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A DISTRIBUTIONAL APPROACH TO TIME-VARYING SENSITIVITY*

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Man, too acute, should perceive
That sensitive hearts have in grown
What's created though varied by time;
Systems are so by construct
But, as with man, little known,
Non sensed, though, in fulfilling man
A theory may guide and conduct.

1. Introduction. The theory of distributions [1], [2] has found wide application in various fields of science as, for example, in relativistic quantum mechanics [3], interaction and scattering of elementary particles [4], and network theory [5], [6], [7], [8]. Still, although results are available concerning systems analysis on a distributional basis [9], [10], little use of the rigorous theory of distributions has been made in the area of control system design. Here we investigate one of the fundamental concepts of control systems, that of sensitivity, obtaining results needed for optimal control design [11] in terms of distributions.

One of the classical problems of control theory is to reduce by feedback the sensitivity of a system to variations in the parameters of the plant. As a consequence a rather extensive literature is available concerning pertinent concepts [12], but little which directly discusses time-variable, as opposed to adjustable parameter, systems. Still time-varying, multiple-input, multiple-output systems are appearing in practical environments, by force of circumstances or as a result of implementing an optimal control law. In terms of distributional kernels we here investigate the question of when the sensitivity performance of such time-variable systems is improved by feedback.

The investigation follows the ideas of Cruz and Perkins as applied to time-invariant systems [13], and later extended by them to cover some time-variable cases [14], by considering the change in the closed loop response versus a change in the open loop response due to plant parameter changes and with the plant input held fixed. The relation between these open and closed loop response changes is linear and, for physical systems,

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describable by a *distributional kernel, the sensitivity* matrix. The main result is that for sensitivity improvement through the application of feedback the sensitivity matrix must be an antecedal contraction mapping of square-integrable vectors into square-integrable vectors. Such a sensitivity matrix is analogous to the scattering matrix of a passive network, and, consequently, many of the results of passive network theory [15] apply to sensitivity problems. This analogy is used to prove the main results, and the paper can then be considered as a solidification, through a complete statement of results, and a generalization, through distribution theory, of [14].

In §2 we review the necessary distributional background with emphasis placed upon distributional kernels. In §3 we discuss the sensitivity concept introducing the sensitivity matrix as well as the return-difference. In §4 the required properties of the sensitivity matrix needed for sensitivity improvement with the application of feedback are discussed, these being obtained by the abovementioned network analogy. For convenience we adhere as closely as possible to the notation of Cruz and Perkins [13].

2. Preliminaries. Here we review and introduce those concepts associated with distributional kernels which are necessary to the sequel. Along with this we discuss the physical constraints placed on kernels used in control theory. We assume as known the basic rudiments of distribution theory [1], [2].

Let $\mathfrak{D}$, $\mathfrak{D}_+$, $\mathfrak{L}_2$, and $\mathfrak{D}'$ denote the spaces of real-valued n-vectors in one real variable with entries which are, respectively, infinitely differentiable functions zero outside a bounded set (i.e., with compact support), infinitely differentiable functions zero until a finite value of the variable (i.e., with support bounded on the left), square-integrable functions on $(-\infty, \infty)$, and distributions. The scalar product between any $y \in \mathfrak{D}'$ and $\varphi \in \mathfrak{D}$ is denoted by $\langle y, \varphi \rangle$ which, on letting $t = \infty$, is the analogue of

$$\langle y, \varphi \rangle_t = \int_{-\infty}^{t} \tilde{y}(\lambda)\varphi(\lambda)\, d\lambda, \tag{2.1a}$$

defined, for instance, when $y, \varphi \in \mathfrak{D}_+$; here the superscript tilde denotes matrix transposition. When defined we also write

$$\| y \|_t^2 = \langle y, y \rangle_t, \tag{2.1b}$$

$$\| y \| = \| y \|_\infty, \tag{2.1c}$$

and observe that $\|\cdot\|$ serves as a norm for the Hilbert space $\mathfrak{L}_2$. The norm of a bounded linear transformation $T[\ ]$ of $u \in \mathfrak{L}_2$ into $T[u] \in \mathfrak{L}_2$ is defined in the customary manner as

$$\| T \| = \sup_{\|u\|=1} \| T[u] \|. \tag{2.2}$$

By a *distributional kernel* $k(t, \tau)$ is meant an $n \times m$ matrix of real-valued distributions in two real variables [16, p. 221]. Any linear continuous map of (m-vectors) $u \in \mathfrak{D}$ (strong topology) into (m-vectors) $y \in \mathfrak{D}'$ (weak topology) defines a distributional kernel k [17, Kernel Theorem, p. 143]:

$$(2.3a) \qquad y = k \cdot u.$$

Conversely, any distributional kernel defines such a map. If we denote the scalar product in two variables by $\langle\langle , \rangle\rangle$, (2.3a) is made precise by the definition [16, p. 221] of k as that distribution which represents the map, for all $u, \varphi \in \mathfrak{D}$, through the equations

$$(2.3b) \qquad \langle k \cdot u, \varphi \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \langle\langle k_{ij}(t, \tau), u_j(\tau) \rangle\rangle, \varphi_i(t) \rangle$$

$$(2.3c) \qquad = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle\langle k_{ij}(t, \tau), \varphi_i(t) u_j(\tau) \rangle\rangle,$$

where, of course, k_{ij}, u_j, φ_i are the entries of k, u, φ. Applying another kernel h to y of (2.3a) we obtain

$$(2.4) \qquad z = h \cdot y = h \cdot (k \cdot u) = (h \circ k) \cdot u,$$

which serves to define [16, p. 229] the Volterra *composition* $h \circ k$ of h and k as the unique kernel mapping u into z, whenever such a mapping exists. Although $h \circ k$ cannot always be formed, we note that it does exist and maps $\mathfrak{D}_+$ into $\mathfrak{D}_+$ whenever h and k both map $\mathfrak{D}_+$ into $\mathfrak{D}_+$. The composition of a number of kernels is not necessarily associative, but a sufficient condition guaranteeing associativity is that all kernels map $\mathfrak{D}_+$ into $\mathfrak{D}_+$ [18, p. 120]. With δ the unit impulse and 1_n the $n \times n$ identity matrix, $\delta 1_n = \delta(t - \tau) 1_n$ acts as the identity map under composition and hence can be composed with any kernel.

In the standard manner one defines the inverse k^{-1} under composition by

$$(2.5) \qquad k^{-1} \circ k = k \circ k^{-1} = \delta 1_n .$$

Depending upon the domain of definition considered one kernel may have several inverses. Consequently, we will assume, unless otherwise stated, that if k is a mapping of $\mathfrak{D}_+$ into $\mathfrak{D}_+$ then k^{-1} is also a mapping of $\mathfrak{D}_+$ into $\mathfrak{D}_+$. For such a mapping (2.5) means that for any $u \in \mathfrak{D}_+$, $k^{-1} \cdot (k \cdot u) = (k^{-1} \circ k) \cdot u = u$.

For intuitive reasoning it is convenient to recall the functional meaning of $\cdot$ and $\circ$:

$$(2.6a) \qquad y = k \cdot u = \int_{-\infty}^{\infty} k(t, \lambda) u(\lambda)\, d\lambda,$$

$$h \circ k = \int_{-\infty}^{\infty} h(t, \lambda)k(\lambda, \tau)\, d\lambda. \tag{2.6b}$$

Also in the standard manner one defines the *adjoint* k^a through

$$\langle u, k^a \cdot \varphi \rangle = \langle k \cdot u, \varphi \rangle \tag{2.7a}$$

for all $u, \varphi \in \mathfrak{D}$. Because of (2.3) [15, §4], one readily finds

$$k^a(t, \tau) = \bar{k}(\tau, t), \tag{2.7b}$$

and, thus, k^a generally will not map $\mathfrak{D}_+$ into $\mathfrak{D}_+$ when k does.

Of special interest are the nonnegative kernels [3, p. 45]. By definition, a real self-adjoint distributional kernel is *nonnegative*, written $k \geqq 0$, if, for all $\varphi \in \mathfrak{D}$,

$$\langle k \cdot \varphi, \varphi \rangle \geqq 0. \tag{2.8}$$

Turning to more physical notions, a system can be conceived as a transformation, here assumed linear, mapping inputs u into outputs y. Because we wish to treat physical systems, we can assume that $u, y \in \mathfrak{D}_+$, [19]. Further, discontinuous transformations seem physically out of the question. Consequently, since $u \in \mathfrak{D} \subset \mathfrak{D}_+$ and $y \in \mathfrak{D}_+ \subset \mathfrak{D}'$, we find by the kernel theorem that a linear physical system is described by a distributional kernel k through $y = k \cdot u$, at least for all $u \in \mathfrak{D}$. But, in actual fact, $k \cdot u$ can be extended to hold for all $u \in \mathfrak{D}_+$ since the original system transformation allows all $u \in \mathfrak{D}_+$, [16, p. 224]. Thus, as we expect also from physical arguments, $y = k \cdot u$ is defined for all $u \in \mathfrak{D}_+$, with $y \in \mathfrak{D}_+$. For some systems $y = k \cdot u$ can be defined for distributional inputs other than $u \in \mathfrak{D}_+$. For example, an extension to $\mathfrak{L}_2$ will be important in the next section while the consideration of impulsive u allows for the physical interpretation of k as an impulse response matrix.

3. The sensitivity matrix. In this section we define and interpret the sensitivity matrix. By the reasoning of §2, all impulse response matrices are assumed to be distributional kernels mapping $\mathfrak{D}_+$ into $\mathfrak{D}_+$.

Consider a fixed linear plant P which takes (m-vector) inputs u_o into (n-vector) outputs y_o and which is subject to variations in a parameter x. Then P is described by its $n \times m$ impulse response matrix p_x, a distributional kernel dependent on x. To obtain desirable transfer characteristics a controller G_1 is customarily inserted before the plant, as shown in Fig. 1, such that actual (p-vector) inputs r are modified by the $m \times p$ controller impulse response matrix g_1 to obtain the plant inputs:

$$y_o = p_x \cdot u_o, \qquad u_o = g_1 \cdot r, \tag{3.1a}$$

or

$$y_o = (p_x \circ g_1) \cdot r. \tag{3.1b}$$

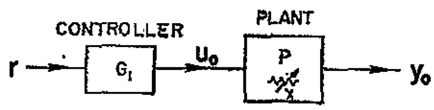

FIG. 1. *The open loop system*

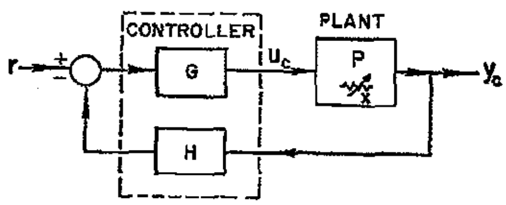

FIG. 2. *The closed loop system*

The $n \times p$ impulse response matrix of the open loop system, Fig. 1, is then $p_x \circ g_1$, and one notes that although y_o depends upon x, u_o does not since r and g_1 are assumed free of such variations. However, classical control theory [20, p. 211] recognizes that a redesign of the controller to incorporate feedback, which will cause the plant input to vary properly with x, can lead to smaller variations in the plant output with x. A general closed loop configuration of this type is shown in Fig. 2, where the controller components G and H are described by their $m \times p$ and $p \times n$ impulse response matrices g and h, also assumed independent of x. We note that

$$(3.2a) \qquad y_c = p_x \cdot u_c, \qquad u_c = g \cdot r - (g \circ h) \cdot y_c,$$

and hence, for the closed loop system,

$$(3.2b) \qquad y_c = [(\delta I_n + p_x \circ g \circ h)^{-1} \circ p_x \circ g] \cdot r.$$

For a meaningful design the open and closed loop controllers are, of course, constructed such that the respective plant outputs are equal, $y_c = y_o$, for a given input r when the parameter x assumes its design value $x = x_d$. This entails, for $x = x_d$, that $u_c = u_o$ or, from (3.1a) and (3.2a),

$$(3.3) \qquad [g - g_1 - g \circ h \circ p_x \circ g_1] \cdot r = 0,$$

which can be used to design g and h. However, the problem of interest here is the determination of the constraints on g and h such that variations in the closed loop output y_c, due to changes in x, are smaller than the corresponding variations in the open loop output y_o, for a given g_1 and p_x.

For such an investigation, in contradistinction to Cruz and Perkins [13, p. 217], let primed quantities denote the designed situation $x = x_d$, and unprimed quantities the situation for general x; thus $p_x' = p_{x_d}$. We then introduce the *open* and *closed loop output errors*, e_o and e_c, through

(3.4a) $$e_o = y_o' - y_o,$$

(3.4b) $$e_c = y_c' - y_c.$$

Then $e_o = e_c + (y_c - y_o)$ and, from (3.1a) and (3.2b), $y_c - y_o = p_x \cdot [g - g_1] \cdot r - p_x \cdot g \cdot h \cdot y_c$, which on subtraction and addition of $p_x \cdot g \cdot h \cdot y_c' = (p_x \circ g \circ h) \cdot p_x' \cdot u_c' = (p_x \circ g \circ h \circ p_x') \cdot u_o = p_x \cdot [g \circ h \circ p_x' \circ g_1] \cdot r$, and the use of (3.3) (primed), yields

(3.5a) $$e_o = [\delta 1_n + p_x \circ g \circ h] \cdot e_c.$$

We note that the feedback factor

(3.5b) $$f = \delta 1_n + p_x \circ g \circ h$$

is the *return-difference* [21, p. 48], that is, the difference between "unit" signal applied to the controller at the input to H and the signal returned to the controller via the feedback path of Fig. 2 when $r = 0$.

Since it is of most interest to evaluate the closed loop changes in terms of the open loop ones, we define the *sensitivity matrix* s as

(3.6a) $$s = [\delta 1_n + p_x \circ g \circ h]^{-1},$$

for which

(3.6b) $$e_c = s \cdot e_o.$$

In summary, given a physical system designed with open and closed loop controllers to obtain a given output-input relationship, a linear transformation exists relating the changes in the open loop output to changes in the closed loop output, due to variations in a plant parameter x, the relationship being represented by an $n \times n$ distributional kernel s, the sensitivity matrix. Being the inverse of the return-difference matrix f, s agrees with the more classical concepts for time-invariant single-input single-output systems [22, p. 121].

4. Sensitivity improvement criteria. Here we show that the closed loop system yields improved sensitivity performance if and only if the sensitivity matrix defines an antecedal map of $\mathfrak{L}_2$ into $\mathfrak{L}_2$ of norm bounded by unity.

We begin by restricting to inputs r in $\mathfrak{D}_+$, in which case we know on physical grounds that $e_o, e_c \in \mathfrak{D}_+$. Consequently, through (2.1), the quadratic performance indices $\langle e_o, e_o\rangle_t$ and $\langle e_c, e_c\rangle_t$ are well defined. A reasonable criteria for improvement of sensitivity performance is then, that for any given $r \in \mathfrak{D}_+$,

(4.1) $$\mathcal{E}(t) = \| e_o \|_t^2 - \| e_c \|_t^2$$

satisfies, for all finite t,

$$\mathcal{E}(t) \geqq 0. \tag{4.2}$$

That is, we will say that *sensitivity is improved* by feedback if at each instant of time the integral of the sum of squared error components is not increased by the application of feedback.

At this point we note that, for a system in which sensitivity is improved, the situation is analogous to that for passive (linear and solvable) n-port networks. Thus, if we consider e_o as incident voltages, v^i, and e_c as reflected voltages, v^r, then s is completely analogous to the scattering matrix of the network with $\mathcal{E}(t)$ the total input energy [15]. Consequently, by choosing $e_o(\lambda) = 0$ for $\lambda < t$, we see that $e_c(\lambda) = 0$ for $\lambda < t$, from (4.1), which implies [15, §4] that s is antecedal, that is, satisfies $s(t, \tau) = 0_n$ for $t < \tau$, where 0_n is the $n \times n$ zero matrix. Further, s can be extended to map $\mathcal{L}_2$ into $\mathcal{L}_2$. To obtain this extension we note that $\| e_o \|$ is well defined for $e_o \in \mathcal{L}_2$, thus implying that $e_c \in \mathcal{L}_2$ by (4.2). A system for which sensitivity is improved therefore defines a map of $e_o \in \mathcal{L}_2$ into $e_c \in \mathcal{L}_2$, this map being represented by s for $e_o \in \mathfrak{D}_+ \cap \mathcal{L}_2$. But $e_c = s \cdot e_o$ is valid [16, p. 224] for all $e_o \in \mathcal{L}_2$, in which case $\mathcal{E}(\infty) \geqq 0$ implies that $\| e_o \| \geqq \| s \cdot e_o \|$, or what is the same, the $\mathcal{L}_2$ into $\mathcal{L}_2$ transformation defined by s has its norm bounded by unity; for notational convenience we write this result as $\| s \| \leqq 1$ since no confusion can arise even though distributions are under consideration. Omitting further particulars which are detailed elsewhere [15, §4], we then have the main result.

THEOREM. *Sensitivity is improved by feedback if and only if the sensitivity matrix s satisfies the following conditions:*

(a) s *maps* $\mathcal{L}_2$ *into* $\mathcal{L}_2$;

(b) $s(t, \tau) = 0_n$ *for* $t < \tau$;

(c) $\| s \| \leqq 1$.

One of the most useful properties that can be determined from the theorem is that $s(t, \tau)$ is a measure (i.e., at most impulsive) in both variables simultaneously over any compact set of the (t, τ)-plane [15, §4]. Another property is seen by writing (4.2) in more detail:

$$\mathcal{E}(t) = \langle e_o, e_o \rangle_t - \langle s \cdot e_o, s \cdot e_o \rangle_t \tag{4.3a}$$

$$= \langle (\delta 1_n - s^a \circ s) \cdot e_o, e_o \rangle_t . \tag{4.3b}$$

We comment that $s^a \circ s$ is a well-defined kernel mapping $\mathcal{L}_2$ into $\mathcal{L}_2$; since s maps $\mathcal{L}_2$ into $\mathcal{L}_2$ so does s^a and consequently also $s^a \circ s$. Thus, letting $t \to \infty$ with $e_o \in \mathfrak{D}$, we see that

$$R = \delta 1_n - s^a \circ s \geqq 0 \tag{4.3c}$$

or R is a nonnegative kernel. Note that, in some sense, the "smaller" R, the less the sensitivity improvement, the limit being for $s^a = s^{-1}$. In terms of the return-difference we also have, from (3.5) and (3.6),

$$(s^a)^{-1} \circ R \circ s^{-1} = f^a \circ f - \delta 1_n \geqq 0. \tag{4.3d}$$

If the system is time-invariant [10], then $s(t, \tau) = s(t - \tau, 0)$, in which case one can take the Laplace transform $\mathfrak{L}[\cdot]$ (see [23], [24]) to obtain

$$S(p) = \mathfrak{L}[s(t, 0)]. \tag{4.4}$$

Again by analogy with the network situation [15], [25, p. 116], $S(p)$ must be *bounded-real*, that is, satisfy the following corollary, where a superscript asterisk denotes complex conjugation.

COROLLARY. *If* $s(t, \tau) = s(t - \tau, 0)$, *then sensitivity is improved by feedback if and only if*:

(a) $S(p)$ *is holomorphic in* $\operatorname{Re} p > 0$;

(b) $S^*(p) = S(p^*)$ *in* $\operatorname{Re} p > 0$;

(c) $1_n - \tilde{S}(p^*)\, S(p)$ *is positive semidefinite in* $\operatorname{Re} p > 0$.

When $S(p)$ is rational this precisely states the results discussed by Cruz and Perkins [13, p. 219].

5. Discussion. By observing the strict equivalence between the scattering matrix of a passive n-port and the sensitivity matrix of an n-output system for which sensitivity is improved by feedback application, the conditions of the theorem have been obtained. The results rest heavily upon the theory of distributions for their formulation with the theorem showing, however, that no "worse" than impulses appear in s. For example, in the case of a system described by differential equations (a differential system) s takes the form

$$s(t, \tau) = A(t)\delta(t - \tau) + \Phi(t)\tilde{\Psi}(\tau)1(t - \tau), \tag{5.1}$$

where $1(\cdot)$ is the unit step function, A has eigenvalues no greater than one, and Φ and Ψ are infinitely differentiable matrices subject to the nonnegative kernel constraint of (4.3c). Although the theory does allow the consideration of any distributional kernel s mapping $\mathfrak{D}_+$ into $\mathfrak{D}_+'$, the results show that for sensitivity improvement only $\mathfrak{L}_2$ maps need be considered, thus justifying the previous metric space assumption [14]. The kernel theorem and related distributional developments have allowed the complete sensitivity improvement conditions of the main result theorem.

If one has a finite dynamical (differential) system with H following the plant in the forward loop and unity feedback (i.e., Fig. 2 with y_c the output of H in place of P), then, under broad conditions, it can be shown that an "optimally" designed linear feedback law leads to sensitivity improvement [11]. Conversely, strict sensitivity improvement means that, for a time-invariant finite dynamical system, there is some quadratic loss function for which the feedback system is optimal [26]. Consequently, the results should be of some practical importance. It should, however, be pointed out that the theory of this paper is based upon creation of the system in the zero state at $t = -\infty$; nevertheless, a finite-dimensional state space is not assumed in the general arguments.

It is clear that the theory is valid for the most general linear systems of interest, but does not cover nonlinear systems, even though many of the concepts should carry over to the latter case. It is not so clear, however, that the variation of the disturbing parameter x should be "nonexistent". That is, x is essentially fixed for all time in the analysis and two "different" systems compared, one with x arbitrary and one with x at its design value x_d. This implied assumption is inherent in all such work and is physically reasonable for slow variations in x.

It is important to recognize the various extensions of kernel domains used in the theory. Although scant mention has been made of the topologies underlying the range and domain spaces, it being felt that these are of minor concern in physical situations, full justifications for the existence and extensions of the kernels can be found in Schwartz [16, p. 224]. However, the study does point out that for greater insight into sensitivity matrices a more detailed study of nonnegative distributional kernels is in order, there being very little presently available [3], [15].

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REFERENCES

[1] L. Schwartz, *Théorie des Distributions*, vol. I, Hermann, Paris, 1957.

[2] ———, *Théorie des Distributions*, vol. II, Hermann, Paris, 1959.

[3] ———, *Application of distributions to the study of elementary particles in relativistic quantum mechanics*, Tech. Rep. 7, NR041-221, Department of Mathematics, University of California, Berkeley, 1961.

[4] W. Güttinger, R. Penzl and E. Pfaffelhuber, *Peratization of unrenormalizable field theories*, Ann. Physics, 33 (1965), pp. 246–271.

[5] R. W. Newcomb, *Hilbert transforms and positive-real functions*, Proc. IRE, 50 (1962), pp. 2516–2517.

[6] A. H. Zemanian, *An N-port realizability theory based on the theory of distributions*, IEEE Trans. Circuit Theory, CT-10 (1963), pp. 265–274.

[7] R. W. NEWCOMB, *The foundations of network theory*, The Institute of Engineers Australia, Electrical and Mechanical Engineering Transactions, EM6 (1964), pp. 7–12.

[8] M. R. WOHLERS AND E. J. BELTRAMI, *Distribution theory as the basis of generalized passive-network analysis*, IEEE Trans. Circuit Theory, CT-12 (1965), pp. 164–170.

[9] V. DOLEŽAL, *Dynamics of Linear Systems*, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964.

[10] R. W. NEWCOMB, *Distributional impulse response theorems*, Proc. IEEE, 51 (1963), pp. 1157–1158.

[11] B. D. O. ANDERSON, *Sensitivity improvement using optimal design*, Proc. IEE, 113 (1966), pp. 1084–1086.

[12] P. V. KOKOTOVIĆ AND R. S. RUTMAN, *Sensitivity of automatic control systems* (Survey), Avtomat. i Telemeh., 26 (1965), pp. 730–750.

[13] J. B. CRUZ, JR. AND W. R. PERKINS, *A new approach to the sensitivity problem in multivariable feedback system design*, IEEE Trans. Automatic Control, AC-9 (1964), pp. 216–223.

[14] W. R. PERKINS AND J. B. CRUZ, JR., *Sensitivity operations for linear time-varying systems*, Sensitivity Methods in Control Theory, L. Radanovic, ed., Pergamon Press, New York, 1966, pp. 67–77.

[15] B. D. O. ANDERSON AND R. W. NEWCOMB, *Functional analysis of linear passive networks*, Internat. J. Engr. Sci., to appear.

[16] L. SCHWARTZ, *Théorie des noyaux*, Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, 1950, vol. I, American Mathematical Society, Providence, 1952, pp. 220–230.

[17] ———, *Espaces des fonctions différentiables à valeurs vectorielles*, J. Analyse Math., 4 (1954–1955), pp. 88–148.

[18] ———, *Théorie des distributions à valeurs vectorielles*, Ann. Inst. Fourier (Grenoble), 7 (1957), Chap. 1, pp. 1–141.

[19] R. W. NEWCOMB, *On the definition of a network*, Proc. IEEE, 53 (1965), pp. 547–548.

[20] H. CHESTNUT, *Systems Engineering Tools*, John Wiley, New York, 1965.

[21] H. W. BODE, *Network Analysis and Feedback Amplifier Design*, Van Nostrand, New York, 1945.

[22] J. G. TRUXAL, *Automatic Feedback Control Systems Synthesis*, McGraw-Hill, New York, 1955.

[23] L. SCHWARTZ, *Transformation de Laplace des distributions*, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.], Tome Supplémentaire, 1952, pp. 196–206.

[24] R. W. NEWCOMB AND R. G. OLIVEIRA, *Laplace transforms-Distributional theory*, Tech. Rep. 2250-3, Stanford Electronics Laboratories, Stanford, California, 1963.

[25] D. C. YOULA, L. J. CASTRIOTA AND H. J. CARLIN, *Bounded real scattering matrices and the foundations of linear passive network theory*, IRE Trans. Circuit Theory, CT-4 (1959), pp. 102–124.

[26] B. D. O. ANDERSON, *The inverse problem of optimal control*, Tech. Rep. 6560-3, Stanford Electronics Laboratories, Stanford, California, 1966.