we adopt a similar procedure of introducing fictitious process and measurement noise covariance matrices, W and V, as in Reference 6 to obtain a GSHF-based compensator. The

Kalman filter in the GSHF-based compensator takes the following form:

$$\hat{x}_d(\overline{k+1}/k) = A_d\hat{x}_d(k/\overline{k-1}) + Gv_d(k) + A_dK_f^f[y_d(k) - C_d\hat{x}_d(k/\overline{k-1})] \tag{8a}$$

$$\hat{x}_d(k/k) = \hat{x}_d(k/\overline{k-1}) + K_f^f[y_d(k) - C_d\hat{x}_d(k/\overline{k-1})] \tag{8b}$$

where K_f^f and P are given by

$$K_f^f = PC_d'(C_dPC_d' + V)^{-1} \tag{9}$$

$$P = A_dPA_d' - A_dPC_d'(C_dPC_d' + V)^{-1}C_dPA_d' + W \tag{10}$$

respectively.

Next, the optimal state estimate feedback law is synthesized using the discrete-time performance index given by

$$J = \sum_{k=0}^{\infty} (v_d'(k)Rv_d(k) + x_d'(k)Qx_d(k)) \tag{11}$$

where $Q = C_d'C_d \geqslant 0$ and $R = I/q^2$ with q being a real number. The optimal state estimate feedback controller is given by

$$v_d(k) = -\tilde{F}\hat{x}_d(k/k) \tag{12}$$

where the state feedback matrix, $\tilde{F}$ is given by

$$\tilde{F} = (G'\tilde{M}G + R)^{-1}G'\tilde{M}A_d \tag{13}$$

and $\tilde{M}$ is the unique positive semidefinite solution of the Riccati equation:

$$\tilde{M} = A_d'\tilde{M}A_d - A_d'\tilde{M}G(G'\tilde{M}G + R)^{-1}G'\tilde{M}A_d + Q \tag{14}$$

Finally, a GSHF-based feedback compensator is synthesized as the series connection of the current-estimation-type Kalman filter given by (8)–(10) and the optimal state estimate feedback controller (12). It is easy to see that the state-space model of the GSHF-based compensator can be written as

$$\zeta_d(k+1) = (A_d - G\tilde{F})(I - K_f^fC_d)\zeta_d(k) + (A_d - G\tilde{F})K_f^fy_d(k) \tag{15a}$$

$$v_d(k) = -\tilde{F}(I - K_f^fC_d)\zeta_d(k) - \tilde{F}K_f^fy_d(k) \tag{15b}$$

where $\zeta_d(k) = \hat{x}_d(k/\overline{k-1})$. It follows that the transfer function of the compensator is given by

$$C_{kf}^{GSHF}(z) = z\tilde{F}[zI - (I - K_f^fC_d)(A_d - G\tilde{F})]^{-1}K_f^f \tag{16}$$

Definition 2.1

When the discrete-time LTR procedure using the proposed GSHF based compensator is used, the output feedback loop transfer function, $H_{kf}^{GSHF}(z)$ is defined as

$$H_{kf}^{GSHF}(z) \triangleq P_{gshf}(z)C_{kf}^{GSHF}(z) \tag{17}$$

where $P_{gshf}(z)$ and $C_{kf}^{GSHF}(z)$ are defined by (7) and (16) respectively.

The following definitions and main result for discrete-time perfect loop recovery for a minimum-phase system developed in Reference 6 are needed for the subsequent development.

Definition 2.2

When the standard discrete-time LTR procedure is used, the output feedback loop transfer function, $H_{kf}^{ZOH}(z)$ is defined as

$$H_{kf}^{ZOH}(z) \triangleq P_{zoh}(z) C_{kf}^{ZOH}(z) \tag{18}$$

where $P_{zoh}(z)$ and $C_{kf}^{ZOH}(z)$ are given by

$$P_{zoh}(z) = C_d(zI - A_d)^{-1} B_d \tag{19}$$

$$C_{kf}^{ZOH} = zF[zI - (I - K_f^f C_d)(A_d - B_d F)]^{-1} K_f^f \tag{20}$$

respectively.

Definition 2.3

The filter's open-loop return ratio (or observer loop transfer function) is defined as

$$H_{ob}(z) \triangleq C_d(zI - A_d)^{-1} A_d K_f^f \tag{21}$$

and the closed-loop transfer function associated with (1) is given by

$$\Phi(z) \triangleq H_{ob}(z)[I + H_{ob}(z)]^{-1} \tag{22}$$

Theorem 2.1[6]

If $P_{zoh}(z)$ given by (19) is minimum-phase, $\det(C_d B_d) \neq 0$ and the standard discrete-time LTR procedure is used to obtain the Kalman filter-based compensator given by (20), then, as $q \to \infty$, perfect loop recovery can be achieved asymptotically at the plant output, i.e. as $q \to \infty$,

$$H_{kf}^{ZOH} \to H_{ob}(z)$$

Now we state the main result concerning discrete-time loop recovery using GSHF based compensator.

Theorem 2.2

Consider the minimum-phase discrete-time $P_{gshf}(z)$ with zero at infinity of order one given by (7). If the prescribed LTR procedure is applied to obtain the GSHF-based compensator given by (16), then, as $q \to \infty$, perfect loop recovery can be achieved asymptotically at the plant output, i.e. as $q \to \infty$,

$$H_{kf}^{GSHF}(z) \to H_{ob}(z)$$

Proof. The result follows from Theorem 2.1 and Lemma 2.2 □

3. ILLUSTRATIVE EXAMPLE

To illustrate the ideas presented, we give an example involving a non-minimum-phase system used in Reference 10 which is given by:

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t)$$

where

$$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \qquad c = [0 \;\; 1]$$

Note that the system has stable poles at -3, -1 and an unstable zero at 5.

Suppose that the goals of our design are to adjust the fictitious process and measurement noise covariance matrices V and W so that the step-response of the closed-loop system (22) meets the following specifications.

(a) Rise time, $t_r \simeq 0{\cdot}1$ s
(b) Settling time, $t_s \leqslant 0{\cdot}5$ s
(c) Maximum overshoot, $M_p \leqslant 15$ per cent

It is well-known that for a second-order discrete-time system without any finite zeros, t_r, t_s, M_p and the phase margin, PM are related to the damping coefficient, ζ and the closed-loop bandwidth, ω_{cl} as follows:[16]

(i) $t_r = \dfrac{1{\cdot}8}{\omega_{cl}}$

(ii) $t_s = \dfrac{4{\cdot}6}{\zeta\omega_{cl}}$

(iii) $\zeta \geqslant 0{\cdot}6\left(1 - \dfrac{M_p}{100}\right)$

(iv) $PM \geqslant 100\zeta$

Even though we are interested in sampled data control of a continuous-time system, the discretization of which has a finite zero, we will use these guidelines as an initial basis for a design. Using guideline (iii), specification (c) implies that the damping coefficient ζ must be $\geqslant 0{\cdot}51$. (Incidentally, this is in line with the recommendation of $0{\cdot}4 \leqslant \zeta \leqslant 0{\cdot}8$ for good design given in Reference 17). Here, we choose $\zeta = 0{\cdot}6$. It follows from specification (b) and guideline (ii) that $\omega_{cl} \geqslant 15{\cdot}3$ rad/s. This in turn implies that $t_r \leqslant 0{\cdot}12$ s according to guideline (i). Also, with $\zeta = 0{\cdot}6$, it follows from guideline (iv) that $PM \geqslant 60°$.

Next, following the recommendations given in Reference 17, the sampling period T_0 is chosen to be $0{\cdot}04$ s. With this value of T_0 and a ZOH interconnecting the continuous-time plant and the Kalman filter-based compensator, the discretized plant becomes

$$x_d(k+1) = A_d x_d(k) + b_d v_d(k)$$
$$y_d(k) = c x_d(k)$$

where

$$A_d = \exp(AT_0) = \begin{bmatrix} 0{\cdot}9977 & -0{\cdot}1108 \\ 0{\cdot}0369 & 0{\cdot}8500 \end{bmatrix}$$

$$b_d = \int_0^{T_0} \exp(At) b \, dt = \begin{bmatrix} -0{\cdot}2021 \\ 0{\cdot}0331 \end{bmatrix}$$

and obviously, the input to the plant, $u(t)$ is related to the output of the compensator, $v_d(k)$ as follows:

$$u(t) = v_d(k) \qquad kT_0 \leqslant t < (k+1)T_0$$

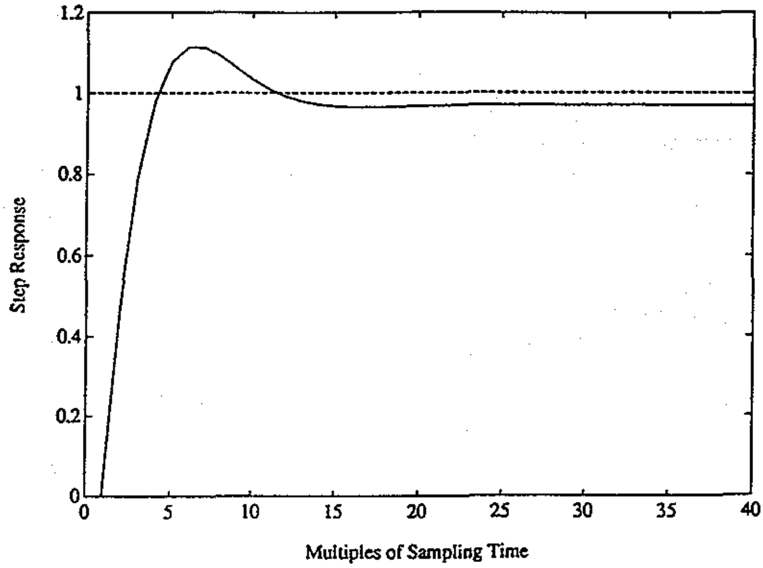

Figure 2. Step response of closed-loop system

It is not difficult to see that the only finite unstable zero is $1\cdot 223$. Thus, the discretized plant is still non-minimum-phase and with the Kalman filter-based compensator given by (20), perfect loop recovery (in fact, even near-perfect recovery) cannot be achieved. We shall now present some simulation results to further illustrate this.

It turns out that with $v = 0\cdot 04$ and

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

the aforementioned specifications for the step response are met for (22), i.e. with no loop recovery introduced and for a discrete-time rather than sampled-data system. The step-response of the closed-loop system is shown in Figure 2. Notice that this is for the discrete-time system, even though the curve is drawn continuously. From Figure 2, t_r, t_s and M_p are found to be $0\cdot 12$ s, $0\cdot 44$ s and $14\cdot 9$ per cent respectively. The steady-state position error of <5 per cent is due to the fact the closed-loop system is of type 0.

Corresponding to the aforementioned values of v and W, the Kalman filter gain is $k_f^f = [3\cdot 8173 \quad 0\cdot 3874]'$ and the filter's open-loop return ratio, $h_{ob}(z)$ is

$$h_{ob}(z) = \frac{0\cdot 4702(z - 0\cdot 702)}{(z - 0\cdot 9608)(z - 0\cdot 8869)}$$

A Nyquist plot of $h_{ob}(z)$ is presented in Figure 3. From the graph, the gain margin and the phase margin are found to be 59° and $13\cdot 3$ dB respectively.

Using the LQ optimization technique with the performance index given by (11) with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $r = 1/q^2$, the optimal state estimate feedback gain f is calculated as q varies. For each q, the Nyquist diagram of $h_{kf}^{ZOH}(z)$ is then evaluated using (18). In Figure 4, the solid curve is

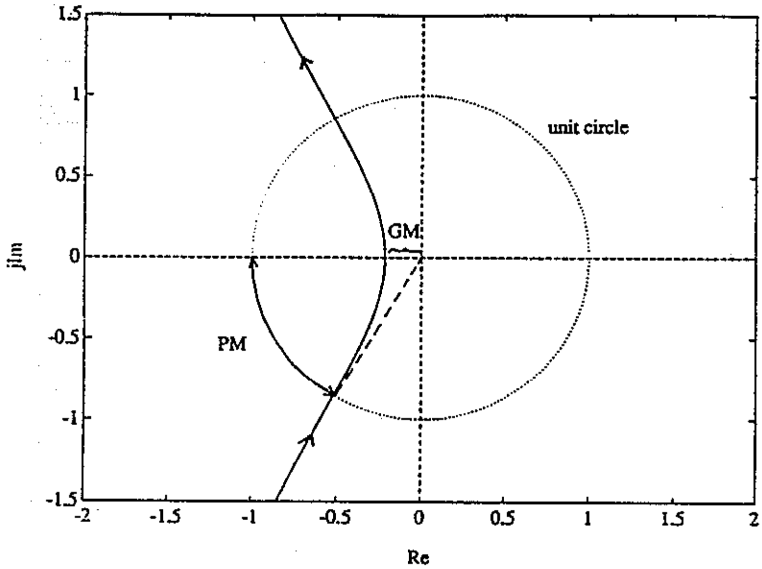

Figure 3. Nyquist plot of $h_{ob}(z)$

the desired response shown in Figure 3. The other curves indicate the Nyquist plots of $h_{kf}^{ZOH}(z)$ as q varies from 3 to 1000. As q increases beyond 1000, there are no noticeable changes in the curves. From Figure 4, it can be seen that the GMs for different q are always smaller than 13·3 dB for all q. The PMs cannot be determined from the plots as $h_{kf}^{ZOH}(z)$ never intersects the unit circle for the frequency range of interest. We further show the corresponding step responses when loop recovery is attempted with the Kalman filter-based compensator as q

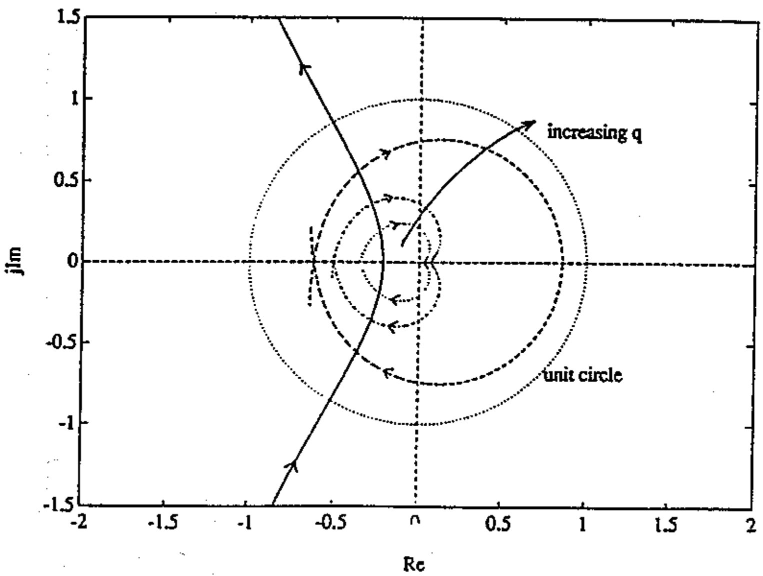

Figure 4. Nyquist plots of $h_{ob}(z)$ and $h_{kf}^{ZOH}(z)$

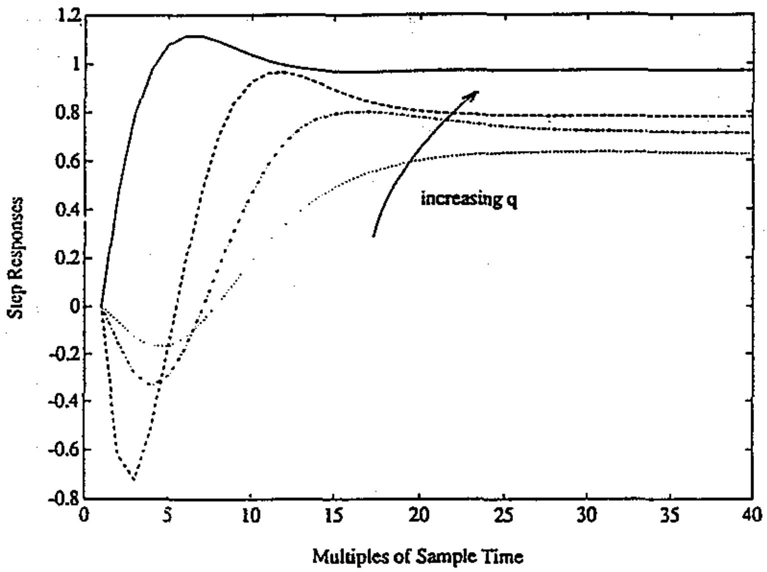

Figure 5. Step responses for loop recovery with $C_{\mathrm{kf}}^{\mathrm{ZOH}}(z)$

varies over the range as before. The simulation results are displayed in Figure 5. In Figure 5, the solid curve is the desired response of Figure 2. The other curves show the step responses for increasing q. The control input for $q = 3$ is displayed in Figure 6. The simulation results clearly demonstrate that perfect loop recovery cannot be achieved using the existing method of discrete-time LTR.

Next, using the procedure of selecting g outlined in the appendix, the discretized plant is

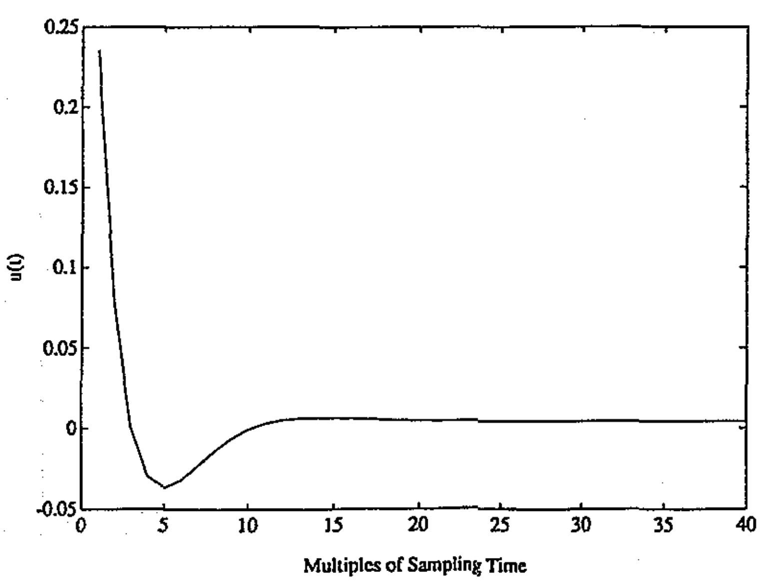

Figure 6. $u(t)$ for loop recovery with $C_{\mathrm{kf}}^{\mathrm{ZOH}}(z)$

made minimum-phase by $g = [27\ 2]'$. Sampling the continuous-time plant with the same T_0 and a GSHF (whose gains are defined later) interconnecting the continuous-time plant and the Kalman filter-based compensator yields

$$x_d(k+1) = A_d x_d(k) + g v_d(k)$$
$$y_d(k) = c x_d(k)$$

It can be readily shown that the finite zero is now at 0·5. The discretized plant from $v_d(k)$ to $y_d(k)$ is evidently minimum phase. Moreover, the product $cg = 2 \neq 0$. Hence, perfect loop recovery is asymptotically achievable according to Theorem 2.2. The input to the plant, $u(t)$ is related to the output of the compensator, $v_d(k)$ as follows:

$$u(t) = f(t) v_d(k) \qquad kT_0 \leqslant t < (k+1)T_0$$

with the GSHF gains associated with g given by:

$$f(t) = \begin{cases} -1957 & 0 \leqslant t < 0{\cdot}02 \\ 1707 & 0{\cdot}02 \leqslant t < 0{\cdot}04 \end{cases}$$

To further demonstrate that perfect recovery is asymptotically achieved using the proposed method, we show some Nyquist plots for the loop transfer function, $h_{kf}^{GSHF}(z)$ when q varies over the same range as before. Following the same procedure of LQ optimization outlined in Section 2, the Nyquist plot of $h_{kf}^{GSHF}(z)$ is displayed in Figure 7. Again, in Figure 7, the solid curve is the desired response shown in Figure 3. The dotted curve shows the Nyquist plot of $h_{kf}^{GSHF}(z)$ for $q = 3$. As q increases beyond 9, the Nyquist plots of $h_{kf}^{GSHF}(z)$ overlap with the desired one. We further present the simulation results for the step responses when loop recovery using the GSHF-based compensator is attempted. The step responses for different values of q are recorded in Figure 8. In Figure 8, the solid curve is the desired response of Figure 2 and is of course a continuous-time step response. It turns out that the step response

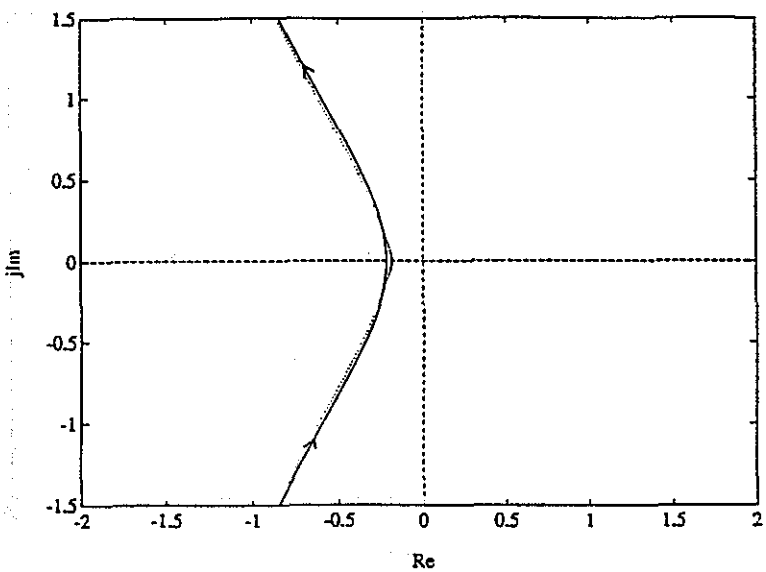

Figure 7. Nyquist plots of $h_{ob}(z)$ and $h_{kf}^{GSHF}(z)$

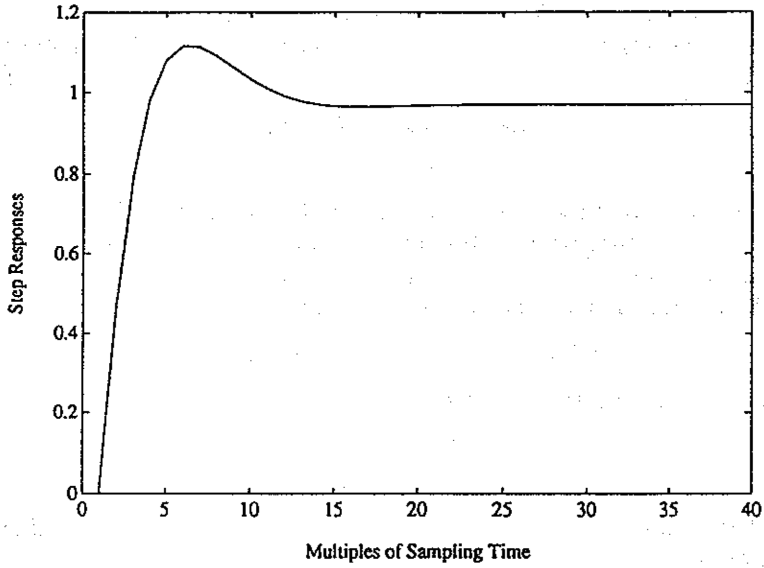

Figure 8. Step responses for loop recovery with $C_{\mathrm{Kf}}^{\mathrm{GSHF}}(z)$

for $q = 3$, which is the dotted curve, coincides with the desired one. The corresponding control input is displayed in Figure 9. Here, the disadvantage of the scheme becomes evident: large values of the control variable are encountered. By choosing a smaller value of q, the control excursion will become smaller, with some deterioration in the step response. Nevertheless, it is clear from the simulation results that perfect loop recovery is asymptotically achievable using the proposed method.

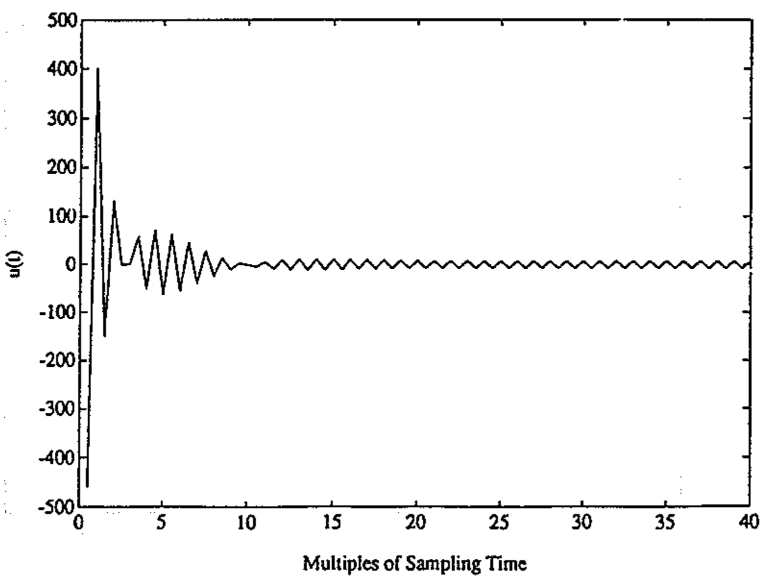

Figure 9. $u(t)$ for loop recovery with $C_{\mathrm{Kf}}^{\mathrm{GSHF}}(z)$

4. CONCLUSIONS

A new approach to discrete-time LQG/LTR is proposed. The idea revolves around the capability of arbitrary zero placement using GSHF. By exploiting this power of GSHF, any arbitrary strictly proper, continuous-time, FDLTI plant can always be discretized to a minimum-phase one. As a consequence, perfect loop recovery is asymptotically achievable using the proposed GSHF-based compensator irrespective of whether the underlying continuous-time plant is minimum-phase or not. This idea is both substantiated by theories and illustrated by an example.

APPENDIX: PROCEDURE FOR CHOOSING G

Using the technique of coprime fractional representation devised in Reference 17 the plant (7) can be written as

$$P_{\text{gshf}}(z) = D_L^{-1}(z) N_L(z) \tag{23}$$

where

$$N_L(z) = C_d(zI - A_d + KC_d)^{-1} G \tag{24}$$

$$D_L(z) = I - C_d(zI - A_d + KC_d)^{-1} K \tag{25}$$

It is well-known that $P_{\text{gshf}}(z)$ is minimum-phase if and only if $N_L(z)$ is. Furthermore, the transmission zeros are not affected by feedback. Hence, to choose a G such that $P_{\text{gshf}}(z)$ is minimum-phase is equivalent to choosing a G so that $N_L(z)$ is. Similarly, $P_{\text{gshf}}(z)$ has a simple zero at infinity if and only if $N_L(z)$ has this property.

2. If (A_d, C_d) is not in the observer form of (30) and (31), transform (A_d, C_d) to observer form as follows:

(a) Let the matrix C_d be partitioned as

$$C_d = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \tag{26}$$

and select a set of linearly independent row vectors starting from $c_1, c_2, \ldots, c_p$, and then $c_1 A_d, c_2 A_d, \ldots, c_p A_d$ and so forth, until n linearly independent vectors are found. The n selected independent vectors are then rearranged as

$$M \triangleq \begin{bmatrix} c_1 \\ \vdots \\ c_1 A_d^{\nu_1 - 1} \\ c_2 \\ \vdots \\ c_2 A_d^{\nu_2 - 1} \\ \vdots \\ c_p \\ \vdots \\ c_p A_d^{\nu_p - 1} \end{bmatrix} \tag{27}$$

where ν_is are termed the observability indices of the plant and

$$\sum_{i=1}^{p} \nu_i = n$$

(b) Define

$$M^{-1} \triangleq [e_{11}\ e_{12}\ \dots\ e_{1\nu_1}\ \dots\ e_{p1}\ e_{p2}\ \dots\ e_{p\nu_p}] \tag{28}$$

and form Q from $e_{i\nu_i}$ $(i = 1, 2, \dots, p)$ as follows

$$Q \triangleq [e_{1\nu_1}\ A_d e_{1\nu_1}\ \dots\ A_d^{\nu_1 - 1} e_{1\nu_1}\ \dots\ e_{p\nu_p}\ A_d e_{p\nu_p}\ \dots\ A_d^{\nu_p - 1} e_{p\nu_p}] \tag{29}$$

(c) Premultiplying and postmultiplying A_d by Q^{-1} and Q respectively, we obtain A_d in the observer form, i.e.

$$\bar{A}_d = Q^{-1} A_d Q$$

$$= \left[\begin{array}{ccccc|ccccc|ccccc}
0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x \\
1 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & x & 0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x \\
\hline
0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & x \\
0 & 0 & \dots & 0 & x & 1 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & & 0 & x & 0 & 0 & \dots & 1 & x & 0 & 0 & \dots & 0 & x \\
\hline
 & & & & & & & & & & \ddots & & & & \\
\hline
0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x \\
0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x & 1 & 0 & \dots & 0 & x \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 0 & x & 0 & 0 & \dots & 1 & x
\end{array}\right] \tag{30}$$

(d) The observer form of C_d is in turn obtained by postmultiplying C_d by Q:

$$\bar{C}_d = C_d Q$$

$$= \left[\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1
\end{array}\right] \tag{31}$$

(e) With this transformation, $P_{gshf}(z)$ becomes

$$P_{gshf}(z) = \bar{C}_d (zI - \bar{A}_d)^{-1} \bar{G} \tag{32}$$

where

$$\bar{G} = Q^{-1} G$$

$$= \left[\begin{array}{cccc}
b_{1,\varepsilon_1} & 0 & \dots & 0 \\
b_{1,\varepsilon_1 - 1} & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots \\
b_{1,2} & 0 & \dots & 0 \\
b_{1,1} & 0 & \dots & 0 \\
\hline
0 & b_{2,\varepsilon_2} & \dots & 0 \\
0 & b_{2,\varepsilon_2 - 1} & \dots & 0 \\
\vdots & \vdots & & \vdots \\
0 & b_{2,\varepsilon_1 + 2} & \dots & 0 \\
0 & b_{2,\varepsilon_1 + 1} & \dots & 0 \\
\hline
0 & 0 & \dots & b_{p,\varepsilon_p} \\
0 & 0 & \dots & b_{p,\varepsilon_p - 1} \\
\vdots & \vdots & & \vdots \\
0 & 0 & \dots & b_{p,\varepsilon_{p-1} + 2} \\
0 & 0 & \dots & b_{p,\varepsilon_{p-1} + 1}
\end{array}\right] \tag{33}$$

3. Since the transmission zeros are not affected by feedback, (32) can be written as

$$\begin{aligned} P_{\text{gshf}}(z) &= \bar{C}_{\text{d}}(zI - \bar{A}_{\text{d}})^{-1}\bar{G} \\ &= \bar{D}_{\text{L}}^{-1}(z)\bar{N}_{\text{L}}(z) \end{aligned} \tag{34}$$

where

$$\bar{N}_{\text{L}}(z) = \bar{C}_{\text{d}}(zI - \bar{A}_{\text{d}} + \bar{K}\bar{C}_{\text{d}})^{-1}\bar{G} \tag{35}$$

$$\bar{D}_{\text{L}}(z) = I - \bar{C}_{\text{d}}(zI - \bar{A}_{\text{d}} + \bar{K}\bar{C}_{\text{d}})^{-1}\bar{K} \tag{36}$$

Further, since $(\bar{A}_{\text{d}}, \bar{C}_{\text{d}})$ is observable, we can select an appropriate $\bar{K}$ such that

$$\bar{A}_{\text{d}} + \bar{K}\bar{C}_{\text{d}} = \text{diag}\left\{\begin{bmatrix} 0 & 0 & \dots & 0 & -a_{i,\varepsilon_{i-1}+1} \\ 1 & 0 & \dots & 0 & -a_{i,\varepsilon_{i-1}+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{i,\varepsilon_i} \end{bmatrix}\right\} \tag{37}$$

where

$$\varepsilon_0 = 0 \tag{38}$$

$$\varepsilon_i = \nu_1 + \nu_2 + \cdots + \nu_i \qquad (i = 1, 2, \dots, p) \tag{39}$$

and $a_{i,j}$s, $i = 1, 2, \dots, p$, $j = \varepsilon_{i-1} + 1, \varepsilon_{i-1} + 2, \dots, \varepsilon_i$ are constants.

It follows from (31), (33), (35) and (37) that

$$\det \bar{N}_{\text{L}}(z) = \prod_{i=1}^{p} N_i(z) \tag{40}$$

where

$$N_i(z) = \frac{b_{i,\varepsilon_i} z^{\nu_i - 1} + \cdots + b_{i,\varepsilon_{i-1}+2} z + b_{i,\varepsilon_{i-1}+1}}{z^{\nu_i} + a_{i,\varepsilon_i} z^{\nu_i - 1} + \cdots + a_{i,\varepsilon_{i-1}+2} z + a_{i,\varepsilon_{i-1}+1}} \tag{41}$$

with $b_{i,j}$, $i = 1, 2, \dots, p$, $j = \varepsilon_{i-1} + 1, \varepsilon_{i-1} + 2, \dots, \varepsilon_i$ being the entries of $\bar{G}$. Hence, the procedure boils down to selecting $b_{i,j}$s, so that (40) is a stable polynomial and with $b_{i,\varepsilon_i} \neq 0$ for each i, to guarantee that $z = \infty$ is a simple zero. The required G that makes $P_{\text{gshf}}(z)$ minimum-phase is then obtained by premultiplying $\bar{G}$ by Q, i.e.

$$G = Q\bar{G} \tag{42}$$

where Q and $\bar{G}$ are given by (29) and (33) respectively.

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