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Hermitian pencils and output feedback stabilization of scalar systems

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Necessary and sufficient semi-algebraic conditions for (*a*) the stabilization of scalar transfer functions, and (*b*) the assignability of real poles by static output feedback, are given in terms of the Weierstrass invariants of an associated hermitian matrix pencil. An explicit graphical test for output feedback stabilizability is derived which is equivalent to the Nyquist criterion.

1. Introduction

In this paper the following two output feedback stabilization problems are considered:

(*a*) given a strictly proper scalar rational transfer function $g = p/q$, when does there exist a constant output feedback gain $k \in \mathbb{R}$ such that $g/(1 + kg)$ is stable?

(*b*) given a strictly proper scalar rational transfer function $g = p/q$, when does there exist a constant output feedback gain $k \in \mathbb{R}$ such that $q + kp$ has only real distinct zeros (or only negative real distinct zeros)?

Of course, problem (*a*) is an old one and amounts to (non-dynamic or) static output feedback stabilization. While graphical approaches to the stability of transfer functions, using root locus techniques or the Nyquist criterion are well known and do appear in almost any classical textbook on control, they are usually not applied to obtain necessary and sufficient conditions for output feedback stabilization of linear systems. Despite some more recent attempts towards a solution of problem (*a*), Byrnes and Crouch (1985), Crouch and Cheng (1989), a necessary and sufficient set of semi-algebraic conditions for static output feedback stabilization appears to be unknown. This is even more true for problem (*b*). While the task of eliminating possible (damped or undamped) oscillations by output feedback is clearly an important task in control, a characterization of the class of transfer functions for which this is possible is apparently unknown. Of course, by the Tarski–Seidenberg theorem, the set of degree n rational transfer functions which can be stabilized by static output feedback is semi-algebraic and tests for a transfer function to belong to the set of stabilizable systems can be given in terms of decision algebra; cf. Anderson *et al.* (1975). Similarly, problem (*b*) can also be cast as a decision algebra problem, which can be solved in principle on a computer, using elimination of quantifiers. However, these tests are extremely time-consuming. There is, therefore, the need to derive more easily verifiable tests for problems (*a*) and (*b*).

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In this paper we develop an algebraic approach to problems (*a*) and (*b*) which is based on an analysis of hermitian matrix pencils. To each transfer function we associate a pencil $K + \rho L$ of hermitian matrices with the property that $K + \rho L$ is positive definite for some real parameters ρ if and only if problem (*a*) or problem (*b*) is solvable. Necessary and sufficient conditions for static output feedback stabilization or for real pole assignability (*b*) are given in terms of the Weierstrass invariants, i.e. the real generalized eigenvalues of $K + \rho L$ and the associated sign characteristics. The conditions are easily verifiable on a computer and it should not be hard to derive numerically efficient tests, using our results. An interpretation of the generalized eigenvalues and sign characteristics in terms of the frequency response of the system leads to a graphical characterization of output feedback stabilizability.

We proceed as follows. Before launching into the theory of hermitian pencils in § 3 we first introduce the output feedback control problems and some of the previous results in more detail in § 2. Using the bezoutian, both problems (*a*) and (*b*) are shown to be equivalent to the positive definiteness of an associated hermitian matrix polynomial $\rho^2 A_0 + \rho A_1 + A_2$ for some real parameter ρ. The invariance properties of the bezoutian are reviewed and preliminary results on the generalized eigenvalues of $\rho^2 A_0 + \rho A_1 + A_2$ are given.

In § 3 we review as well as extend some of the early results on the classification and signature distribution of pencils of hermitian matrices. One of the main results in this section is Theorem 3.4, which expresses the signature values of a hermitian matrix polynomial $\rho^2 A_0 + \rho A_1 + A_1$ in terms of the generalized eigenvalues and sign characteristics of an associated hermitian pencil $K + \rho L$.

The results of § 3 are then applied in § 4 to solve the output feedback problems (*a*) and (*b*). Necessary and sufficient conditions are derived in terms of the generalized eigenvalues and sign characteristics of an associated $2n \times 2n$ hermitian pencil $K + \rho L$ (Theorems 4.2, 4.6). The generalized eigenvalues and sign characteristics of $K + \rho L$ which play such a crucial role in our analysis also have an appealing system-theoretic interpretation: they refer to the real intersection points and directions in which the frequency response curve crosses the real axis. This leads to a simple graphical test for output feedback stabilization, which is equivalent to the Nyquist test.

2. The output feedback control problems

To explain why the theory of pencils is important to the output feedback control problems we first review some elementary facts about the bezoutian as well as explaining the connection to output feedback invariants, such as the breakaway points. For a more detailed discussion on the subsequent material we refer to Fuhrmann and Helmke (1988), Helmke and Fuhrmann (1989).

Given two polynomials

$$p(s) = \sum_{i=0}^{n} p_i s^i \quad \text{and} \quad q(s) = \sum_{i=0}^{n} q_i s^i$$

the $(n \times n)$ bezoutian of p and q is the $n \times n$ symmetric matrix

$$B(p, q) = (b_{ij})_{i,j=0}^{n-1} \tag{2.1}$$

defined by

$$\frac{p(z)q(w) - p(w)q(z)}{w - z} = \sum_{i,j=0}^{n-1} b_{ij} z^{i-1} w^{j-1} \tag{2.2}$$

If $g = p/q$ is the unique irreducible representation of a rational function g with $q(s)$ monic, then we write $B(g)$ for $B(p, q)$.

Thus $B(p, q)$ is linear in p and q, furthermore $B(p, q)$ is invertible if and only if $\max(\deg p, \deg q) = n$ and p, q are coprime. If $p = q'$ is the derivative of the degree n polynomial q, then the signature of $B(q', q)$ gives the number of distinct real roots of q, and hence is always non-negative.

An important property of the bezoutian $B(g)$ of a transfer function is its invariance under output feedback:

$$B\left(\frac{g}{1 - kg}\right) = B(g)$$

for all $k \in \mathbb{R}$. Moreover, for strictly proper transfer functions g, h, the bezoutian is a complete invariant for static output feedback

$$B(g) = B(h) \Leftrightarrow h = \frac{g}{1 - kg} \quad \text{for some} \quad k \in \mathbb{R} \tag{2.3}$$

(Fuhrmann and Helmke 1988, Helmke and Fuhrmann 1989).

Another equivalent set of invariants for static output feedback is defined by the zeros and breakaway points of a transfer function. The *breakaway points* of a real transfer function $g(s) = p(s)/q(s) \in \mathbb{R}(s)$ are defined as the real and complex roots of

$$p'(s)q(s) - q'(s)p(s) = 0 \tag{2.4}$$

These points are the branching points of the corresponding root loci. If $s_0 \in \mathbb{C}$ is a breakaway point for g then $g(s_0)$ is called a *breakaway value*.

The relevance of the breakaway points is clarified by the following result of Byrnes and Crouch (1985), see also Fuhrmann and Helmke (1988) and Helmke and Fuhrmann (1989).

Theorem 2.1 (Byrnes and Crouch 1985): *The real and complex zeros and breakaway points of a strictly proper transfer function $g(s) \in \mathbb{R}(s)$ are a complete set of invariants for the full group of static output feedback transformations*

$$g(s) \mapsto \frac{\alpha g(s)}{1 - kg(s)}, \quad \alpha \in \mathbb{R} - \{0\}, \quad k \in \mathbb{R}$$

For us the following lemma will be useful.

Lemma 2.2: *Let $g = p/q \in \mathbb{R}(s)$ be strictly proper of McMillan degree n and with relative degree $r \geq 1$. Then (2.4) has $2n - r - 1$ roots, counted with multiplicities. $s_0 \in \mathbb{C}$ is a simple breakaway point if and only if*

$$g'(s_0) = 0, \quad g''(s_0) \neq 0 \tag{2.5}$$

The set of transfer functions of McMillan degree n which has only simple breakaway points is a generic class.

Proof: The first statement follows from the fact that $p'q - pq'$ has degree $2n - r - 1$. The second statement follows from the identity

$$g'(s) = \frac{p'(s)q(s) - p(s)q'(s)}{q(s)^2} \tag{2.6 a}$$

$$g'' = \frac{(p''q - pq'')q - 2(p'q - pq')q'}{q^3} \tag{2.6 b}$$

Finally, the genericity result follows since the class of transfer functions $g(s)$ with non-degenerate critical points (2.5) is open and dense. □

2.1. *Output feedback stabilization*

The classical output feedback stabilization problem is this:

> Given a strictly proper real rational transfer function $g = p/q$, when does there exist $k \in \mathbb{R}$ such that $g/(1 + kg)$ is stable?

Of course, this problem is an old one in systems theory and graphical approaches to the stability problem involving, for example, the root locus or the Nyquist criterion, are well known.

For example, a well-known sufficient root locus condition for output feedback stabilizability is that the system is minimum phase and has relative degree 1, i.e. the system has $n - 1$ zeros in the open left half-plane, where $n = \deg q$. In this case, for sufficiently large positive or negative values of k, the zeros of $q + kp$ are also in the left half-plane.

Furthermore, by the Tarski–Seidenberg theorem, the set of rational transfer functions $g = p/q$ of fixed McMillan degree which can be stabilized by constant output feedback $k \in \mathbb{R}$ is semialgebraic and thus is described by finitely many polynomial equations and inequalities in the coefficients of p and q see Anderson *et al.* (1975). The polynomials describing the semi-algebraic set of stabilizable systems can be found using a technique from decision algebra called elimination of quantifiers. Thus, the output feedback stabilization problem can be cast as a decision algebra problem. Elimination of quantifiers is, however, in general an extremely time consuming task and there is therefore the need to develop more specific and more easily verifiable tests for output feedback stabilizability. This will be done in § 4 where we develop an algebraic approach which is based on the analysis of Hermitian pencils, given in § 3.

For the sake of definiteness we deal only with the continuous time case, where $q + kp$ is required to be Hurwitz. The results hold *mutatis mutandis* also for discrete-time systems.

For $\omega \in \mathbb{R}$ let

$$q(\mathrm{i}\omega) = q_+(\omega) + \mathrm{i}q_-(\omega) \tag{2.7}$$

Thus $q_+(\omega)$ and $q_-(\omega)$ are even and odd polynomials of ω respectively. By the Hurwitz test, a monic polynomial $q(s) \in \mathbb{R}[s]$ of degree n is Hurwitz if and only if the bezoutian $B(q_+, q_-)$ is positive definite. Moreover, the signature of $B(q_+, q_-)$ is equal to the number of zeros of $q(s)$ with negative real part minus the number of zeros of $q(s)$ with positive real part; see for example Heinig and Rost (1984).

A property of the bezoutian $B(q_+, q_-)$ which is crucial to our approach is the linearity of $B(q_+, q_-)$ in q_+ and q_-. It is this linearity property of the bezoutian which allows us to apply the theory of matrix pencils to the above output feedback problems.

For any $k \in \mathbb{R}$ we consider the family of symmetric matrices

$$\begin{aligned} B(k) &:= B(q_+ + kp_+, q_- + kp_-) \\ &= B(q_+, q_-) + k(B(q_+, p_-) + B(p_+, q_-)) + k^2 B(p_+, p_-) \end{aligned} \tag{2.8}$$

Thus, $B(k)$ is a degree 2 polynomial in k and $B(k) > 0$ if and only if $q + kp$ is Hurwitz. This shows that problem (a) is solvable if and only if the quadratic pencil (2.8) is positive definite for some $k \in \mathbb{R}$. The following remarks and notations will be useful in § 4.

Remark: Without loss of generality we can assume in the sequel that $B(0) = B(q_+, q_-)$ is invertible, i.e. that $q(s)$ has no zeros on the imaginary axis.

Given a real rational function $g(s) = p(s)/q(s)$ and $\omega \in \mathbb{R}$ we have a decomposition

$$g(i\omega) = g_+(\omega) + ig_-(\omega) \tag{2.9}$$

with rational functions

$$g_+(s) = \frac{p_+(s)q_+(s) + p_-(s)q_-(s)}{q_-(s)^2 + q_+(s)^2} \tag{2.10 a}$$

$$g_-(s) = \frac{p_-(s)q_+(s) - p_+(s)q_-(s)}{q_-(s)^2 + q_+(s)^2} \tag{2.10 b}$$

□

Definition:

(*a*) $s \in \mathbb{C}$ is called a ***generalized breakaway point*** if and only if $g_-(s) = 0$, i.e.

$$p_-(s)q_+(s) - p_+(s)q_-(s) = 0 \tag{2.11}$$

(*b*) The values $g(s)$ at generalized breakaway points $s \in \mathbb{C}$ are called ***generalized breakaway values***. □

Thus $\omega \in \mathbb{R}$ is a real breakaway point if and only if $g(i\omega)$ is real. The degree of $p_-q_+ - p_+q_-$ depends in a rather complicated way on the coefficients of p and q; for generic transfer functions the degree is given in the following table (Fig. 1, $r := \deg p$).

$r = \deg p$, $n = \deg q$	r even	r odd
n even	$n + r - 1$	$n + r$
n odd	$n + r$	$n + r - 1$

Figure 1. Degree of $p_-q_+ - p_+q_-$.

Remark: The generalized breakaway points are certainly invariants for output feedback. However it is not true that the real and complex zeros and generalized breakaway points form a complete set of invariants for output feedback. In fact, consider any transfer function $g = p/q$ with p and q even polynomials and hence $p = p_+$, $q = q_+$. Then $p_-q_+ - p_+q_-$ is the zero polynomial and the transfer functions p/q, with p, q arbitrary even polynomials and p fixed, all correspond to the same p and $p_-q_+ - q_-p_+$. Thus the situation here is more complicated. □

2.2. *Real pole assignability*

Here we are interested in the following problem:

> Given a strictly proper real rational transfer function $g = p/q$ of McMillan degree n, when does there exist a real gain $k \in \mathbb{R}$ such that $q + kp$ has only real distinct zeros (where q is assumed to be monic)?

Again, by the Tarski–Seidenberg theorem, the set of such transfer functions is semi-algebraic and tests for a transfer function to belong to this class can be developed using elimination of quantifiers. However, the Bezoutian enables us again to find equivalent but more easily verifiable tests.

It is a classical theorem of Sylvester and Hermite that a monic degree n polynomial q has n real roots if and only if the bezoutian $B(q', q)$ is positive definite. Thus, in our case of interest, $q + kp$ has n real roots if and only if the $n \times n$ bezoutian $B(q' + kp', q + kp)$ is positive definite; more generally the signature of $B(q' + kp', q + kp)$ gives the number of real *distinct* roots of $q(s) + kp(s) = 0$. Let

$$B(k) := B(q' + kp', q + kp) \tag{2.12}$$

Hence

$$B(k) = A_0 + A_1k + A_2k^2 \tag{2.13}$$

with real symmetric $n \times n$ matrices

$$A_0 = B(q', q) \tag{2.14 a}$$

$$A_1 = B(p', q) + B(q', p) \tag{2.14 b}$$

$$A_2 = B(p', q) \tag{2.14 c}$$

Thus again, we are asked to find the conditions under which the degree 2 polynomial $B(k)$ is positive definite for some $k \in \mathbb{R}$. We conclude this paragraph with a number of remarks and results which will be useful later on.

Remark 1: If we are only interested in $q + kp$ having all real zeros, with repeated zeros being permitted, then the condition is that $A_0 + kA_1 + k^2A_2 \geqslant 0$ for some $k \in \mathbb{R}$; the rank is the number of distinct zeros. □

Remark 2: Without loss of generality we can assume that $A_0 = B(q, q')$ is non-singular. For if not, i.e. $\det B(q, q') = 0$, q and q' have a common zero. Since p and q are coprime, an easy argument involving Gauss' Lemma shows that $q(s) + kp(s) \in \mathbb{R}(k)[s]$ is irreducible and hence separable in $\mathbb{R}(k)[s]$ (Brockett 1983). Thus the discriminant $\det B(q + kp, q' + kp') \in \mathbb{R}(k)$ is non-zero and therefore for almost all $k \in \mathbb{R}$, $q + kp$ has only simple roots. Then choose

$k_0 \in \mathbb{R}$ such that $q + k_0 p$ has simple zeros and consider the shifted matrix polynomial

$$\tilde{B}(k) = B(k + k_0)$$

□

From now on we assume that A_0 is invertible, i.e. that $q(s)$ has only simple (real or complex) zeros.

In order to see how the signature of $B(k)$ changes as k goes from $-\infty$ to $+\infty$ it is obviously of interest to analyse the zeros of $\det B(k)$. The following results are important for our analysis in § 4.

Lemma 2.3: $\det B(k) = 0$ *if and only if* $-1/k$ *is a breakaway value.* $g = p/q$ *has at most* $2 \deg p$ *breakaway values, counted with multiplicities.* g *has exactly* $2 \deg p$ *breakaway values if and only if* p *and* p' *are coprime. The number of breakaway points is equal to the number of breakaway values only if* g *has relative degree* 1.

Proof: $\det B(k) = 0$ if and only if $q' + kp'$ and $q + kp$ have a common zero $s_0 \in \mathbb{C}$. Thus,

$$k = -\frac{q(s_0)}{p(s_0)} = -\frac{q'(s_0)}{p'(s_0)}$$

or equivalently, $g(s_0) = -1/k$ and $q'(s_0)p(s_0) - p'(s_0)q(s_0) = 0$. To show that there are at most $2 \deg p$ breakaway values, consider

$$B(k) = A_0 + A_1 k + A_2 k^2$$

where

$$A_2 = \begin{bmatrix} A_{11}^{(2)} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{12}^{(1)\mathrm{T}} & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ A_{12}^{(0)\mathrm{T}} & A_{22}^{(0)} \end{bmatrix}$$

where $A_{11}^{(2)}$ is equal to the $\deg p \times \deg p$ bezoutian of (p', p). Thus

$$\deg \det B(k) = 2 \operatorname{rank} A_{11}^{(2)} \leqslant 2 \deg p$$

with equality if and only if p' and p are coprime. Thus the number of distinct zeros of p is equal to one half the number of breakaway values, counted with multiplicities. By Lemma 2.2, g has $n + \deg p - 1$ breakaway points and thus

$$2 \deg p \geqslant 2 rk A_{11}^{(2)} = n + \deg p - 1$$

implies that p/q has relative degree one (since g is strictly proper!). □

Lemma 2.3 suggests making a change of variables

$$\rho = 1/k \tag{2.15}$$

and to consider the polynomial matrix function in ρ

$$\begin{aligned}\Delta(\rho) &= \rho^2 A_0 + \rho A_1 + A_2\\ &= \rho^2 B(1/\rho)\end{aligned} \tag{2.16}$$

In the following lemma a few basic properties of $\Delta(\rho)$ are summarized.

Lemma 2.4:

(*a*) *$\rho = 0$ is a multiple root of* $\det \Delta(\rho) = 0$. *The multiplicity of $\rho = 0$ is twice the relative degree of g.*

(*b*) Let $\rho \neq 0$. *Then* $\det \Delta(\rho) = 0$ *if and only if $-\rho$ is a breakaway value of g.*

(*c*) $\det \Delta(\rho)$ *has* $2n - 1$ *distinct roots if and only if q has* $2n - 2$ *distinct breakaway values.*

Proof: For $\rho \neq 0$, Lemma 2.3 implies

$$\begin{aligned}\det \Delta(\rho) &= \rho^{2n} \det B(1/\rho)\\ &= c \cdot \rho^{2n-2\deg p} + \text{higher order terms}\end{aligned}$$

Hence the multiplicity of zero is $2(n - \deg p)$. This proves (*a*). (*b*) and (*c*) follow immediately from Lemma 2.3. □

3. Signatures of $\rho^2 A_0 + \rho A_1 + A_2$

In this section we derive a general result for the signature values of a hermitian matrix polynomial $\rho^2 A_0 + \rho A_1 + A_2$, where A_0, A_1, A_2 are complex hermitian $n \times n$-matrices and ρ is a real parameter. The result of main interest for output feedback stabilization is Theorem 3.4, which expresses the signature of $\rho^2 A_0 + \rho A_1 + A_2$ in terms of the Weierstrass invariants of an associated hermitian matrix pencil. *Throughout this section we assume that A_0 is non-singular.*

Associated with $\rho^2 A_0 + \rho A_1 + A_2$ is the regular pencil of Hermitian $2n \times 2n$ matrices

$$\begin{aligned}H(\rho) &= \begin{bmatrix} A_2 + \rho A_1 & \rho A_0 \\ \rho A_0 & -A_0 \end{bmatrix}\\ &= K + \rho L\end{aligned} \tag{3.1}$$

where

$$K = \begin{bmatrix} A_2 & 0 \\ 0 & -A_0 \end{bmatrix}, \quad L = \begin{bmatrix} A_1 & A_0 \\ A_0 & 0 \end{bmatrix} \tag{3.2}$$

Note that L is congruent to

$$\begin{bmatrix} A_0 & 0 \\ 0 & -A_0 \end{bmatrix}$$

and therefore has signature zero.

By the identity

$$H(\rho)=\begin{bmatrix} I & -\rho I \\ 0 & I \end{bmatrix}\begin{bmatrix} \rho^2A_0+\rho A_1+A_2 & 0 \\ 0 & -A_0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -\rho I & I \end{bmatrix} \tag{3.3}$$

$$H(\rho) \quad \text{and} \quad \begin{bmatrix} \rho^2A_0+\rho A_1+A_2 & 0 \\ 0 & -A_0 \end{bmatrix}$$

are congruent and therefore must have the same signature:

$$\boxed{\operatorname{sign}(\rho^2A_0+\rho A_1+A_2)=\operatorname{sign}H(\rho)+\operatorname{sign}A_0} \tag{3.4}$$

for all $\rho \in \mathbb{R}$.

Recall that the *generalized eigenvalues* of a pencil $K+\rho L$ are defined as (minus) the real or complex eigenvalues of the associated complex operator $A=L^{-1}K$. If K, L are defined by (3.2) then

$$L^{-1}K=\begin{bmatrix} 0 & -I_n \\ A_0^{-1}A_2 & A_0^{-1}A_1 \end{bmatrix} \tag{3.5 a}$$

which is similar to

$$A:=\begin{bmatrix} 0 & I \\ -A_0^{-1}A_2 & A_0^{-1}A_1 \end{bmatrix} \tag{3.5 b}$$

The classification of regular pencils of hermitian matrices is due to Weierstrass and is summarized in the following theorem (see Gohberg *et al.* 1983).

Theorem 3.1: *Let K, L be hermitian complex $m\times m$ matrices with L non-singular. Let $\mathcal{J}=\mathcal{J}_r\oplus\mathcal{J}_c\oplus\bar{\mathcal{J}}_c$ denote the Jordan matrix of $L^{-1}K$ with*

$$\mathcal{J}_r=\operatorname{diag}(\mathcal{J}(\lambda_1),\dots,\mathcal{J}(\lambda_k)) \tag{3.6 a}$$

$$\lambda_1\geqslant\cdots\geqslant\lambda_k \text{ real} \tag{3.6 b}$$

$$\mathcal{J}_c=\operatorname{diag}(\mathcal{J}(\lambda_{k+1}),\dots,\mathcal{J}(\lambda_{k+l})) \tag{3.6 c}$$

$$\operatorname{Re}\lambda_{k+1}\geqslant\cdots\geqslant\operatorname{Re}\lambda_{k+l} \tag{3.6 d}$$

$$\operatorname{Im}\lambda_{k+i}>0,\quad i=1,\dots,l \tag{3.6 e}$$

$$\mathcal{J}(\lambda_i)=\begin{bmatrix} \lambda_i & 1 & 0 \\ \ddots & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix} \quad i=1,\dots,k+l \tag{3.6 f}$$

Then there exists a non-singular $m\times m$ matrix $M\in GL(m,\mathbb{C})$ such that

$$MLM^*=\Sigma_{\varepsilon,\mathcal{J}},\quad MKM^*=\Sigma_{\varepsilon,\mathcal{J}}\cdot\mathcal{J} \tag{3.7}$$

where $\sum_{\varepsilon,\mathcal{J}}$ is a canonical matrix defined by $\mathcal{J}$ and a sign characteristic ε in the following way:

$$\sum_{\varepsilon,\mathcal{J}} = \begin{bmatrix} S_r & 0 & 0 \\ 0 & 0 & S_c \\ 0 & S_c & 0 \end{bmatrix} \tag{3.7 a}$$

with

$$S_r = \operatorname{diag}(\varepsilon_1\Sigma_1, \ldots, \varepsilon_k\Sigma_k) \tag{3.7 b}$$

$$S_c = \operatorname{diag}(\Sigma_{k+1}, \ldots, \Sigma_{k+l}) \tag{3.7 c}$$

Here

$$\Sigma_i = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix} \tag{3.7 d}$$

is of the same size as the Jordan block $\mathcal{J}(\lambda_i)$ and $\varepsilon_1, \ldots, \varepsilon_k$ are each equal to $+1$ or -1. The tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ is called the sign characteristic of (K, L) and is uniquely determined to within arbitrary permutations of signs corresponding of equal blocks $\mathcal{J}(\lambda_i)$.

It follows that the congruence classes of regular hermitian pencils $K + \rho L$ are completely classified by the generalized eigenvalues and the sign characteristic ε. Note that the ith sign characteristic ε_i is associated with every *real* generalized eigenvalue $\rho_i = -\lambda_i$ of $K + \rho L$. (There is no sign characteristic associated with the complex eigenvalues.) We say that a real Jordan block $\mathcal{J}(\lambda_i)$ is of *positive type* (*negative type*) if the corresponding sign characteristic ε_i is positive (respectively $\varepsilon_i = -1$).

As an immediate consequence of the above theorem we obtain the following lemma.

Lemma 3.2: *Let $K + \rho L$ be a pencil of $m \times m$ hermitian matrices with L non-singular. Let $(MLM^*, MKM^*) = (\Sigma_{\varepsilon,\mathcal{J}}, \Sigma_{\varepsilon,\mathcal{J}} \cdot \mathcal{J})$ be the Weierstrass canonical form* (3.7). *Then*

(i) $\operatorname{sign} S_r = \operatorname{sign} L$ (3.8)

(ii) *For all* $\rho \in \mathbb{R}$

$$\operatorname{sign}(K + \rho L) = \operatorname{sign} S_r(\rho I + \mathcal{J}_r) \tag{3.9}$$

Remark: By (3.6 *a*), (3.7 *b*),

$$S_r(\rho I + \mathcal{J}_r) = \operatorname{diag}(\varepsilon_1\Sigma_1\mathcal{J}(\lambda_1 + \rho), \ldots, \varepsilon_r\Sigma_k\mathcal{J}(\lambda_k + \rho)) \tag{3.9 a}$$

with

$$\Sigma_i\mathcal{J}(\lambda_i + \rho) = \begin{bmatrix} 0 & & \lambda_i + \rho \\ \cdot & & 1 \\ \lambda_i + \rho & 1 & 0 \end{bmatrix} \tag{3.9 b}$$

of size $n_i \times n_i$, $i = 1, \ldots, k$. Distinguishing between the cases where n_i is even or odd we obtain

$$\operatorname{sign} \Sigma_i \mathcal{J}(\lambda_i + \rho) = \begin{cases} \operatorname{sign}(\lambda_i + \rho) & \text{if } n_i \text{ odd} \\ 0 & \text{if } n_i \text{ even} \end{cases}$$

Thus

$$\operatorname{sign} S_r(\rho I + \mathcal{J}_r) = \sum_{i=1}^{k} \frac{1 - (-1)^{n_i}}{2} \varepsilon_i \operatorname{sign}(\lambda_i + \rho) \tag{3.9 c}$$

which specializes to

$$\operatorname{sign} S_r(\rho I + \mathcal{J}_r) = \sum_{i=1}^{k} \varepsilon_i \operatorname{sign}(\rho + \lambda_i) \tag{3.9 d}$$

if $n_1 = \cdots = n_k = 1$. □

Corollary 3.3: *Let A_0 be non-singular and let $\rho_1 < \cdots < \rho_r$ denote the distinct real roots of $\det(\rho^2 A_0 + \rho A_1 + A_2) = 0$. Let n_{ij}, $j = 1, \ldots, v_i$, denote the sizes of the real Jordan blocks $\mathcal{J}(\lambda_i)$, $\lambda_i = -\rho_i$, occurring in the Jordan matrix of $L^{-1}K$, with K, L as in* (3.2).

Let ε_{ij} denote the corresponding sign characteristics. Then for all $\rho \in \mathbb{R}$

$$\boxed{\operatorname{sign}(\rho^2 A_0 + \rho A_1 + A_2) = \operatorname{sign} A_0 + \sum_{i=1}^{r} \sum_{j=1}^{v_i} \varepsilon_{ij} \frac{1 - (-1)^{n_{ij}}}{2} \operatorname{sign}(\rho - \rho_i)} \tag{3.10}$$

Furthermore

$$\boxed{\sum_{i=1}^{r} \sum_{j=1}^{v_i} \varepsilon_{ij} \frac{1 - (-1)^{n_{ij}}}{2} = 0} \tag{3.11}$$

Proof: (3.10) follows immediately from (3.4) and (3.9). Obviously for A_0 non-singular for $|\rho|$ large enough, $\rho^2 A_0 + \rho A_1 + A_2$ has the same signature as A_0. Equation (3.11) thus follows from (3.10). □

A more precise result on the distribution of the signature values of $\rho^2 A_0 + \rho A_1 + A_2$ is given by the following result. Let

$$\Delta(\rho) := \rho^2 A_0 + \rho A_1 + A_2 \tag{3.12}$$

and let $\rho_1 < \cdots < \rho_r$ denote the real roots of $\det \Delta(\rho) = 0$. Set

$$\delta_i := \sum_{j=1}^{v_i} \varepsilon_{ij} \frac{1 - (-1)^{n_{ij}}}{2} \tag{3.13}$$

where the n_{ij} denote the sizes of the Jordan blocks of the eigenvalues $\lambda_i = -\rho_i$ of $L^{-1}K$ and ε_{ij} denote the associated sign characteristic.

Corollary 3.3 and the following result are the main algebraic tools for the solutions of the output feedback problems (a), (b) given in § 4.

Let $B(q', q)$ be invertible and let $r = \operatorname{sign} B(q', q) \geq 0$ denote the number of real zeros of $q(s)$. Then

(*a*) *For $1/k < \rho_1$ or $1/k > \rho_m$, $q(s) + kp(s)$ has exactly r real roots.*

(*b*) *For $\rho_i < 1/k < \rho_{i+1}$, $q(s) + kp(s)$ has exactly $r + 2(\delta_1 + \cdots + \delta_i)$ real zeros, $i = 1, \ldots, m - 1$.*

(*c*) *There exists $k \in \mathbb{R}$ such that $q + kp$ has n real distinct roots if and only if*

$$n - r = 2\max(\delta_1, \delta_1 + \delta_2, \ldots, \delta_1 + \cdots + \delta_r) \tag{4.2}$$

The above theorem holds without assuming conditions (GA). The main reason for imposing (GA) is to have the simple relation between generalized eigenvalues of (4.1) and breakaway values as expressed in Lemma 4.1. Using Lemma 4.1 and the conditions we therefore obtain the following theorem.

Theorem 4.2′: *Assume (GA) holds. Let $\lambda_1 > \cdots > \lambda_m$ denote the real (non-zero) breakaway values of p/q, $m < 2n$, and let $(\varepsilon_1, \ldots, \varepsilon_m)$, $\sum_1^m \varepsilon_i = 0$, denote the associated sign characteristic of the pencil (4.1). Let $r \geq 0$ denote the number of real zeros of $q(s)$. Then*

(*a*) *For $1/k > -\lambda_m$ or $1/k < -\lambda_1$, $q(s) + kp(s)$ has exactly r real zeros.*

(*b*) *For $-\lambda_i < 1/k < -\lambda_{i+1}$, $q(s) + kp(s)$ has exactly $r + 2(\varepsilon_1 + \cdots + \varepsilon_i)$ real zeros, $i = 1, \ldots, m - 1$.*

(*c*) *There exists $k \in \mathbb{R}$ such that $q + kp$ has n real distinct roots if and only if*

$$n - r = 2\max(\varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \cdots + \varepsilon_m) \tag{4.2′}$$

Corollary 4.3: *A necessary condition for the existence of $k \in \mathbb{R}$ such that $q + kp$ has n real zeros is that p/q has at least $n - r$ real distinct breakaway points, where r is the number of real roots of $q(s) = 0$.*

Remark: If the breakaway values λ_i and the sign characteristics ε_i are known, which are in any case explicitly computable, then Theorem 4.2 implies that the set of points $k \in \mathbb{R}$ for which $q + kp$ has n real distinct roots is a union of intervals given by

$$-\lambda_i < \frac{1}{k} < -\lambda_{i+1}$$

where $n - r = 2(\varepsilon_1 + \cdots + \varepsilon_i)$. In the closed intervals $-\lambda_i \leq 1/k \leq \lambda_{i+1}$, $q + kp$, has n (possibly not distinct) real zeros. □

Theorem 4.4: *Let $g = p/q$ be a transfer function of McMillan degree n, satisfying (GA). Let $r \geq 0$ denote the number of real distinct poles of g and let $\lambda_1 > \cdots > \lambda_m$ and $(\varepsilon_1, \ldots, \varepsilon_m)$ denote the real breakaway values of g and the associated sign characteristic. Then there exists a constant output feedback $k \in \mathbb{R}$ such that $g/(1 + kg)$ has n real poles on the negative real axis if and only if there exist an $1 \leq i \leq 2m$ such that the following conditions are satisfied:*

$$n - r = 2(\varepsilon_1 + \cdots + \varepsilon_i) \tag{4.3 a}$$

$$\textit{The coefficients of } q(s) + kp(s) \textit{ are all positive for } -\lambda_i < \frac{1}{k} < -\lambda_{i+1} \tag{4.3 b}$$

Proof: By Theorem 4.2 there exists $1 \leq i \leq m$ such that (4.3 *a*) holds if and only if one can place the n poles of $g(s)$ somewhere on the real axis. The result now follows from the following simple fact. □

Fact: *Let $\alpha(s)$ be a monic polynomial all of whose roots are real. Then the roots of $\alpha(s)$ are all negative if and only if the coefficients of α are all positive.*

4.2. *Output feedback stabilization*

In order to simplify the subsequent analysis we will sometimes impose the following genericity assumption on g (cf. § 2.1).

Genericity assumptions (GA 2):

(*a*) *q and p have no roots on the imaginary axis, the relative degree is* $\deg q - \deg p = 1$.

(*b*) *The real generalized breakaway values of $g(s)$ are simple.*

Following the analysis of §§ 2 and 3 we consider the pencil of symmetric matrices

$$H(\rho) = K + \rho L$$

$$= \begin{bmatrix} A_2 + \rho A_1 & \rho A_0 \\ \rho A_0 & -A_0 \end{bmatrix} \tag{4.4}$$

where

$$A_2 = B(p_+, p_-) \tag{4.5 a}$$

$$A_1 = B(q_+, p_-) + B(p_+, q_-) \tag{4.5 b}$$

$$A_0 = B(q_+, q_-) \tag{4.5 c}$$

and

$$q(\mathrm{i}\omega) = q_+(\omega) + \mathrm{i}q_-(\omega)$$

$$p(\mathrm{i}\omega) = p_+(\omega) + \mathrm{i}p_-(\omega)$$

Lemma 4.5: $\det H(\rho) = 0$ *if and only if* $\rho = 0$ *or* $\rho \neq 0$ *and* $-\rho$ *is a generalized breakaway value of g. If (GA 2) holds, then* $\rho = 0$ *is a simple zero of* $\det H(\rho)$.

Proof: Since

$$\det H(\rho) = (-1)^n \det A_0 \det(\rho^2 A_0 + \rho A, + A_2)$$
$$= (-1)^n \det A_0 \rho^{2n} \det B(1/\rho)$$

ρ is a non-zero generalized eigenvalue of (K, L) if and only if $\det B(1/\rho) = 0$. But $\det B(1/\rho) = 0$ if and only if there exists $s \in \mathbb{C}$ such that $\rho q_+(s) + p_+(s) = \rho q_-(s) + p_-(s) = 0$, i.e.:

$$\rho = -\frac{p_+(s)}{q_+(s)} = -\frac{p_-(s)}{q_-(s)}$$

Hence s is a generalized breakaway point. Since

$$\frac{p(s)}{q(s)} = \frac{p_+(s)q_+(s) + p_-(s)q_-(s)}{q_+(s)^2 + q_-(s)^2} + \frac{p_+(s)q_-(s) - p_-(s)q_+(s)}{q_+^2(s) + q_+(s)^2}$$

$$= \frac{p_+(s)q_+(s) + p_-(s)q_-(s)}{q_+(s)^2 + q_-(s)^2}$$

$$= -\rho$$

$-\rho$ is a generalized breakaway value. If (GA 2) holds then rank of A_2 is $n-1$. The result follows. □

The following result is an immediate consequence of Theorem 3.4 and formulates the new necessary and sufficient conditions for output feedback stabilization. Here we do not assume (GA 2).

Theorem 4.6: *Let $\rho_1 < \cdots < \rho_m$ denote the real generalized eigenvalues of the pencil* (4.4) *and let n_{ij}, $j = 1, \ldots, v_i$, denote the sizes of the Jordan blocks $J(\lambda_i)$, $\lambda_i = -\rho_i$, occurring in the Jordan matrix of $L^{-1}K$, with K, L as in* (4.4), (4.5). *Let ε_{ij} denote the corresponding sign characteristics and let*

$$\delta_i = \sum_{j=1}^{v_i} \varepsilon_{ij} \frac{1-(-1)^{n_{ij}}}{2}, \quad i = 1, \ldots, m$$

Let $B(q_+, q_-)$ be invertible and let $r = \operatorname{sign} B(q_+, q_-)$ denote the number of zeros of q in the open left half-plane $\mathbb{C}^-$ minus the number of zeros in the open right half-plane $\mathbb{C}^+$. Then

(*a*) *For $1/k < \rho_1$ or $1/k > \rho_m$, $q(s) + kp(s)$ has exactly $(n + r)/2$ eigenvalues in $\mathbb{C}^-$.*

(*b*) *For $\rho_i < 1/k < \rho_{i+1}$, $i = 1, \ldots, m-1$, $q(s) + kp(s)$ has exactly $\delta_1 + \cdots + \delta_i + (n + r)/2$ zeros in $\mathbb{C}^-$.*

(*c*) *$q(s) + kp(s)$ is Hurwitz if and only if there exists $1 \leq i \leq m$ with $\rho_i < 1/k < \rho_{i+1}$ and $\delta_1 + \cdots + \delta_i = (n - r)/2$.*

It is well known that a minimum phase system of relative degree 1 can be stabilized by a sufficiently large positive or negative $k \in \mathbb{R}$. The following more precise version of this result is a simple corollary of Theorem 4.6 and Lemma 4.5.

Corollary 4.7: *Let $g(s) = (p(s)/q(s))$ be a strictly proper rational transfer function satisfying (GA 2) such that $B(p_+, p_-) \geq 0$. Then $q(s) + kp(s)$ is Hurwitz for $k < -(1/\lambda_v) < 0$ if $\varepsilon_{v+1} = -1$, or for $k > (1/\lambda_{v+2}) > 0$ if $\varepsilon_{v+1} = 1$. Here λ_v, λ_{v+2} are the smallest positive and largest negative generalized breakaway values, respectively.*

Proof: With the notation as in Theorem 4.6 we have (see Lemma 3.3) with $\lambda_i := -\rho_i$

$$\sum_{i=1}^{m} \varepsilon_i = 0 \tag{4.6}$$

$$\begin{aligned} \operatorname{sign} K &= \operatorname{sign} A_2 - \operatorname{sign} A_0 \\ &= \sum_{i=1}^{m} \varepsilon_i \operatorname{sign} \lambda_i \\ &= \sum_{i=1}^{v} \varepsilon_i - \sum_{i=v+2}^{m} \varepsilon_i \end{aligned} \tag{4.7}$$

By substituting (4.6) into (4.7)

$$2\sum_{i=1}^{v} \varepsilon_i + \varepsilon_{v+1} = \operatorname{sign} A_2 - \operatorname{sign} A_0$$
$$= n - r - 1$$

Suppose $\varepsilon_{v+1} = -1$. Then $2 \cdot \sum_{i=1}^{v} \varepsilon_i = n - r$ and Theorem 4.6(*c*) implies that $q + kp$ is Hurwitz for $k < -(1/\lambda_v) < 0$. If $\varepsilon_{v+1} = +1$, then $2(\varepsilon_1 + \cdots + \varepsilon_{v+1}) = n - r$ and therefore $q + kp$ is Hurwitz for $k > -(1/\lambda_{v+2}) > 0$. □

Remark: The result indicates that $\varepsilon_{v+1} = -\operatorname{sign}(p_{n-1})$, where p_{n-1} is the leading coefficient of $p(s)$. □

Theorem 4.6 gives a necessary and sufficient algebraic condition for output feedback stabilization, expressed in terms of the sign characteristic of the hermitian pencil (4.4). We now express the sign characteristics of the pencil (4.4) in terms of the frequency response $g(\mathrm{i}\omega)$ of the transfer function.

Let $g(s) = (p(s)/q(s))$ be a strictly proper transfer function and let $\psi_\rho(s) = p(s) + \rho q(s)$ for $\rho \in \mathbb{R}$. Suppose $s_0 \in \mathbb{C}$ is a simple zero of $\psi_{\rho 0}$. By the implicit function theorem, as ρ varies, a root $s = s(\rho)$ of $\psi_\rho(s) = 0$ varies continuously from s_0, and

$$s'(\rho_0) = -\frac{q(s_0)}{\psi'_{\rho 0}(s_0)} \tag{4.8 a}$$

$$s(\rho_0) = s_0 \tag{4.8 b}$$

Now suppose that $s_0 = \sigma_0 + \mathrm{i}\omega_0$ and that we consider pairs (s_0, ρ) where $\sigma_0 = 0$. Let $\psi_\rho(\mathrm{i}s) = \psi_+(s) + \mathrm{i}\psi_-(s)$. Using the Cauchy–Riemann equations, (4.8) implies ($\mathrm{i} = \sqrt{-1}$)

$$\frac{\partial \sigma_0}{\partial \rho} + \mathrm{i}\,\frac{\partial \omega_0}{\partial \rho} = -\,\frac{q_+(\omega_0) + \mathrm{i}q_-(\omega_0)}{\psi'_-(\omega_0) - \mathrm{i}\psi'_+(\omega_0)} \tag{4.9}$$

where the derivates on the left-hand side of (4.9) are to be evaluated at ρ_0.

By Theorem 4.6(*b*) the *i*th sign characteristic ε_i is determined by whether, for ρ increasing across ρ_i, one zero of ψ_ρ moves from the right half-plane to the left half-plane, or *vice versa*. Thus for $1 \leqslant i \leqslant m$

$$\varepsilon_i = -\operatorname{sign}\left|\frac{\partial \sigma_0}{\partial \rho}\right|_{\rho=\rho i} \tag{4.10}$$

Now assume condition (GA 2) holds. Let $\rho_1 < \cdots < \rho_m$, $\rho_v = 0$, denote the m real generalized eigenvalues of the pencil (4.4). Then $\lambda_i = -\rho_i$, $i = 1, \ldots, m$, are the generalized breakaway values of the transfer function $g(s)$ and there exist real distinct negative $\omega_1, \ldots, \omega_{2m}$ with

$$g(\mathrm{i}\omega_j) = \rho_j, \quad j = 1, \ldots, m \tag{4.11 a}$$

$$\omega_v = -\infty \tag{4.11 b}$$

Thus the real generalized breakaway values of $g(s)$ are given by the points where the frequency response curve

$$\omega \mapsto g(\mathrm{i}\omega), \quad \omega \in \mathbb{R}_- \cup \{\infty\}$$

crosses the real axis.

Evidently, the direction in which $\omega \mapsto g(i\omega)$, $\omega \leqslant 0$, crosses the real axis is given by the derivative $(\partial g_-/\partial\omega)$, where

$$g_-(\omega) = \frac{p_-(\omega)q_+(\omega) - p_+(\omega)q_-(\omega)}{q_-(\omega)^2 + q_+(\omega)^2} \tag{4.12}$$

If $g_-(\omega_0) = 0$ then, since $q_-^2 + q_+^2 > 0$ on $\mathbb{R}$ (GA 2(a)),

$$\begin{aligned}\operatorname{sign} \frac{\partial g_-}{\partial\omega}\bigg|_{\omega_0} &= \operatorname{sign} \frac{\partial}{\partial\omega}[p_-q_+ - p_+q_-]\bigg|_{\omega_0} \\ &= \operatorname{sign}[p'_-(\omega_0)q_+(\omega_0) + p_-(\omega_0)q'_+(\omega_0) \\ &\quad - p'_+(\omega_0)q_-(\omega_0) - p_+(\omega_0)q'_-(\omega_0)]\end{aligned} \tag{4.13}$$

From (4.9)

$$\begin{aligned}\operatorname{sign}\left(\frac{\partial\sigma_0}{\partial\rho}\right) &= \operatorname{sign}\operatorname{Re}[(-q_+ - iq_-)(\psi'_- + i\psi'_+)] \\ &= \operatorname{sign}[-q_+(p'_- + \rho q'_-) + q_-(p'_+ + \rho q'_+)]\end{aligned}$$

Now at a generalized breakaway point $i\omega_0$

$$p_+(\omega_0) + \rho q_+(\omega_0) = 0, \quad p_-(\omega_0) + \rho q_-(\omega_0) = 0$$

So

$$\begin{aligned}&\operatorname{sign}\left|\left(\frac{\partial\sigma_0}{\partial\rho}\right)\right. \\ &= \operatorname{sign}[-q_+(\omega_0)p'_-(\omega_0) + p_+(\omega_0)q'_-(\omega_0) + q_-(\omega_0)p'_+(\omega_0) - p_-(\omega_0)q'_+(\omega_0)] \\ &= -\operatorname{sign}\left(\frac{\partial g_-}{\partial\omega}\right)\bigg|_{\omega=\omega_0}\end{aligned} \tag{4.14}$$

Thus we have proved the following result.

Lemma 4.8: *Assume (GA 2) holds. Let* $\rho_1 < \cdots < \rho_{v-1} < \rho_v = 0 < \cdots < \rho_m$ *denote the m real generalized eigenvalues of* (4.4). *Let* $\varepsilon_1, \ldots, \varepsilon_m$ *denote the associated sign characteristics of the pencil* (4.4). *Then there exist m distinct negative real numbers* $\omega_1, \ldots, \omega_m$ $(\omega_v = -\infty)$ *with*

$$g(i\omega_j) = -\rho_j, \quad j = 1, \ldots, m \tag{4.15 a}$$

$$\varepsilon_j = \operatorname{sign} g'_-(\omega_j), \quad j = 1, \ldots, m \tag{4.15 b}$$

Note that the existence of m real distinct numbers $\omega_1, \ldots, \omega_m$, $\omega_v = \infty$, which satisfy (4.15 a) is an immediate consequence of the genericity condition GA 2(b).

Lemma 4.8 and Theorem 4.6 together imply the following necessary and sufficient condition for output feedback stabilization in terms of the frequency response curve $\omega \mapsto g(i\omega)$, which is equivalent to the Nyquist criterion (cf. Oppenheim and Willsky 1983).

Corollary 4.9: *Let $g(s)$ be a strictly proper transfer function of McMillan degree n which satisfies the genericity conditions (GA 2). Let* $\lambda_1 > \cdots > \lambda_m$ *denote the intersection points of the frequency response curve* $\omega \mapsto g(i\omega)$ *with the real axis* *and let signs* $\varepsilon_1, \ldots, \varepsilon_m$ *be defined such that* $\varepsilon_j = +1 (\varepsilon_j = -1)$ *if* $\omega \mapsto g(i\omega)$, $\omega \leq 0$, *crosses the axis at* λ_j *from the lower half-plane (respectively from the upper half-plane to the lower half-plane). Let* $r = \operatorname{sign} B(q_+, q_-)$ *denote the number of zeros of* $q(s)$ *in the open left half-plane minus the number of zeros of* $q(s)$ *in the open right half-plane. Then there exists* $k \in \mathbb{R}$ *such that* $q(s) + kp(s)$ *is Hurwitz if and only if there exists* $1 \leq i \leq m$ *such that* $\varepsilon_1 + \cdots + \varepsilon_i = (n - r)/2$.

Remark: The above conditions (GA) and (GA 2) were assumed mainly in order to simplify the relation between generalized eigenvalues and breakaway values. Our main results, Theorems 4.2 and 4.6, did not require such assumptions. It would be desirable to have general versions of Lemma 4.8 and Corollary 4.9 without requiring (GA 2). □

Example: To illustrate the above result (Corollary 4.9) let us consider the strictly proper transfer function of McMillan degree 6

$$g(s) = \frac{(s + 0{\cdot}1)(s + 0{\cdot}5)(s + 3)(s + 4)(s - 1)}{(s - 0{\cdot}2)(s + 1)(s + 2)(s + 5)(s + 6)(s - 2)} \tag{4.16}$$

$g(s)$ has four stable and two unstable poles, so its difference is $r = 2$ and thus $(n - r)/2 = 2$. The Nyquist plot $\omega \mapsto g(i\omega)$, $\omega \in \mathbb{R}$ is shown in Fig. 2 and we obtain the following (approximate) values of the real generalized breakaway values $\lambda_1 > \lambda_2$ and sign characteristics ε_1, ε_2:

$$\lambda_1 = 0{\cdot}025 \qquad \varepsilon_1 = 1$$

$$\lambda_2 = 0 \qquad \varepsilon_2 = -1$$

Thus the condition of Corollary 4.9 is not satisfied and we conclude that $g(s)$ is not stabilizable by static output feedback. Applying Theorem 4.6 we find that $g(s)$ has four stable poles for $k > -40$ and five stable poles for $k < -40$.

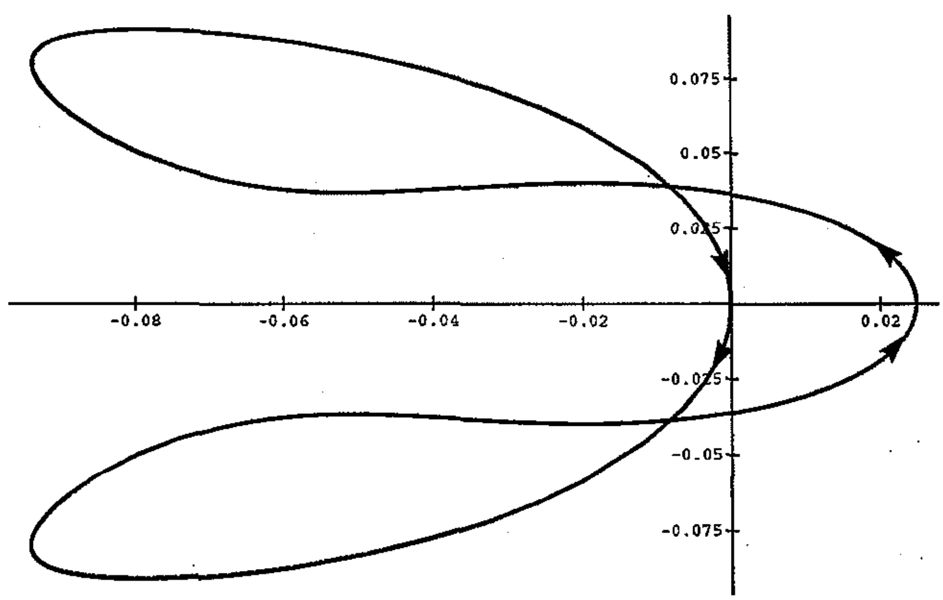

Figure 2. Nyquist plot for $g(s)$.

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