Matrix-Fraction Description From Frequency Samples

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ABSTRACT

It is shown how the controllability indices in a minimal state-space realization of a real rational transfer matrix may be calculated from evaluations of this transfer matrix at a sufficient number of discrete points in the frequency domain. Subsequently a procedure for obtaining coprime matrix-fraction descriptions of the transfer matrix is given. An example is used to illustrate the results.

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1. INTRODUCTION

Given a set of frequency-response samples from an unknown linear multivariable system, characterized by a proper rational transfer matrix $H(z)$, we consider the problem of obtaining a matrix-fraction description of the system, of minimal dimension consistent with the data. [When $H(z)$ is scalar, the task is to obtain polynomials $a(z), b(z)$ of least degree such that $b(z)/a(z)$ interpolates the data (see [1]); we shall subsequently define the multivariable generalization of this problem.] The problem stands in contrast to the various time-domain techniques of rational modeling and system identification. Such classical methods may make use of the Hankel matrix structure [2] to determine various system characteristics, such as the McMillan degree and the controllability indices, which form a special set of integers needed to construct canonical state-space models [2]. In some signal-processing applications, such as seismic modeling and stellar imaging, frequency samples are the natural *a priori* information.

It is known that a certain matrix, called a Loewner matrix, formed from interpolation data, is able to reveal the order of a minimal-degree rational function which interpolates the data [3]. It is our purpose to extend the properties of Loewner matrices to show that not only the degree but also the controllability indices are obtainable. A method is then developed to obtain a coprime matrix-fraction description of an interpolating transfer matrix. A few mathematical preliminaries are in order before we proceed.

2. PRELIMINARIES

Loewner Matrices

Let $H(z)$ be an $r \times m$ proper real rational matrix taking a finite value at each point of $S = (s_1, s_2, \ldots, s_p)$ and $T = (t_1, t_2, \ldots, t_q)$ where S and T are two mutually disjoint, distinct sets of points in the complex plane C. (Generalization to the case when the points are not distinct is possible, but naturally much more complicated. With repeated-point data, we would be given values of H and derivatives of H at the point or points in question.) We define a $pr \times qm$ matrix L, called a Loewner matrix, associated with the values assumed by $H(z)$ on the set $S \cup T$, by

$$L_{ij} = \frac{H(s_i) - H(t_j)}{s_i - t_j} \qquad (i = 1, 2, \ldots, p \quad j = 1, 2, \ldots, q), \quad (2.1)$$

where L_{ij} is an $r \times m$ matrix located in the ith block row and the jth block column of L. Consult [4] for a detailed treatment of these matrices formed from scalar rational functions. We recall now how the Loewner matrix may be used to obtain the McMillan degree and show in the next section how it can also be used to obtain controllability indices (and of course, by duality, the observability indices) of $H(z)$. [The McMillan degree of $H(z)$ (see [2]), is the least dimension of a state-variable realization of $H(z)$; thus if $H(\infty) < \infty$, the McMillan degree is the least dimension of the square matrix A in all decompositions of $H(z)$ as $D + C(zI - A)^{-1}B$. The concept of controllability indices, also explained in [2], is reviewed further below.]

There are several key facts to recall concerning the Loewner matrix. This appears in [3] and [5] for the matrix case. Related results appear in [1]. We shall use ρX to denote the rank of a matrix X.

PROPOSITION 2.1. *Let L be a Loewner matrix associated with interpolating values of a proper rational transfer matrix $H(z)$ and two sets S and T as defined above. Suppose that $\delta[H(z)] \leqslant \min(p, q)$, where $\delta[H(z)]$ denotes the McMillan degree of $H(z)$. Then*

$$\rho L = \delta[H(z)]. \tag{2.2}$$

This theorem says nothing about the construction or the existence of an $H(z)$ satisfying the McMillan-degree constraint when one starts with L. In [3], an existence result (which is extendible to a construction procedure) is established.

In order to present the construction result, we need several definitions. Let L^* denote the Loewner matrix obtained by reassigning the last element of T, viz. t_q, to become an additional and new last element of S. [Thus L^* has one more block row and one less block column than L and the upper left (block p) $\times$ (block $q - 1$) submatrices of L and L^* are identical.] Also, if

$$L = [L_1 \quad L_2] \tag{2.3}$$

with L_1 the first block column of L, define

$$Q = L_2 - [L_1 \quad L_1 \quad \cdots \quad L_1], \tag{2.4}$$

$$R = L_2 \operatorname{diag}[t_2 I \; t_3 I \ldots t_q I] - t_1[L_1 \quad L_1 \quad \cdots \quad L_1]. \tag{2.5}$$

We introduce the following assumptions:

ASSUMPTION 2.1. If $\rho L = n$, then $n \leqslant p$, $n < q$.

ASSUMPTION 2.2. All $n \times n$ block submatrices of L and L^* have rank n.

ASSUMPTION 2.3. $\rho Q = n$.

ASSUMPTION 2.4. $\operatorname{Ker} Q \subset \operatorname{Ker} R$.

Then the key result is:

PROPOSITION 2.2. *Let the Loewner matrix L be defined using interpolation data as above, and suppose Assumptions 2.1 through 2.4 hold. Then there exists a proper transfer-function matrix $H(z)$ of McMillan degree n interpolating the data, and this is the only transfer-function matrix of this or lower degree with the interpolation property. Moreover, if L is formed from a proper transfer-function matrix $H(z)$ of degree n and if Assumption 2.1 holds, then Assumptions 2.2 through 2.4 hold.*

Further theory, of no interest to us here, is available to characterize other interpolants (with nonminimal degree).

Controllability Indices

Let (A, B, C, D) be a minimal state-space description of a real rational transfer-function matrix $H(z)$ [i.e., $H(z) = D + C(zI - A)^{-1}B$ with A of minimal dimension], and let P be the associated controllability matrix, where

$$P = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \qquad \text{with} \quad n = \dim A. \tag{2.6}$$

Upon resolving B into its columns and searching the individual columns of P from left to right, we write down all columns which are linearly independent of the predecessors and discard the remainder. We then reorder the remaining columns to construct a matrix of the form

$$\bar{P} = \big[b_1 \quad Ab_1 \quad \cdots \quad A^{\mu_1 - 1}b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{\mu_2 - 1}b_2 \quad \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{\mu_m - 1}b_m\big]. \tag{2.7}$$

The indices $\mu_1, \ldots, \mu_m$ are called the controllability indices, and

$$\mu = \max_i \{ \mu_i \} \tag{2.8}$$

is termed "the" controllability index. In case some b_i is linearly dependent on $b_1, \ldots, b_{i-1}$, we set $\mu_i = 0$. Since we assume $\{A, B, C, D\}$ to be a minimal description of $H(z)$, the rank of P is known to be $\delta[H(z)]$, the McMillan degree of $H(z)$ [2]. For further discussions of controllability indices, consult References [2, pp. 431–434] and [6, pp. 187–192]. The following easily established fact allows us to compute the controllability indices of a minimal realization of $H(z)$ (which are independent of the particular minimal realization [2, 6]).

PROPOSITION 2.3. *Let P_i represent the partial controllability matrix*

$$P_i = [B \quad AB \quad A^2B \quad \cdots \quad A^{i-1}B], \qquad i = 1, 2, \ldots, n, \tag{2.9}$$

and let α_i be the number of controllability indices of value i. Then

$$\alpha_i = (\rho_i - \rho_{i-1}) - (\rho_{i-1} - \rho_i), \qquad i = 1, 2, \ldots, n, \tag{2.10}$$

where $\rho_i = \rho P_i$ and α_0 is the nullity of B.

We can also obtain the controllability indices by looking at the ranks of similarly defined nested Hankel matrices, the entries of which are defined by the Markov parameters $H(z)$, which are the coefficients in the Laurent series expansion of $H(z)$. As argued in [1], a Hankel matrix is a particular type of Loewner matrix. Therefore it should be no surprise that, as we show in the next section, the controllability indices can be obtained from the ranks of nested Loewner matrices.

The controllability matrices have a second important role, in matrix-fraction descriptions of $H(z)$ [2]. Let $A(z)$ and $B(z)$ be polynomial matrices in z of dimension $m \times m$ and $r \times m$ respectively, and such that

$$H(z) = B(z) A^{-1}(z). \tag{2.11}$$

In case $[B'(z) \quad A'(z)]$ has full rank for all z [or the pair $A(z), B(z)$ is right coprime], deg det $A(z)$ is the McMillan degree of $H(z)$. Among all fractions with minimal determinantal degree for $A(z)$, there are those termed column-proper. In this case, the column degrees of $A(z)$ are minimal, and are given (with arbitrary ordering) by the set of controllability indices

$\mu_1, \mu_2, \ldots, \mu_m$. There holds (subject to appropriate ordering of the columns of A, and this is commented upon below)

$$\lim_{z\to\infty} A(z)\,\mathrm{diag}[z^{-\mu_i}] = \Gamma \tag{2.12}$$

with Γ of full rank. If Σ is a permutation matrix, $B(z)A^{-1}(z) = [B(z)\Sigma][A(z)\Sigma]^{-1}$, so that the ordering of the columns of $A(z)$ is free; typically, the ordering is chosen so that the column degrees are ascending or descending.

The maximum controllability index is evidently the degree of the matrix polynomial $A(z)$ when the pair $A(z), B(z)$ is right-coprime and $A(z)$ is column-proper.

3. CONTROLLABILITY INDICES AND THE LOEWNER MATRIX

In this section, we present a method to compute the controllability indices of (a minimal realization of) an unknown rational transfer matrix from frequency-response samples. This is accomplished through the use of the Loewner matrix. We will show that the ranks of partial or nested Loewner matrices equal those of the corresponding partial controllability matrices. Hence, the controllability indices may be computed from the Loewner matrix through (2.10).

THEOREM 3.1. *Let a Loewner matrix with p block rows and q block columns be defined using interpolation data as described in Section* 2 (*see* 2.1)), *and suppose that Assumptions* 2.1 *through* 2.4 *hold. Further, suppose that the transfer-function matrix $H(z)$ giving rise to the interpolation data has McMillan degree less than* $\min(p, q-1)$, *so that by Proposition* 2.2, *$H(z)$ is the unique interpolating transfer-function matrix of degree ρL. Let m be the column dimension of $H(z)$, and α_i the number of controllability indices with value i. Define $\rho_i = \rho L_i$, where L_i is the partial Loewner matrix*

$$L_i = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1i} \\ L_{21} & L_{22} & \cdots & L_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ L_{p1} & L_{p2} & \cdots & L_{pi} \end{bmatrix} \quad \text{for} \quad i = 1, 2, \ldots, q.$$

Then

$$\alpha_i = (\rho_i - \rho_{i-1}) - (\rho_{i+1} - \rho_i) \quad \text{for} \quad i = 1, 2, \ldots, q \tag{3.1}$$

with

$$\rho_0 = 0, \qquad \rho_j = n \ \textit{for} \ j > n, \quad \textit{and} \quad \alpha_0 = m - \sum_{k=1}^{q} \alpha_k. \tag{3.2}$$

Proof. Assume a minimal state-space description for $H(z)$ as $\{A, B, C, D\}$ with $\dim A = n = \delta[H(z)]$. The sets S and T are disjoint from the set of eigenvalues of A, since by assumption $H(z)$ is finite at all points of S and T. Observe the following rewriting of a Loewner matrix entry:

$$\begin{aligned} L_{ij} &= \frac{H(s_i) - H(t_j)}{s_i - t_j} \\ &= \frac{C(s_i I - A)^{-1}B - C(t_j I - A)^{-1}B}{s_i - t_j} \\ &= \frac{C(s_i I - A)^{-1}[(t_j I - A) - (s_i I - A)](t_j I - A)^{-1}B}{s_i - t_j} \\ &= -C(s_i I - A)^{-1}(t_j I - A)^{-1}B. \end{aligned} \tag{3.3}$$

This suggests the following decomposition for L, see [3]:

$$L = -\begin{bmatrix} C(s_1 I - A)^{-1} \\ C(s_2 I - A)^{-1} \\ \vdots \\ C(s_p I - A)^{-1} \end{bmatrix} \times \left[(t_1 I - A)^{-1}B \quad (t_2 I - A)^{-1}B \quad \cdots \quad (t_q I - A)^{-1}B \right]. \tag{3.4}$$

Thus, the Loewner matrix has a decomposition similar to a Hankel matrix [2]. The two matrices appearing in the product are termed generalized observability and controllability matrices, and they have rank equal to that of L, and thus full column and row ranks, respectively [3]. We now show that ranks of the partial controllability matrices are the same as the ranks of the partial Loewner matrices.

Set $T_i = t_i I - A$. Because the generalized observability matrix has full column rank, it follows that

$$\begin{aligned}
\rho L_i &= \rho\big[T_1^{-1}B \quad \cdots \quad T_{i-1}^{-1}B \quad T_i^{-1}B\big] \\
&= \rho\big[T_i^{-1}\big]\big[T_i T_1^{-1}B \quad \cdots \quad T_i T_{i-1}^{-1}B \quad B\big] \\
&= \rho\big[T_i T_1^{-1}B \quad \cdots \quad T_i T_{i-1}^{-1}B \quad B\big] \\
&= \rho\big[(T_i - t_1 I + t_1 I)T_1^{-1}B \quad \cdots \quad (T_i - t_{i-1}I + t_{i-1}I)T_{i-1}^{-1}B \quad B\big] \\
&= \rho\big[((t_i - t_1)I + T_1)T_1^{-1}B \quad \cdots \quad ((t_i - t_{i-1})I + T_{i-1})T_{i-1}^{-1}B \quad B\big] \\
&= \rho\big[(t_i - t_1)T_1^{-1}B + B \quad \cdots \quad (t_i - t_{i-1})T_{i-1}^{-1}B + B \quad B\big] \\
&= \rho\big[T_1^{-1}B \quad \cdots \quad T_{i-1}^{-1}B \quad B\big].
\end{aligned} \tag{3.5}$$

Repeating the same procedure yields

$$\rho L_i = \rho\big(T_1^{-1}B \quad T_2^{-1}B \quad \cdots \quad B \quad AB\big).$$

Continuing in a like manner, we finally obtain

$$\rho L_i = \rho(B \quad AB \quad \cdots \quad A^{i-1}B) = \rho P_i.$$

We have shown that the ranks of the partial Loewner matrices can be used to obtain the ranks of the partial controllability matrices. We have therefore proven the theorem, except for some minor details. The first concerns how to choose ρ_i for $i > n$. Since the rank of $P_n = P$ is n, and $\operatorname{rank} P_i$ cannot increase for $i > n$, it is apparent that $\rho_i = n$ for $i > n$. Secondly, it is known that the m controllability indices correspond to the column degrees in a column-reduced right-coprime matrix-fraction decomposition. Since it is conceivable that the strictly proper part of $H(z)$ is singular, which means α_0 has a nonzero value, it is apparent that we should choose α_0

after we compute all the other controllability indices. Namely,

$$\alpha_0 = m - \sum_{i=1}^{q} \alpha_i,$$

and the proof is complete. ■

REMARK 3.1. One can easily check that $\alpha_0 = m - \rho_1$.

REMARK 3.2. Like the Hankel matrix, the Loewner matrix can be used to obtain the McMillan degree and the controllability indices of a proper rational transfer matrix. If we similarly look at the ranks of partial block rows of the Loewner matrix, we can obtain the observability indices.

4. MATRIX-FRACTION DESCRIPTION

In [3], the problem was studied of passing from interpolation data, arranged in a Loewner matrix, to a minimal *state-variable realization* of a minimal-degree interpolating transfer-function matrix. In this section, we study a parallel problem: passing from the Loewner matrix to a right-coprime matrix-fraction description of a minimal-degree interpolating matrix with column-proper denominator. In [3] and here, we adopt the assumptions of Section 2 on the Loewner matrix guaranteeing uniqueness of the minimal-degree interpolating matrix. The coprimeness and column-properness properties here parallel the minimality of the state-variable description in [3]. Another approach to the problem treated in this section can be found in [7]. This requires the easy determination of a controllable pair $[F, G]$ from the data, the computation of a right-coprime polynomial pair $W(z), \Theta(z)$ with

$$(zI - F)^{-1}G = W(z)\Theta^{-1}(z)$$

and $\Theta(z)$ column-proper, and the construction from $\Theta(z)$, through constant transformation, of $\tilde{\Theta}_f(z)$, submatrices of which yield the desired matrix-fraction description.

Our construction procedure will rest on the following proposition.

PROPOSITION 4.1. *Consider interpolation data and an associated Loewner matrix L defined as per* (2.1). *Let Assumptions* 2.1 *through* 2.4 *hold. Further, let* $\bar{H}(z) = \bar{B}(z)\bar{A}^{-1}(z)$ *be any transfer-function matrix interpolat-*

ing the data with deg $\bar{A} < q$, deg $\bar{B} < q$, *and* $\bar{A}(s_i)$ *and* $\bar{A}(t_j)$ *nonsingular for all* i *and* j *(conditions which are satisfied by a coprime column-proper fractional representation of the minimal-degree interpolating fraction). Define*

$$l_j(z) = \frac{\prod_{i=1}^{q}(z - t_i)}{z - t_j} \tag{4.1}$$

and

$$\begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} = \operatorname{diag}\left[l_j^{-1}(t_j)\right] \begin{bmatrix} I & t_1 I & \cdots & t_1^{q-1} I \\ I & t_2 I & \vdots & t_2^{q-1} I \\ \vdots & \vdots & & \vdots \\ I & t_q I & \vdots & t_q^{q-1} I \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_q \end{bmatrix}, \tag{4.2}$$

where

$$\bar{A}(z) = \sum_{j=1}^{q} \bar{A}_j z^{j-1}. \tag{4.3}$$

Then

(i) *one has*

$$l_j(t_j) E_j = \bar{A}(t_j); \tag{4.4}$$

(ii) *one has*

$$\bar{A}(z) = \sum_{j=1}^{q} E_j l_j(z); \tag{4.5}$$

(iii) *one has*

$$\bar{B}(z) = \sum_{j=1}^{q} \bar{H}(t_j) E_j l_j(z); \tag{4.6}$$

(iv) E_j *is nonsingular;*

(v) *one has*

$$\sum_{j=1}^{q} \frac{\overline{H}(z) - \overline{H}(t_j)}{z - t_j} E_j = 0; \tag{4.7}$$

(vi) *one has*

$$L \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} = 0. \tag{4.8}$$

Proof. (i): From (4.2), $\overline{A}(t_j) = E_j l_j(t_j)$.

(ii): In view of the definition (4.1), we see that if $A(z) = \sum_{j=1}^{q} E_j l_j(z)$, then $\overline{A}(t_j) = A(t_j)$ for $j = 1, 2, \ldots, q$; since deg $A(z) < q$, deg $\overline{A}(z) < q$, it follows that $\overline{A}(z) = A(z)$, i.e., (4.5) holds

(iii): Since $\overline{B}(t_j) = \overline{H}(t_j)\overline{A}(t_j) = \overline{H}(t_j)E_j l_j(t_j)$, an argument identical to that for (ii) yields (iii).

(iv) follows from (4.4) and the fact that $\overline{A}(t_j)$ is nonsingular.

(v):

$$\sum_{j=1}^{q} \frac{\overline{H}(z) - \overline{H}(t_j)}{z - t_j} E_j = \frac{1}{\prod_{j=1}^{q}(z - t_j)} \sum_{j=1}^{q} \left[\overline{H}(z) l_j(z) E_j - \overline{H}(t_j) l_j(z) E_j \right]$$

$$= \frac{1}{\prod_{j=1}^{q}(z - t_j)} \left[\overline{H}(z) \overline{A}(z) - \overline{B}(z) \right]$$

by (ii) and (iii)

$$= 0.$$

(vi): Immediate from (4.7) with $z = s_1, s_2, \ldots$. ∎

We remark that results like these for the case of scalar $\overline{H}(z)$ are to be found in [1] and [8–10].

We shall also need a form of converse of Proposition 4.1.

PROPOSITION 4.2. *Consider interpolation data defined at distinct points* $s_1, \ldots, s_p, t_1, \ldots, t_q$ *by quantities* $H(s_i)$ *and* $H(t_j)$, *and let* L *be the associated Loewner matrix defined as per* (2.1). *Let Assumptions* 2.1 *through* 2.4 *hold. Then there exist square matrices* E_i *satisfying* (4.8) *such that* E_i *is* *nonsingular for each i and $\sum_{j=1}^{q} E_j(s_i - t_j)^{-1}$ is nonsingular for each i. For any such E_i, define*

$$\bar{A}(z) = \sum_{j=1}^{q} E_j l_j(z), \tag{4.9}$$

$$\bar{B}(z) = \sum_{j=1}^{q} H(t_j) E_j l_j(z), \tag{4.10}$$

and

$$\bar{H}(z) = \bar{B}(z)\bar{A}^{-1}(z). \tag{4.11}$$

Then $\bar{H}(z)$ solves the interpolation problem, i.e.,

$$\bar{H}(s_i) = H(s_i), \qquad i = 1, \dots, p, \tag{4.12a}$$

$$\bar{H}(t_j) = H(t_j), \qquad j = 1, \dots, q. \tag{4.12b}$$

Proof. By proposition 2.2, there exists a proper transfer-function matrix $\tilde{H}(z)$ of McMillan degree n interpolating the data. By Assumption 2.1, $n < q$ and so $\tilde{H}(z)$ has a coprime column-proper fractional representation $\tilde{B}(z)\tilde{A}^{-1}(z)$ with $\deg \tilde{A} < q$, $\deg \tilde{B} < q$. By Proposition 4.1, there exist nonsingular $\tilde{E}_i$, defined using (4.2) but with $\tilde{A}_i$ replacing $\bar{A}_i$, such that (4.8) holds.

Because $\tilde{H}(z)$ interpolates the data, $\tilde{H}(s_i)$, $i = 1, \dots, p$, is finite. Because the description $\tilde{B}(z)\tilde{A}^{-1}(z)$ is minimal, $\tilde{A}(s_i)$, $i = 1, \dots, p$, is finite. Now $\tilde{A}(z) = \sum_{j=1}^{q} \tilde{E}_j l_j(z)$ [see (4.5)], and

$$\sum_{j=1}^{q} \tilde{E}_j \frac{\prod_{k=1}^{q}(s_i - t_k)}{s_i - t_j}$$

is nonsingular, i.e., $\sum_{j=1}^{q} \tilde{E}_j(s_i - t_j)^{-1}$ is nonsingular. This proves the existence claim.

Now let E_i be *any* set of matrices satisfying (4.8), with E_i for $i = 1, \ldots, q$ and $\sum_{j=1}^{q} E_j(s_i - t_j)^{-1}$ for $i = 1, \ldots, p$ nonsingular, i.e., not necessarily those defined by $\bar{H}(z)$. By (4.9) through (4.11), $\bar{B}(t_j) = \bar{H}(t_j)\bar{A}(t_j)$, or

$$\sum_{j=1}^{q} H(t_j)E_j l_j(t_j) = \bar{H}(t_j)\sum_{j=1}^{q} E_j l_j(t_j),$$

or

$$H(t_j)E_j = \bar{H}(t_j)E_j,$$

or

$$H(t_j) = \bar{H}(t_j).$$

Next, we must check that $H(s_i) = \bar{H}(s_i)$ for $i = 1, 2, \ldots, p$. Observe that

$$\begin{aligned}
&\sum_{j=1}^{q} \frac{\bar{H}(z) - \bar{H}(t_j)}{z - t_j} E_j \\
&\quad = \frac{1}{\prod_{j=1}^{q}(z - t_j)} \sum_{j=1}^{q} \left[\bar{H}(z) l_j(z) E_j - \bar{H}(t_j) l_j(z) E_j\right] \\
&\quad = \frac{1}{\prod_{j=1}^{q}(z - t_j)} \left[\bar{H}(z)\bar{A}(z) - \sum_{j=1}^{q} H(t_j) l_j(z) E_j\right] \qquad \text{by (4.12b)} \\
&\quad = 0 \qquad \text{by (4.10) and (4.11)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=1}^{q} \frac{\bar{H}(s_i)E_j}{s_i - t_j} &= \sum_{j=1}^{q} \frac{\bar{H}(t_j)E_j}{s_i - t_j} \\
&= \sum_{j=1}^{q} \frac{H(t_j)E_j}{s_i - t_j} \qquad \text{by (4.12b)} \\
&= \sum_{j=1}^{q} \frac{H(s_i)E_j}{s_i - t_j} \qquad \text{by (4.8)}.
\end{aligned}$$

Since $\sum_{j=1}^{l} E_j(s_i - t_j)^{-1}$ is nonsingular, (4.12a) follows. ∎

The machinery is available now to explain the construction of a coprime, column-proper fractional description of the minimal-degree interpolating transfer-function matrix.

The point to focus on above in the first instance is (4.8) and (4.2). Objects in the kernel of L [see (4.8)] define the coefficients of the denominator polynomial of an interpolating $\overline{H}(z)$, via (4.2). Evidently, there holds for any $\overline{H}(z)$ satisfying the restrictions of Proposition 4.1

$$L \operatorname{diag}\left[l_j^{-1}(t_j)\right] V \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \\ \vdots \\ \overline{A}_q \end{bmatrix} = 0, \tag{4.13}$$

where V is the block Vandermonde matrix set out in (4.2). Such an equality necessarily holds for the denominator of the unique minimal-degree interpolating transfer-function matrix, when this matrix is represented by a coprime column-proper fractional description.

Conversely, if we can find $\overline{A}_i$ such that (4.13) holds, with $\overline{A}(s_k)$ and $\overline{A}(t_j)$ nonsingular for all s_k and t_j, and such that $\overline{A}(z)$ is column-proper with the correct set of column indices, we can construct the minimal-degree interpolating function: we use (4.3) to define $\overline{A}(z)$, (4.2) to define E_j, and (4.6) to define $\overline{B}(z)$; nonsingularity of $\overline{A}(s_k)$ and $\overline{A}(t_j)$ is equivalent to nonsingularity of $\sum_{j=1}^{q} E_j(s_k - t_j)^{-1}$ and E_j respectively.

Finding the $\overline{A}_i$ is a standard task in linear systems. Rewrite (4.13) as

$$\Sigma M_i \overline{A}_i = 0 \tag{4.14}$$

with obvious definition of M_i. Define $M(z) = \sum_{i=1}^{q} M_i z^{q-i}$. Then (4.14) is equivalent to

$$M(z)\overline{A}(z) = 0. \tag{4.15}$$

The determination of a minimal-column-degree solution to (4.15) is standard; see e.g. [2, pp. 381–386], where some of the argument is couched in a dual framework. Solutions are unique to within right multiplication by certain constant nonsingular matrices. Since the solution defined by the minimal-degree interpolating function has nonsingular $\overline{A}(s_k)$ and $\overline{A}(t_j)$, any minimal-column-degree solution of (4.15) has this property.

It is straightforward to express these ideas using constant rather than polynomial matrices. Suppose the column degrees of $\bar{A}(z)$ are $\mu_1, \mu_2, \ldots, \mu_m$ (in *ascending* order). Then (4.13) will have the structure

$$L\,\mathrm{diag}\big[l_j^{-1}(t_j)\big]\,V \begin{bmatrix} \underline{a}_{01} & \underline{a}_{02} & \cdots & \underline{a}_{0m} \\ \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1m} \\ \vdots & \vdots & & \cdot \\ \underline{a}_{\mu_1 1} & \underline{a}_{\mu_1 2} & & \cdot \\ 0 & \vdots & & \cdot \\ \cdot & \underline{a}_{\mu_2 2} & & \cdot \\ \cdot & 0 & & \cdot \\ \cdot & \vdots & & \underline{a}_{\mu_m m} \end{bmatrix} = 0. \qquad (4.16)$$

Knowing the values of the μ_i, it is a trivial matter to determine $\underline{a}_{ij}$ from $\ker\{L\,\mathrm{diag}[l_j^{-1}(t_j)]V\}$.

The column-properness property of $\bar{A}(z)$ holds if and only if $[\underline{a}_{\mu_1 1}\ \underline{a}_{\mu_2 2}\ \cdots\ \underline{a}_{\mu_m m}]$ is nonsingular. As argued above, it is not necessary to verify (or otherwise allow for) the condition that the $\bar{A}(s_k)$ and $\bar{A}(t_j)$ are nonsingular when the minimal-column-degree property for $\bar{A}(z)$ is secured.

Example. Let the following transfer matrix $H(z)$ be used to generate frequency samples on two sets S and T:

$$H(z) = \begin{bmatrix} \dfrac{1}{z+2} & \dfrac{2z}{z+2} \\ \dfrac{1}{z-2} & \dfrac{z+1}{z+2} \end{bmatrix}$$

with $S = \{1, -1\}$ and $T = \{3, -3, 4\}$. If we form a Loewner matrix using frequency samples of $H(z)$, we can compute the controllability indices, and thus the McMillan degree of $H(z)$. There results

$$\mu = \{1, 2\} \quad \text{and} \quad \delta[H(z)] = 3.$$

The first step toward obtaining the parameters of a matrix-fraction decomposition for $H(z)$ involves forming the system of equations in (4.16). If we form this system of equations, then scale to obtain integer elements, we obtain

$$\begin{bmatrix} 1 & -4 & -2 & 8 & 4 & -16 \\ -9 & -1 & -18 & 2 & -36 & -4 \\ 3 & -12 & -6 & 24 & 12 & -48 \\ -3 & -3 & -6 & 6 & -12 & -12 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \\ 0 & \beta_5 \\ 0 & \beta_6 \end{bmatrix} = 0.$$

We need to identify

$$\underline{a}_{01} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \qquad \underline{a}_{11} = \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \qquad \underline{a}_{02} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad \text{etc.}$$

and

$$\bar{A}_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}, \qquad \bar{A}_2 = \begin{pmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{pmatrix}, \qquad \bar{A}_3 = \begin{pmatrix} 0 & \beta_5 \\ 0 & \beta_6 \end{pmatrix}.$$

It is easy to verify that the following is a solution, with the condition that

$$\begin{pmatrix} \alpha_3 & \beta_5 \\ \alpha_4 & \beta_6 \end{pmatrix}$$

is nonsingular:

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \\ 0 & \beta_5 \\ 0 & \beta_6 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 2 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So

$$\bar{A}(z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & -4 \\ 2 & 0 \end{pmatrix}.$$

We can check that $\det \bar{A}(z) = -(z+2)^2(z-2)$.]

In order to compute $\bar{B}(z)$, we need the quantities E_j. We have

$$l_1(z) = (z+3)(z-4), \quad l_1(t_1) = -6,$$
$$l_2(z) = (z-3)(z-4), \quad l_2(t_2) = 42,$$
$$l_3(z) = (z+3)(z-3), \quad l_3(t_3) = 7.$$

It is straightforward to obtain from (4.4)

$$E_1 = \begin{bmatrix} 0 & -\frac{5}{6} \\ \frac{5}{6} & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & \frac{5}{42} \\ -\frac{1}{42} & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & \frac{12}{7} \\ \frac{6}{7} & 0 \end{bmatrix},$$

and then from (4.6)

$$\bar{B}(z) = \begin{bmatrix} -1 & -\frac{1}{6} \\ -\frac{2}{3} & -\frac{5}{6} \end{bmatrix} l_1(z) + \begin{bmatrix} -\frac{1}{7} & -\frac{5}{42} \\ -\frac{1}{21} & -\frac{1}{42} \end{bmatrix} l_2(z) + \begin{bmatrix} \frac{8}{7} & \frac{2}{7} \\ \frac{5}{7} & \frac{6}{7} \end{bmatrix} l_3(z)$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}.$$

[It is trivial to check that $H(z) = \bar{B}(z)\bar{A}^{-1}(z)$.]

5. CONCLUSIONS

Time-domain modeling techniques, such as Prony's method and the Padé approximation, all use time-domain data in the generation of rational models, the typical information being samples of the impulse-response matrix. In this work, we have demonstrated the use of Loewner matrices to tackle the problem of rational modeling using frequency samples. The Loewner matrix has been shown to possess useful information, such as the controllability indices of a proper rational matrix function, and allows construction of a matrix-fraction representation of the minimal degree interpolating transfer-function matrix when this exists.

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