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Design of reduced-order multirate input compensators for output injection feedback laws

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Output injection feedback is a special kind of pole-positioning mechanism whereby linear combinations of the output measurements are fed directly into the plant's state. Using this mechanism, arbitrary closed-loop pole assignment can be achieved so long as the plant is completely observable. In the event that output injection feedback is not possible, a dual-observer-based compensator can be used to realize the pole-positioning effect of output injection. In this paper, we consider discrete-time systems and derive the equivalent dual-observer-based compensator, herein termed a single-rate input compensator. Further, we explore the concept of multirate input sampling and show that a multirate input compensator (employing multirate sampling of the plant input) of dimension much smaller than that of the single-rate input compensator (employing single-rate input sampling of the plant input) can be designed. Necessary and sufficient conditions for the existence of both types of compensators are found. Design procedures for constructing these compensators are also outlined.

1. Introduction

Output injection feedback is a special kind of pole-positioning mechanism whereby linear combinations of the output measurements are fed directly into the plant's state. Using output injection feedback, arbitrary closed-loop pole assignment can be achieved so long as the plant is completely observable. Nevertheless, this mechanism is, in general, impractical because inputs to the plant normally have to be applied through the input matrix rather than directly to the plant's state. In order to secure, at least approximately, the pole-positioning effect of output injection feedback while applying feedback at the correct input point, the dual-observer-based compensator introduced by Luenberger (1971) is used. The dual-observer-based compensator is essentially a linear dynamical system whose input and output are the plant's output and input respectively. Its implementation positions the closed-loop system poles at the eigenvalues of the compensator and also those assigned via output injection feedback. In effect, it circumvents the problem of feeding back to an inaccessible point, namely the plant state, and allows the implementation of output injection feedback. A review of the concept of a dual-observer based compensator will be given in § 2.

Since the seminal work of Luenberger (1966, 1971), the search for reduced-order compensator designs has been an ongoing process, see Er and Anderson (1992), Fortmann and Williamson (1972), Hagiwara *et al.* (1990), Pearson and Ding (1969) and Tse and Athans (1970). Of these, the results of Er and Anderson (1992) and Hagiwara *et al.* (1990) are particularly interesting. In

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Hagiwara *et al.* (1990), the authors studied controllers employing multirate sampling of the plant output and showed that arbitrary pole assignment is possible by generalized multirate-output controllers of reduced-order. Their approach essentially follows the discrete-time version of designing dynamic compensators as developed by Pearson and Ding (1969). An augmented system is formed by connecting L delay elements, as opposed to L differentiators, to each input of the discrete-time system. A multi-frame control law is then devised by employing the concept of multirate output sampling and it is shown to be equivalent to realizing the dynamic control law for the augmented system in the absence of measurement of the plant's state. One of the main results is that for a single-input system, the order of the proposed controller, L, the output-rate multiplicity, N_i^0 and the observability index, n_i^0 are related by $LN_i^0 \geqslant n_i^0$. This implies that, in order to achieve arbitrary pole assignment for an nth-order single-input single-output (SISO) system, the smallest order of such a controller is n with single-rate sampling while, with multirate output sampling, the smallest order becomes $[n/N^0]$, with N^0, of course, the ratio of the output sampling rate to the input sampling rate.

In Er and Anderson (1992), it was shown that in the case of estimating a single (but pre-specified) linear functional of a system's state, a multirate output linear functional observer (employing multirate sampling of the plant output) has an advantage over a single-rate output linear functional observer (employing single-rate sampling of the plant output). To be precise, it was shown that by exploring the multirate output sampling mechanism developed by Hagiwara and Araki (1988), one can design a multirate output linear functional observer of dimension much smaller than that of the single-rate output linear functional observer. The controller therein can be regarded as a combined estimator and state feedback law, which is not really the same as the controller in Hagiwara, *et al.* (1990). In the controller of Er and Anderson (1992), the dynamics of the estimator can be chosen separately from those of the closed-loop system with true state feedback. Nevertheless, the order of the controller is the same as that in Hagiwara *et al.* (1990).

In view of the substantial order reduction achievable by the multirate output linear functional observer in implementing linear state feedback laws, it is natural to ask whether the same result could be achieved in output injection feedback laws. In this paper, we show that this is indeed possible. First, we consider discrete-time systems and derive the equivalent dual-observer-based compensator, herein termed a single-rate input compensator. Next, by exploring the mechanism of multirate input sampling developed by Araki and Hagiwara (1986), we show that, in the case of realizing the pole-positioning effect of output injection feedback, a multirate input compensator (employing multirate input sampling of the plant input) of dimension much smaller than that of a single-rate input compensator (employing single-rate sampling of the plant input) can be designed. At this juncture, it is important to point out that N_i^{I} in Araki and Hagiwara (1986) satisfies $N_i^{\mathrm{I}} \geqslant n_i^{\mathrm{c}} (i = 1, \ldots, m)$ where N_i^{I} and n_i^{c} are the input-rate multiplicity and controllability indices respectively. In our scheme, N_i^{I} satisfies $1 \leqslant N_i^{\mathrm{I}} < n_i^{\mathrm{c}}$. Furthermore, the multirate input sampling employed here has uniform input-rate multiplicity, i.e. $N_1^{\mathrm{I}} = N_2^{\mathrm{I}} = \cdots = N_m^{\mathrm{I}} = N^{\mathrm{I}}$.

The structure of the paper is as follows: a review of the concept of a dual-observer-based compensator appears in § 2. The next section considers

SISO continuous-time linear time-invariant (LTI) systems and derives the structures and design procedures for the single-rate input compensators and multirate input compensator. Results pertaining to the orders of both compensators are also presented. In § 4, the results from the previous section are extended to the multiple-input single-output (MISO) case. The structures and design procedures for the two types of compensators together with results concerning the orders of the compensators for this case are also presented here. An example of a SISO system appears in § 5 to illustrate the ideas and methods described; § 6 contains concluding remarks.

2. Review of the concept of a dual-observer-based compensator

In this section, we shall review the concept of a dual-observer-based compensator introduced in Luenberger (1971). We recall first the notion of output injection feedback. There is prescribed a continuous-time LTI minimal plant of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(0) = x_0 \tag{2.1 a}$$

$$y(t) = Cx(t) \tag{2.1 b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. Output injection feedback is the process of postulating a feedback from the output directly to the state derivative, so that (2.1 *a*) is replaced by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + K_e y(t)\\ &= (A + K_e C)x(t) + Bu(t)\end{aligned} \tag{2.2}$$

For observable (A, C), it is always possible to select a K_e, termed output injection gain, so that the eigenvalues of $(A + K_e C)$ take prescribed values.

As foreshadowed in the introduction, output injection would be a desirable way of repositioning the poles of a system, except that, in general, it is impractical: inputs to the plant normally have to be applied 'through' B, rather than directly, as in (2.2). The question therefore arises: can we secure, at least approximately, the pole-positioning effect of output injection feedback, while applying feedback at the correct input point? As will be shown in the following, this is possible using a dual-observer-based compensator.

To facilitate the following development, we shall use the concept of a dual system.

Definition 2.1 (O'Reilly 1983): The continuous-time LTI system

$$\dot{\xi}(t) = F'\xi(t) + H'\eta(t) \tag{2.3 a}$$

$$w(t) = G'\xi(t) + E'\eta(t) \tag{2.3 b}$$

is said to be the *dual of the system* (and vice versa)

$$\dot{x}(t) = Fx(t) + Gu(t) \tag{2.4 a}$$

$$y(t) = Hx(t) + Eu(t) \tag{2.4 b}$$

□

To understand what a dual-observer-based compensator is, let us first recall the concept of an observer-based compensator. There is prescribed a plant in state variable form given by (2.1). To this plant, we wish to apply linear state feedback

$$u(t) = Kx(t) + v(t) \tag{2.5}$$

to position the closed-loop poles. (Here, $v(t)$ is an external input.) Because of the unavailability of the plant state, $x(t)$, as a measured signal, an estimator or observer of $x(t)$, or perhaps more efficiently, $Kx(t)$ is used. An observer of $Kx(t)$, termed a linear functional observer, is itself a linear system, driven by $u(t)$ and $y(t)$, of the form

$$\dot{z}(t) = Fz(t) + Gy(t) + Eu(t) \tag{2.6 a}$$

$$w(t) = Py(t) + Qz(t) \tag{2.6 b}$$

and a necessary and sufficient condition for $w(t)$ to estimate $Kx(t)$, in the sense that $w(t) - Kx(t) \to 0$ as $t \to \infty$ for all $v(t)$, $x(0)$ and $z(0)$, is that

$$\operatorname{Re} \lambda_i(F) < 0 \tag{2.7 a}$$

$$TA - FT = GC \tag{2.7 b}$$

$$E = TB \tag{2.7 c}$$

$$K = PC + QT \tag{2.7 d}$$

where T is a linear transformation such that $z(t)$ estimates $Tx(t)$. Procedures for selecting F, G etc to satisfy (2.7), including procedures which determine the dimension of F, can be found in Murdoch (1973). We note that $(z(t) - Tx(t)) \to 0$ at a rate determined by the eigenvalues of F, and that $\dim(z(t)) = \dim(x(t))$ (the Kalman filter-type observer), $\dim(z(t)) = \dim(x(t)) - \dim(y(t))$ (the Luenberger reduced-order observer, given independent outputs of (2.1)) and $\dim(z(t)) =$ observability index (given scalar $u(t)$, and linear functional observer design).

The compensator resulting from the above design results in replacing (2.5) by

$$u(t) = w(t) + v(t) \tag{2.8}$$

and is

$$\dot{z}(t) = (F + EQ)z(t) + (G + EP)y(t) + Ev(t) \tag{2.9 a}$$

$$w(t) = Py(t) + Qz(t) \tag{2.9 b}$$

(Its inputs are $y(t)$ and $v(t)$ and output is $w(t)$.)

When the compensator is implemented, the closed-loop transfer function matrix, obtainable from the combined system equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + BPC & BQ \\ (G + EP)C & F + EQ \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ E \end{bmatrix} v(t) \tag{2.10 a}$$

$$y(t) = [C \quad 0] \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{2.10 b}$$

is precisely $C(sI - A - BK)^{-1}B$, as would result from the use of (2.5). The combined system (2.10) has uncontrollable modes with eigenvalues $\lambda_i(F)$.

We now outline how a dual-observer-based compensator for (2.1) is obtained. (Details will be given subsequently.) First, the dual system of (2.1), with transfer function matrix $B'(sI - A')^{-1}C'$ is found. Next, we design an observer-based compensator for this dual plant. Then, we take the dual of the compensator and use it on the original plant. It turns out that the associated closed-loop system transfer function matrix is of the form $C(sI - A - K_d C)^{-1}B$, which has the form that would result if output injection feedback were applied to (2.1).

To understand the details, consider the dual plant

$$\dot{\xi}(t) = A'\xi(t) + C'\eta(t) \tag{2.11 a}$$

$$w(t) = B'\xi(t) \tag{2.11 b}$$

and suppose we wish to apply a linear feedback

$$\eta(t) = K'_d\xi(t) + \zeta(t) \tag{2.12}$$

to reposition the closed-loop poles. Here, $\zeta(t)$ is an external input. With $\xi(t)$ unavailable for measurement, we construct an observer $K'_d\xi(t)$, of the form

$$\dot{\lambda}(t) = F_d\lambda(t) + G_d w(t) + E_d\eta(t) \tag{2.13 a}$$

$$\mu(t) = P_d w(t) + Q_d\lambda(t) \tag{2.13 b}$$

with

$$\operatorname{Re}\lambda_i(F_d) < 0 \tag{2.14 a}$$

$$T_d A' - F_d T_d = G_d B' \tag{2.14 b}$$

$$E_d = T_d C' \tag{2.14 c}$$

$$K'_d = P_d B' + Q_d T_d \tag{2.14 d}$$

where T_d is a linear transformation such that $\lambda(t)$ estimates $T_d\xi(t)$. The associated compensator, obtained like (2.9) by combining the state feedback law (2.12) with the estimator (2.13), is

$$\dot{\lambda}(t) = (F_d + E_d Q_d)\lambda(t) + (G_d + E_d P_d)w(t) + E_d\zeta(t) \tag{2.15 a}$$

$$\mu(t) = P_d\omega(t) + Q_d\lambda(t) \tag{2.15 b}$$

It follows also that the closed-loop transfer function matrix of the combined equations

$$\begin{bmatrix}\dot{\xi}(t)\\ \dot{\lambda}(t)\end{bmatrix} = \begin{bmatrix} A' + C'P_d B' & C'Q_d \\ (G_d + E_d P_d)B' & F_d + E_d Q_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \lambda(t)\end{bmatrix} + \begin{bmatrix}C'\\ E_d\end{bmatrix}\zeta(t) \tag{2.16 a}$$

$$w(t) = [B' \quad 0]\begin{bmatrix}\xi(t)\\ \lambda(t)\end{bmatrix} \tag{2.16 b}$$

is $B'(sI - A' - C'K'_d)^{-1}C'$. By taking the dual of (2.16), we see that the equation set

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix} A + BP'_d C & B(G'_d + P'_d E'_d) \\ Q'_d C & F'_d + Q'_d E'_d\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}v(t) \tag{2.17 a}$$

$$\bar{y}(t) = [C \quad E'_d]\begin{bmatrix}x(t)\\ z(t)\end{bmatrix} \tag{2.17 b}$$

has transfer function matrix $C(sI - A - K_d C)^{-1}B$. Observe now that the set (2.17) can be regarded as the *interconnection* of the original system (2.1) together with a second system, defined by

$$\dot{z}(t) = (F'_d + Q'_d E'_d)z(t) + Q'_d y(t) \quad (2.18\,a)$$

$$u(t) = (G'_d + P'_d E'_d)z(t) + P'_d y(t) + v(t) \quad (2.18\,b)$$

At the same time, through (2.17 *b*), a new output, $\bar{y}(t)$ is defined by

$$\bar{y}(t) = y(t) + E'_d z(t) \quad (2.19)$$

The second system, with input $y(t)$ and output $u(t)$ (with $v(t)$ temporarily equal to zero), is the dual-observer-based compensator. Its implementation positions the closed-loop system poles at the eigenvalues of F_d and of $A + K_d C$. Note that the transfer function from $v(t)$ to $\bar{y}(t)$ rather than from $v(t)$ to $y(t)$ is $C(sI - A - K_d C)^{-1}B$; the closed-loop modes attributable to F_d are unobservable from $\bar{y}(t)$, but, in general, are observable from $y(t)$.

It is instructive to consider the direct dual of (2.15). (Notice that (2.18) was obtained by splitting up the dual of an interconnection of (2.15) with (2.11).) The direct dual of (2.15), which has two inputs ($w(t)$ and $\zeta(t)$) and one output, necessarily has one input and two outputs and is

$$\dot{z}(t) = (F'_d + Q'_d E'_d)z(t) + Q'_d y(t) \quad (2.20\,a)$$

$$\bar{u}(t) = (G'_d + P'_d E'_d)z(t) + P'_d y(t) \quad (2.20\,b)$$

$$\tilde{u}(t) = E'_d z(t) \quad (2.20\,c)$$

Of course, $\bar{u}(t)$ is used as the feedback part of $u(t)$, while $\tilde{u}(t)$ is combinable with $y(t)$ to yield $\bar{y}(t)$. Further understanding of $\bar{y}(t)$ can be achieved by noting that in a (conventional) observer-based compensator, the term $Ev(t)$ in (2.9 *a*) or $E_d\zeta(t)$ in (2.15 *a*) is normally inserted. This is a second input to the observer-based compensator. In the dual-observer-based compensator, the dual of this new input becomes a second output, namely $E'_d z(t)$. By deleting the input $Ev(t)$ from (2.9 *a*), we can obtain another compensator such that the open-loop system has unchanged eigenvalues while the modes associated with $\lambda_i(F_d)$ are now observable from the output. This is akin to the fact that with the dual-observer-based compensator, the modes associated with $\lambda_i(F_d)$ show up in the transfer function from $u(t)$ to $y(t)$, but not $u(t)$ to $y(t) + E'_d z(t)$.

Apart from the appearance of the distinction between $y(t)$ and $\bar{y}(t)$, we see that the dual-observer-based compensator, in effect, allows implementation of output injection feedback (corresponding to replacement of $C(sI - A)^{-1}B$ by $C(sI - A - K_d C)^{-1}B$ for some choice of K_d). The dynamics of the dual observer in effect circumvent the problem of feeding back to an inaccessible point, the plant state, (which is apparently needed for output injection feedback), just like the dynamics of a normal observer circumvent the problem of feeding back from an inaccessible point (again the plant's state).

In some circumstances, one type of observer may be much more attractive than the other when it comes to pole positioning. Thus, for a MISO system, a much lower dimension dual-observer-based compensator is likely to be possible. It is this idea which will be exploited in the later material, since multirate input sampling in some ways is like having extra inputs available.

3. SISO case

3.1. *Single-rate input compensator*

In this section, we begin by presenting the general structure of, and the design procedure for, the single-rate input compensator which is the discrete-time equivalent of the dual-observer-based compensator. It also serves as the basis for deriving the multirate input compensator and comparing the reduction in the dimension of the compensator presented in the later sections. The treatment is restricted here to single-input systems, and extended to multiple-input systems in § 4.

3.1.1. *Structure of the compensator.* Without loss of generality, we assume that the SISO discretized plant

$$x_d(k+1) = A_s x_d(k) + b_s u_d(k) \quad x_d(0) = x_0 \qquad (3.1\,a)$$

$$y_d(k) = c x_d(k) \qquad (3.1\,b)$$

inherits the controllability and observability properties of its continuous-time counterpart. Here,

$$A_s = \exp(AT_0), \quad b_s = \int_0^{T_0} \exp(At) b \, dt \qquad (3.2)$$

where the triple (A, b, c) represents the continuous-time plant.

In the previous section, we saw that to find a dual-observer-based compensator to realize the pole-positioning effect of output injection feedback is equivalent to finding a linear functional observer to estimate the single linear functional, $k_d'\xi(t)$ for the dual system and implementing the feedback. It follows that to find a single-rate input compensator to realize the pole positioning effect of output injection feedback for (3.1) is equivalent to finding a discrete-time linear functional observer to estimate the single linear functional, $k_e'\xi_d(k)$ where k_e and $\xi_d(k)$ are the output injection gain and the state of the dual of (3.1) respectively. In view of this, we obtain the following structure for the single-rate input compensator:

$$z_d(k+1) = (F' + q'e')z_d(k) + q'y_d(k) \qquad (3.3\,a)$$

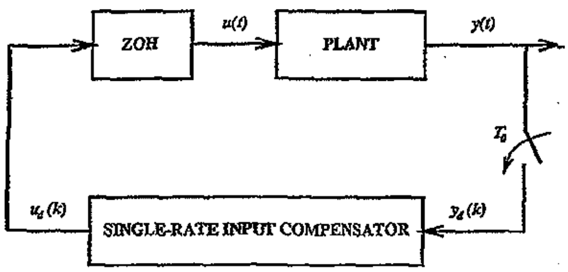
$$u_d(k) = (g' + p'e')z_d(k) + p'y_d(k) \qquad (3.3\,b)$$

where the relations of F, g, e, p and q to the triple (A_s, b_s, c) are given in Lemma 3.1. The connection of the plant and the single-rate input compensator is shown in the Figure below.

The necessary and sufficient conditions for the existence of the single-rate input compensator are precisely those for the existence of the discrete-time linear functional observer for the dual system. These conditions are contained in the following lemma.

Lemma 3.1: *The single-rate input compensator given by (3.3) exists if and only if the following conditions hold:*

Condition a:	$\lvert\lambda_i(F)\rvert < 1$
Condition b:	$k_e' = pb_s' + qS$
Condition c:	$SA_s' - FS = gb_s'$
Condition d:	$e = Sc'$



Connection of plant and single-rate input compensator.

where S is a linear transformation such that $\lambda_d(k)$ estimates $S\xi_d(k)$ with $\lambda_d(K)$ and $\xi_d(k)$ being the states of the duals of (3.1) and (3.3) respectively.

Proof: The proof is virtually identical with that for the continuous-time result. □

Remark 3.1: Note that here p is a scalar and q is a row vector. Hence, solvability of the equation in condition b for p and q, given b_s, S and k_e is guaranteed if $\text{rank}[b_s\ S'\ k_e] = \text{rank}[b_s\ S'] = n$, which is the dimension of the system's state.

3.1.2. *Order of the compensator.* From the preceding material, we see that the single-rate input compensator is obtained by taking the dual of the discrete-time linear functional observer (plus feedback law). Further, the latter is designed for the dual of (3.1) and has dimension $(\nu - 1)$ where ν is the observability index of the dual of (3.1). Now, it is well-known that the discrete-time plant (3.1) is controllable with controllability index, $\mu = n$ and its dual is observable with observability index, $\nu = \mu = n$. Hence, the order of the compensator is $(n - 1)$ for the SISO case. A more general result for the MISO case will be given in a later section.

3.1.3. *Design procedure for the compensator.* The problem at hand boils down to finding F, g, e, p and q such that the conditions for the existence of the compensator are fulfilled. To facilitate the construction of the compensator, we outline here a design procedure for the single-rate input compensator.

(1) Select T_0 according to the recommendations given in Åström and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990).

(2) Discretize the continuous-time plant with the selected T_0. Let the discretized plant be represented by (A_s, b_s, c).

(3) Take the dual of the plant and let it be denoted by (A_s', c', b_s').

(4) Choose a stable F (for simplicity, choose F to be diagonal with distinct eigenvalues) and $q = [1\ 1\ \ldots\ 1] \in \mathbb{R}^{1\times(n-1)}$. Solve for $p \in \mathbb{R}^1$, $g \in \mathbb{R}^{(n-1)}$ and $S \in \mathbb{R}^{(n-1)\times n}$ from $SA_s' - FS = gb_s'$ and $k_e' = pb_s' + qS$, using the algorithm of Murdoch (1973). Note that there always exists a unique triple p, g and S solving these equations if A_s and F do not have common eigenvalues.

(5) Compute $e = Sc'$.

(6) Construct the required single-rate input compensator via (3.3).

3.2. *Multirate input compensator*

For a single-input system, multirate input sampling allows N^1 successive and independent values of the input during each time interval $[(i-1)T_0, iT_0]$, $i = 1, 2, 3, \ldots$, i.e. a new value every T_0/N^I seconds. Intuitively, this is like maintaining the original T_0 but increasing the input dimension and the column rank of the input matrix, thereby reducing the controllability index of the discretized plant. This indicates that further reduction in the order of the compensator should be possible with multirate input sampling, and it motivates us to study the structure of the multirate input compensator and a design procedure for its construction.

3.2.1. *Multirate input sampling.* Before we deal with the structure and design procedure for a multirate input compensator, let us briefly review the concept of multirate input sampling. For single-input systems, multirate input sampling means that the plant input is changed N^I times over the time interval $[kT_0, \overline{k+1}T_0)$, $k = 0, 1, 2, \ldots$ where the integer N^I is termed the input-rate multiplicity i.e.

$$u(t) = u(kT_0 + jT) \triangleq u_d\left(k + \frac{j}{N^I}\right) \tag{3.4}$$

$$kT_0 + jT \leqslant t < kT_0 + (j+1)T$$

$$(j = 0, 1, \ldots, N^I - 1;\ k = 0, 1, 2, \ldots)$$

with T defined by

$$T \triangleq \frac{T_0}{N^I} \tag{3.5}$$

As a result of multirate input sampling, the discretized plant becomes

$$x_d(k+1) = A_s x_d(k) + \bar{B}\bar{u}_d(k) \tag{3.6 a}$$

$$y_d(k) = c x_d(k) \tag{3.6 b}$$

where A_s is given by (3.2),

$$\bar{B} = [b_m \quad A_m b_m \quad \ldots \quad A_m^{(N^I-1)} b_m] \tag{3.7 a}$$

$$A_m = \exp(AT_0/N^I) \tag{3.7 b}$$

$$b_m = \int_0^{T_0/N^I} \exp(At) b \, dt \tag{3.7 c}$$

Also, $\bar{u}_d(k)$ is an N^I-vector, with entries given by the N^I different values assumed by $u(.)$ in the interval $[kT_0, \overline{k+1}T_0]$.

3.2.2. *Structure of compensator.* In the same spirit as the single-rate input compensator, we propose the following structure for the multirate input compensator:

$$z_d(k+1) = (F' + q'e')z_d(k) + q'y_d(k) \tag{3.8 a}$$

$$\bar{u}_{\rm d}(k) = (g' + p'e')z_{\rm d}(k) + p'y_{\rm d}(k) \tag{3.8 b}$$

where the relations of F, g, e, p and q to the triple $(A_s, \bar{B}, c)$ are given in Lemma 3.2.

The conditions for the existence of the multirate input compensator are noted in the following trivial variant on Lemma 3.1.

Lemma 3.2: *The multirate input compensator given by (3.8) exists if and only if the following conditions hold:*

Condition a: $|\lambda_i(F)| < 1$

Condition b: $k_{\rm e}' = p\bar{B}' + qS$

Condition c: $SA_{\rm s}' - FS = g\bar{B}'$

Condition d: $e = Sc'$

where S is a linear transformation such that $\lambda_{\rm d}(k)$ estimates $S\xi_{\rm d}(k)$ with $\lambda_{\rm d}(k)$ and $\xi_{\rm d}(k)$ being the states of the duals of (3.6) and (3.8) respectively.

3.2.3. *Order of the compensator.* We define the order of the compensator (3.8) to be the dimension of $z_{\rm d}(k)$. Then we have the following theorem concerning the order of the compensator.

Theorem 3.1: *The order of the multirate input compensator, whose existence is assured by Lemma 3.2, required to realize the pole-positioning effect of output injection feedback is $\geq [n/N^{\rm I}] - 1$ where n and $N^{\rm I}$ are the dimensions of the state and the input-rate multiplicity respectively.*

Note that there might exist specific values of the output injection feedback law that could be achievable with a lower order compensator. The bound given in the Theorem statement applies irrespective of the output injection feedback law.

Proof: The dual of (3.6) is

$$\xi_{\rm d}(k+1) = A_{\rm s}'\xi_{\rm d}(k) + c'\eta_{\rm d}(k) \tag{3.9 a}$$

$$\bar{\omega}_{\rm d}(k) = \bar{B}'\xi_{\rm d}(k) \tag{3.9 b}$$

where $\xi_{\rm d}(k) \in \mathbb{R}^n$, $\eta_{\rm d}(k) \in \mathbb{R}^1$ and $\bar{\omega}_{\rm d}(k) \in \mathbb{R}^{N^{\rm I}}$ correspond to the state, input and output of the dual system. Now, it is well known that for almost all choices of T_0,

$$\operatorname{rank}(b_{\rm s}) = \operatorname{rank}\left(\int_0^{T_0} \exp(At)b\,{\rm d}t\right) = \operatorname{rank}(b) \tag{3.10 a}$$

$$\operatorname{rank}(b_{\rm m}) = \operatorname{rank}\left(\int_0^{T_0/N^{\rm I}} \exp(At)b\,{\rm d}t\right) = \operatorname{rank}(b) \tag{3.10 b}$$

and $(b' \exp(A'T_0/N^{\rm I}))$ will be observable. As a consequence, we have

$$\operatorname{rank}\begin{bmatrix} b_{\rm m}' \\ b_{\rm m}' \exp(A'T_0/N^{\rm I}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} b' \\ b' \exp(A'T_0/N^{\rm I}) \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} b' \\ b'A' \end{bmatrix} \tag{3.11 a}$$

$$\operatorname{rank}\begin{bmatrix} b'_m \\ b'_m \exp(A'T_0/N^I) \\ b'_m \exp(2A'T_0/N^I) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} b' \\ b' \exp(A'T_0/N^I) \\ b' \exp(2A'T_0/N^I) \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} b' \\ b'A' \\ b'(A')^2 \end{bmatrix} \qquad (3.11\,b)$$

etc and the rank of these successive matrices depends on the observability indices of the pair (b', A') or controllability indices of the pair (A, b). In any case, the row rank of $\bar{B}'$ can never exceed the number of rows or exceed n. Hence, column rank $\bar{B} \leqslant N^I$ and the result of the theorem follows. □

Remark 3.2: Generically, (3.9) will have observability index $\lceil n/N^I \rceil$ and thus the equality sign will hold in the theorem statement. □

3.2.4. *Design procedure for the compensator.* From the proofs of the preceding lemmas, we summarize here the design procedure for the multirate input compensator for a SISO system.

(1) Select T_0 according to the recommendations given in Astrom and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990).

(2) Choose the input-rate multiplicity N^I and discretize the continuous-time plant with T_0 obtained in (1). Let the discretized plant be represented by $(A_s, \bar{B}, c)$.

(3) Using Theorem 3.1, the smallest order of the compensator, whose existence is assured by Lemma 3.2, is $(\lceil n/N^I \rceil - 1)$.

(4) Take the dual of the plant and let it be denoted by $(A'_s, c', \bar{B}')$.

(5) Choose a stable F (again for simplicity, choose F to be diagonal with distinct eigenvalues) and $q = [1 \;\; 1 \;\; \ldots 1] \in \mathbb{R}^{1\times(\lceil n/N^I \rceil - 1)}$. Solve for $p \in \mathbb{R}^{1\times N^I}$, $g \in \mathbb{R}^{(\lceil n/N^I \rceil - 1)\times N^I}$ and $S \in \mathbb{R}^{(\lceil n/N^I \rceil - 1)\times n}$ from $SA'_s - FS = g\bar{B}'$ and $k'_e = p\bar{B}' + qS$, using the algorithm of Murdoch (1973). Again, provided that A_s and F do not have common eigenvalues, there always exists a triple p, g and S.

(6) Compute $e = Sc'$.

(7) Steps (4), (5) and (6) result in a causal multirate output linear functional observer. See Er and Anderson (1992) for details of a multirate output linear functional observer. The required multirate input compensator (3.8) is obtained by taking the dual of the constructed multirate output observer (plus feedback law).

4. MISO case

In this section, we indicate briefly the changes applying when the original system is multiple input.

4.1. *Single-rate input compensator*

When the original system is multiple-input, it turns out that the general structure of the single-rate input compensator is a direct extension from the

SISO case. For completeness, the structure of the compensator to realize the pole-positioning effect of output injection feedback and the conditions for its existence are summarized here.

The structure of the single-rate input compensator for a MISO system is

$$z_d(k+1) = (F' + q'e')z_d(k) + q'y_d(k) \tag{4.1 a}$$

$$u_d(k) = (g' + p'e')z_d(k) + p'y_d(k) \tag{4.1 b}$$

and the conditions for its existence are the solvability for a stable F of the equation

$$SA_s' - FS = gB_s' \tag{4.2}$$

together with the satisfaction of the following equations:

$$e = Sc' \tag{4.3}$$

$$k_e' = pB_s' + qS \tag{4.4}$$

where S is a linear transformation such that $\lambda_d(k)$ estimates $S\xi_d(k)$ with $\lambda_d(k)$ and $\xi_d(k)$ being the states of the duals of the original discrete-time system and (4.1) respectively.

The design procedure is the same as that for the SISO case except that the smallest dimension of the proposed single-rate input compensator is $(\mu - 1)$ where μ is the controllability index of the discretized plant.

4.2. *Multirate input compensator*

As mentioned earlier in the introduction, the multirate input sampling scheme employed here has *uniform* input-rate multiplicity i.e. $N_1^{\mathrm{I}} = N_2^{\mathrm{I}} = \ldots = N_m^{\mathrm{I}} = N^{\mathrm{I}}$. The structure of the multirate input compensator with uniform input-rate multiplicity and the associated design procedure also turn out to be a direct extension from the SISO case. The structure of the multirate input compensator is thus given by

$$z_d(k+1) = (F' + q'e')z_d(k) + q'y_d(k) \tag{4.5 a}$$

$$\bar{u}_d(k) = (g' + p'e')z_d(k) + p'y_d(k) \tag{4.5 b}$$

with F stable and S satisfying

$$SA_s' - FS = g\bar{B}' \tag{4.6}$$

There also holds

$$e = Sc' \tag{4.7}$$

$$k_e' = p\bar{B}' + qS \tag{4.8}$$

$$\bar{B} = [B_m \quad A_m B_m \quad \ldots \quad A_m^{(N^{\mathrm{I}}-1)} B_m] \tag{4.9}$$

$$A_m = \exp(AT_0/N^{\mathrm{I}}) \tag{4.10}$$

$$B_m = \int_0^{T_0/N^{\mathrm{I}}} \exp(At)B\,dt \tag{4.11}$$

The design procedure for a multirate input compensator turns out to be the same as that for a SISO system. The smallest dimension of the multirate input compensator, whose existence is assured by the satisfaction of (4.6)–(4.11), is contained in the following corollary.

Corollary 4.1: *The order of the multirate input compensator, whose existence is assured by the satisfaction of (4.6)–(4.11), required to realize the pole-positioning effect of output injection feedback for a multiple-input single-output system* (A, B, c) *is* $\geqslant [n/N^{\mathrm{I}}r] - 1$ *where n, r and* N^{I} *are the dimensions of the state, the column rank of the input matrix B and the uniform input-rate multiplicity respectively.*

Proof: The proof follows the same approach as that of Theorem 3.1. □

5. Illustrative example

To illustrate the ideas presented, we give an example involving a linear model for the equations of motion of a double mass-spring system used in Franklin *et al.* (1990). This system is used to study the flexible structure of some mechanical systems, for example, a communications satellite with a three-axis attitude-control. The state-space description for a particular set-up is given by:

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$y(t) = cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0{\cdot}91 & -0{\cdot}036 & 0{\cdot}91 & 0{\cdot}036 \\ 0 & 0 & 0 & 1 \\ 0{\cdot}091 & 0{\cdot}0036 & -0{\cdot}091 & -0{\cdot}0036 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c = [1 \quad 0 \quad 0 \quad 0]$$

5.1. *Single-rate input compensator*

A sampling period of $T_0 = 0{\cdot}4$ is used in Franklin *et al.* (1990) so that the sampling frequency is 15 times faster than the closed-loop bandwidth of $1\,\mathrm{rad\,s^{-1}}$. With single-rate sampling of the plant input, the discretized plant becomes

$$x_{\mathrm{d}}(k+1) = A_{\mathrm{s}}x_{\mathrm{d}}(k) + b_{\mathrm{s}}u_{\mathrm{d}}(k)$$

$$y_{\mathrm{d}}(k) = cx_{\mathrm{d}}(k)$$

where

$$A_{\mathrm{s}} = \begin{bmatrix} 0{\cdot}9285 & 0{\cdot}3876 & 0{\cdot}0715 & 0{\cdot}0124 \\ -0{\cdot}3516 & 0{\cdot}9146 & 0{\cdot}3516 & 0{\cdot}0854 \\ 0{\cdot}0071 & 0{\cdot}0012 & 0{\cdot}9929 & 0{\cdot}3988 \\ 0{\cdot}0352 & 0{\cdot}0085 & -0{\cdot}0352 & 0{\cdot}9915 \end{bmatrix}$$

$$b_s = \begin{bmatrix} 0{\cdot}0013 \\ 0{\cdot}0124 \\ 0{\cdot}0799 \\ 0{\cdot}3988 \end{bmatrix}$$

The open-loop poles of the discretized plant are $0{\cdot}9137 \pm 0{\cdot}3865i$ and 1 with multiplicity 2. In Franklin *et al.* (1990), the desired closed-loop poles are $0{\cdot}8000 \pm 0{\cdot}4000i$ and $0{\cdot}9000 \pm 0{\cdot}0500i$ and the pole-positioning is achieved via a state feedback law. Suppose that our desire is to implement output injection feedback rather than state feedback to position the closed-loop poles. It turns out that this is achievable via output injection feedback gain, $k_e' = [-0{\cdot}4275 \ \ -0{\cdot}1911 \ \ -0{\cdot}2537 \ \ -0{\cdot}0247]$. Nevertheless, we know that output injection is not feasible because inputs to the plant should be applied through b_s. Hence, we shall attempt to design a minimal order compensator to realize the pole-positioning effect of output injection feedback. As will be shown in the following, this is accomplished using a single-rate input compensator of order 3. However, using a multirate input compensator, the order becomes 1.

The structure of the single-rate input compensator is given by

$$z_d(k+1) = (F' + q'e')z_d(k) + q'y_d(k)$$

where

$$SA_s' - FS = gb_s'$$
$$e = Sc'$$

Choose

$$F = \begin{bmatrix} 0{\cdot}1 & 0 & 0 \\ 0 & 0{\cdot}2 & 0 \\ 0 & 0 & 0{\cdot}3 \end{bmatrix}$$

and

$$q = [1 \ \ 1 \ \ 1]$$

Since F and A_s have no common eigenvalues, there is a unique solution p, g and S to $SA_s' - FS = gb_s'$ and $k_e' = pb_s' + qS$. Using the algorithm of Murdoch (1973), we get

$$p = -12{\cdot}08$$

$$g = \begin{bmatrix} 415{\cdot}27 \\ -587{\cdot}68 \\ 199{\cdot}61 \end{bmatrix}$$

$$S = \begin{bmatrix} -1{\cdot}0263 & 6{\cdot}0365 & -45{\cdot}01 & 183{\cdot}96 \\ 4{\cdot}0588 & -16{\cdot}48 & 87{\cdot}75 & -292{\cdot}20 \\ -3{\cdot}4437 & 10{\cdot}40 & -42{\cdot}02 & 113{\cdot}02 \end{bmatrix}$$

Further,

$$e = Sc' = \begin{bmatrix} -1{\cdot}0263 \\ 4{\cdot}0588 \\ -3{\cdot}4437 \end{bmatrix}$$

Hence, the desired single-rate input compensator is

$$\begin{bmatrix} z_{1d}(k+1) \\ z_{2d}(k+1) \\ z_{3d}(k+1) \end{bmatrix} = \begin{bmatrix} -0\cdot9263 & 4\cdot0588 & -3\cdot4437 \\ -1\cdot0263 & 4\cdot2588 & -3\cdot4437 \\ -1\cdot0263 & 4\cdot0588 & -3\cdot1437 \end{bmatrix} \begin{bmatrix} z_{1d}(k) \\ z_{2d}(k) \\ z_{3d}(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y_d(k)$$

$$u_d(k) = [427.67 \quad -636.71 \quad 241.21] \begin{bmatrix} z_{1d}(k) \\ z_{2d}(k) \\ z_{3d}(k) \end{bmatrix} - 12.08 y_d(k)$$

Note that the compensator is open-loop stable.

5.2. *Multirate input compensator*

Using a multirate input compensator with the same T_0 and input-rate multiplicity $N^I = 2$, we obtain the following discretized plant:

$$x_d(k+1) = A_s x_d(k) + \bar{B}\bar{u}_d(k)$$
$$y_d(k) = c x_d(k)$$

where

$$\bar{u}_d(k) = \begin{bmatrix} u_d(k) \\ u_d(k+1/2) \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0\cdot0001 & 0\cdot0012 \\ 0\cdot0019 & 0\cdot0105 \\ 0\cdot0200 & 0\cdot0599 \\ 0\cdot1998 & 0\cdot1990 \end{bmatrix}$$

From Theorem 3.1, the order of the compensator is 1. Choose $f = 0\cdot1$ and $q = 1$. Solving $sA_s' - fs = g\bar{B}'$ and $k_e' = p\bar{B}' + qs$ for the triple p, g and s, we obtain

$$p = [806\cdot78 \quad -279\cdot88]$$
$$g = [334\cdot52 \quad -808\cdot93]$$
$$s = [-0\cdot1696 \quad 1\cdot1935 \quad 0\cdot3774 \quad -105\cdot54]$$

Also,

$$e = sc' = -0\cdot1696$$

Hence, the desired multirate input compensator is given by

$$z_d(k+1) = -0\cdot0696 z_d(k) + y_d(k)$$
$$\bar{u}_d(k) = \begin{bmatrix} 197\cdot67 \\ -761\cdot46 \end{bmatrix} z_d(k) + \begin{bmatrix} 806\cdot78 \\ -279\cdot88 \end{bmatrix} y_d(k)$$

which is again open-loop stable.

6. Conclusions

In this paper, we have given a new insight into using multirate input sampling in designing reduced-order compensators for realizing the pole-positioning effect of output injection feedback. Specifically, we have shown via theory and examples that a reduction in the order of the compensator is possible using a multirate input compensator with uniform input-rate multiplicity for single-output systems. It turns out that the order of the compensator only depends on the controllability index of the discretized plant induced via sampling of the continuous-time plant. The same type of ideas could be extended to achieve order reduction in the MIMO case. The algorithm of Murdoch (1974) would be relevant in this context.

Two caveats are necessary. First, our results concerning the order of the compensator irrespective of the choice of output injection feedback law—special choices may allow lower-dimension compensators again, with the methods of this paper providing no guidance to their construction. Second, we have adopted an apparently harmless definition of compensator order. Note that in any discrete-time system, the input which is applied to the plant must also be maintained by some information storage element, and if the input is multirate or multidimensional (which is also the same thing), a collection of input values needs to be stored. One could argue that they should be counted in assessing the state dimension.

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